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GENERALIZATIONS OF MILNE'S $U(n+1)$ q-CHU-VANDERMONDE SUMMATION

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Abstract. We derive two identities for multiple basic hyper-geometric series associated with the unitary $U(n+1)$ group. In order to get the two identities, we first present two known q-exponential operator identities which were established in our earlier paper. From the two identities and combining them with the two $U(n+1)$ q-Chu-Vandermonde summations established by Milne, we arrive at our results. Using the identities obtained in this paper, we give two interesting identities involving binomial coefficients. In addition, we also derive two nontrivial summation equations from the two multiple extensions.

Keywords: $U(n+1)$ group; multiple basic hypergeometric series; basic hypergeometric series

MSC 2010: 33D80, 33D70, 33C80, 11B65, 15A09

1. Introduction and main results

The importance of the q-analogue of the basic hypergeometric series in $U(n)$ was first discussed by Andrews in [1]. Since the multiple basic hypergeometric series associated with the unitary $U(n+1)$ group was systematically studied by Milne [16], it has been studied by many researchers, who have produced much literature about it. For instance, the authors ([2], [6], [11], [12], [13], [15], [17], [18], [22], [21]) made a systematic study on it. Wang [23] applied the q-Beta integral transformation to obtain several generalizations of Milne's $U(n+1)$ q-binomial theorems. Zhang [24] gave

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several $U(n+1)$ generalizations of the Kalnins-Miller transformations by applying q-exponential operators which were constructed by Rogers [19], [20], and developed by Carlitz [4], Chen and Liu [5], Liu [14] and Bowman [3]. Mainly inspired by [15], [23], [24], we will focus on the generalizations of the following Milne's $U(n+1)$ q-Chu-Vandermonde formulas which were presented as Theorem 5.12 and Theorem 5.36 (cf. [15]):

Let b, c and $x_1,\dots,x_n$ be indeterminate, and let N_i be nonnegative integers for $i=1,2,\dots,n$; $e_2(y_1,\dots,y_n)$ is the second elementary symmetric function of $\{y_1,\dots,y_n\}$, and we suppose that none of the denominators vanishes:

$$\tag{1} \prod_{i=1}^{n}\frac{(\frac{x_i}{x_n}\frac{c}{b};q)_{N_i}}{(\frac{x_i}{x_n}c;q)_{N_i}} = \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\dots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \times\prod_{i=1}^{n}\Big(\frac{x_i}{x_n}\Big)^{y_i}\Big(\frac{cq^{\mathcal{N}_n}}{b}\Big)^{\mathcal{Y}_n}\prod_{r,s=1}^{n}\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(\frac{x_r}{x_s}q;q)_{y_r}} \times\prod_{i=1}^{n}\Big(\frac{x_i}{x_n}c;q\Big)_{y_i}^{-1}(b;q)_{\mathcal{Y}_n}q^{y_2+2y_3+\dots+(n-1)y_n-e_2(y_1,\dots,y_n)}\Bigg\}$$

and

$$\tag{2} \prod_{i=1}^{n}\Big(\frac{x_n}{x_i}\frac{cq^{N_n-N_i}}{b};q\Big)_{N_i} = \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\dots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \times\prod_{i=1}^{n}\Big(\frac{x_n}{x_i}\Big)^{y_i}\Big(\frac{cq^{\mathcal{N}_n}}{b}\Big)^{\mathcal{Y}_n}\prod_{r,s=1}^{n}\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(\frac{x_r}{x_s}q;q)_{y_r}} \times\prod_{i=1}^{n}\frac{(\frac{x_n}{x_i}cq^{N_n-N_i}q^{\mathcal{Y}_n-y_i};q)_{N_i}}{(\frac{x_n}{x_i}cq^{N_n-N_i}q^{\mathcal{Y}_n-y_i};q)_{y_i}} \times(b;q)_{\mathcal{Y}_n}q^{y_2+2y_3+\dots+(n-1)y_n+e_2(y_1,\dots,y_n)}\Bigg\}.$$

We adopt the notation used in [10]. Throughout the paper unless otherwise stated we assume that $0<|q|<1$. For any complex parameter a, the q-shifted factorials are defined as

$$\tag{3} (a;q)_0=1,\quad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\ \ n=1,2,\dots,\quad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k).$$

For brevity, we also use the notation

$$\tag{4} (a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\dots(a_m;q)_n,\quad \mathcal{N}_n=\sum_{i=1}^{n}N_i,\mathcal{Y}_n=\sum_{i=1}^{n}y_i.$$

The q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}. \tag{5}$$

The q-differential operator D_q and the q-shifted operator η, acting on the variable a, are defined as (cf. [5], [6], [7], [8], [9], [14], [19], [20], [24])

$$D_q\{f(a)\} = \frac{f(a)-f(aq)}{a} \quad \text{and} \quad \eta\{f(a)\} = f(aq). \tag{6}$$

The basic hypergeometric series ${}_s\Phi_t$ is given as

$${}_s\Phi_t\left(\begin{matrix} a_1,\ a_2,\ \ldots,\ a_s \\ b_1,\ b_2,\ \ldots,\ b_t \end{matrix};\ q,\ x\right) = \sum_{k=0}^{\infty} \frac{(a_1,a_2,\ldots,a_s;q)_k}{(q,b_1,\ldots,b_t;q)_k} \big[(-1)^k q^{\binom{k}{2}}\big]^{1+t-s} x^k, \tag{7}$$

where $s,t = 0,1,2,\ldots$ The main results of this paper are stated as follows:

Theorem 1.1. *Let b, c, d, e, x, y and $x_1,\ldots,x_n,a_1,\ldots,a_{2t}$ be indeterminate, let N_i be nonnegative integers for $i=1,2,\ldots,n$ with $n \geqslant 1$, and suppose that none of the denominators in (8) vanishes. For $|e| < \min\{|x|,|y|\}$, $|a_{2j}| < 1$, $j=1,2,\ldots,t$, $e_2(y_1,\ldots,y_n)$ being the second elementary symmetric function of $\{y_1,\ldots,y_n\}$, we have*

$$\begin{aligned} &\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}} \Bigg\{ \prod_{1\leqslant r<s\leqslant n} \frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \prod_{r,s=1}^{n} \frac{\left(\frac{x_r}{x_s}q^{-N_s};q\right)_{y_r}}{\left(\frac{x_r}{x_s}q;q\right)_{y_r}} \prod_{i=1}^{n} \left(\frac{x_i}{x_n}cx;q\right)_{y_i}^{-1} \\ &\quad\times \prod_{i=1}^{n} \left(\frac{x_i}{x_n}\right)^{y_i} (q^{\mathcal{N}_n})^{\mathcal{Y}_n} q^{y_2+2y_3+\ldots+(n-1)y_n-e_2(y_1,\ldots,y_n)} \\ &\quad\times \frac{\left(\frac{x}{b},\frac{x}{d},\frac{x}{a_1},\ldots,\frac{x}{a_{2t-1}};q\right)_{\mathcal{Y}_n}}{\left(\frac{x}{e},\frac{x}{a_2},\frac{x}{a_4},\ldots,\frac{x}{a_{2t}};q\right)_{\mathcal{Y}_n}} \left(\frac{cbda_1a_3\ldots a_{2t-1}}{ea_2a_4\ldots a_{2t}}\right)^{\mathcal{Y}_n} \Bigg\} \\ &= \prod_{i=1}^{n} \frac{\left(\frac{x_i}{x_n}cy;q\right)_{N_i}}{\left(\frac{x_i}{x_n}cx;q\right)_{N_i}} \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}} \Bigg\{ \prod_{1\leqslant r<s\leqslant n} \frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \prod_{r,s=1}^{n} \frac{\left(\frac{x_r}{x_s}q^{-N_s};q\right)_{y_r}}{\left(\frac{x_r}{x_s}q;q\right)_{y_r}} \\ &\quad\times \prod_{i=1}^{n} \left(\frac{x_i}{x_n}cy;q\right)_{y_i}^{-1} (cbq^{\mathcal{N}_n})^{\mathcal{Y}_n} q^{y_2+2y_3+\ldots+(n-1)y_n-e_2(y_1,\ldots,y_n)} \left(\frac{y}{b};q\right)_{\mathcal{Y}_n} \\ &\quad\times \sum_{0\leqslant j\leqslant \mathcal{Y}_n} \frac{\left(q^{-\mathcal{Y}_n},\frac{x}{b},\frac{d}{e};q\right)_j q^j}{\left(q,\frac{x}{e},\frac{y}{b};q\right)_j} \sum_{0\leqslant j_t\leqslant\ldots\leqslant j_0} \prod_{i=1}^{t} \frac{\left(q^{-j_{i-1}},\frac{a_{2i-1}}{a_{2i}},\frac{x}{a_{2i-3}};q\right)_{j_i} q^{j_i}}{\left(q,\frac{x}{a_{2i}},\frac{q^{1-j_{i-1}}a_{2i-2}}{a_{2i-3}};q\right)_{j_i}} \Bigg\}, \end{aligned} \tag{8}$$

where $a_{-1}=d$, $a_0=e$, $j_0=j$, and t is a nonnegative integer.

Theorem 1.2. *Let b, c, d, e, x, y and $x_1,\ldots,x_n,a_1,\ldots,a_{2t}$ be indeterminate, let N_i be nonnegative integers for $i=1,2,\ldots,n$ with $n\geqslant 1$, and suppose that none of the denominators in (9) vanishes. For $|e|<\min\{|x|,|y|\}$, $|a_{2j}|<1$, $j=1,2,\ldots,t$, $e_2(y_1,\ldots,y_n)$ being the second elementary symmetric function of $\{y_1,\ldots,y_n\}$, we have*

$$\begin{aligned}(9)\quad &\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\prod_{i=1}^{n}\Big(\frac{x_n}{x_i}\Big)^{y_i}(q^{N_n})^{\mathcal{Y}_n}\prod_{r,s=1}^{n}\frac{\big(\frac{x_r}{x_s}q^{-N_s};q\big)_{y_r}}{\big(\frac{x_r}{x_s}q;q\big)_{y_r}}\\ &\times\prod_{i=1}^{n}\frac{\big(\frac{x_n}{x_i}cxq^{N_n-N_i}q^{\mathcal{Y}_n-y_i};q\big)_{N_i}}{\big(\frac{x_n}{x_i}cxq^{N_n-N_i}q^{\mathcal{Y}_n-y_i};q\big)_{y_i}}q^{y_2+2y_3+\ldots+(n-1)y_n+e_2(y_1,\ldots,y_n)}\\ &\times\frac{\big(\frac{x}{b},\frac{x}{d},\frac{x}{a_1},\ldots,\frac{x}{a_{2t-1}};q\big)_{\mathcal{Y}_n}}{\big(\frac{x}{e},\frac{x}{a_2},\frac{x}{a_4},\ldots,\frac{x}{a_{2t}};q\big)_{\mathcal{Y}_n}}\Big(\frac{cbda_1a_3\ldots a_{2t-1}}{ea_2a_4\ldots a_{2t}}\Big)^{\mathcal{Y}_n}\Bigg\}\\ =&\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\prod_{i=1}^{n}\Big(\frac{x_n}{x_i}\Big)^{y_i}(bcq^{N_n})^{\mathcal{Y}_n}\prod_{r,s=1}^{n}\frac{\big(\frac{x_r}{x_s}q^{-N_s};q\big)_{y_r}}{\big(\frac{x_r}{x_s}q;q\big)_{y_r}}\\ &\times\prod_{i=1}^{n}\frac{\big(\frac{x_n}{x_i}cyq^{N_n-N_i}q^{\mathcal{Y}_n-y_i};q\big)_{N_i}}{\big(\frac{x_n}{x_i}cyq^{N_n-N_i}q^{\mathcal{Y}_n-y_i};q\big)_{y_i}}q^{y_2+2y_3+\ldots+(n-1)y_n+e_2(y_1,\ldots,y_n)}\Big(\frac{y}{b};q\Big)_{\mathcal{Y}_n}\\ &\sum_{0\leqslant j\leqslant\mathcal{Y}_n}\frac{\big(q^{-\mathcal{Y}_n},\frac{x}{b},\frac{d}{e};q\big)_j q^j}{\big(q,\frac{x}{e},\frac{y}{b};q\big)_j}\sum_{0\leqslant j_t\leqslant\ldots\leqslant j_0=j}\prod_{i=1}^{t}\frac{\big(q^{-j_{i-1}},\frac{a_{2i-1}}{a_{2i}},\frac{x}{a_{2i-3}};q\big)_{j_i}q^{j_i}}{\big(q,\frac{x}{a_{2i}},\frac{q^{1-j_{i-1}}a_{2i-2}}{a_{2i-3}};q\big)_{j_i}}\Bigg\},\end{aligned}$$

where $a_{-1}=d$, $a_0=e$, and t is a nonnegative integer.

Remark. Throughout the paper, convergence of the series is no issue at all because they are terminating series.

2. Lemmas and proofs

In this section, we will apply the q-exponential operator

$$(10)\qquad W(b;c\theta):={}_1\Phi_0\left(\begin{matrix}b\\ -\end{matrix};q,-c\theta\right)=\sum_{n=0}^{\infty}\frac{(b;q)_n(-c\theta)^n}{(q;q)_n}$$

which is constructed by us (cf. [7], [8], [9]) to obtain the results. For convenience, we will use $W(b;c\theta)_a$ to denote the operator (10) acting on the variable a in this paper.

In order to complete our proof, we need to use the following known identity which was established in our earlier papers [8], [9]:

Lemma 2.1 ([9], Theorem 1.1 or [8], Lemma 2.1). *If* $|cst/\omega| < 1$, $s/\omega = q^{-n}$, *and* n *is a nonnegative integer, then*

$$W(b;c\theta)_a\Big\{\frac{(as,at;q)_\infty}{(a\omega;q)_\infty}\Big\} = \frac{(as,at,bct;q)_\infty}{(a\omega,ct;q)_\infty}\,{}_3\Phi_2\left(\begin{matrix} b, \frac{s}{\omega}, \frac{q}{at} \\ \frac{q}{ct}, \frac{q}{a\omega}\end{matrix};q,q\right). \tag{11}$$

Taking $n = 0$ in the above lemma, then replacing s by t, we have

Lemma 2.2. *If* $|cs| < 1$, *then*

$$W(b;c\theta)_a\{(as;q)_\infty\} = \frac{(as,bcs;q)_\infty}{(cs;q)_\infty}. \tag{12}$$

Proof. We will start our proof by the following steps.

Proof of Theorem 1.1. Replacing (b,c) by (bx,cx) and (by,cy) in (1), then comparing the two identities obtained, we get

$$\begin{aligned}(13)\quad &\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\dots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\prod_{r,s=1}^{n}\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{(\frac{x_r}{x_s}q;q)_{y_r}}\prod_{i=1}^{n}\Big(\frac{x_i}{x_n}cx;q\Big)^{-1}_{y_i}\\
&\times\prod_{i=1}^{n}\Big(\frac{x_i}{x_n}\Big)^{y_i}\Big(\frac{cq^{\mathcal{N}_n}}{b}\Big)^{\mathcal{Y}_n}(bx;q)_{\mathcal{Y}_n}q^{y_2+2y_3+\ldots+(n-1)y_n-e_2(y_1,\dots,y_n)}\Bigg\}\\
&=\prod_{i=1}^{n}\frac{\big(\frac{x_i}{x_n}cy;q\big)_{N_i}}{\big(\frac{x_i}{x_n}cx;q\big)_{N_i}}\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\dots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\prod_{r,s=1}^{n}\frac{\big(\frac{x_r}{x_s}q^{-N_s};q\big)_{y_r}}{\big(\frac{x_r}{x_s}q;q\big)_{y_r}}\\
&\times\prod_{i=1}^{n}\Big(\frac{x_i}{x_n}cy;q\Big)^{-1}_{y_i}\prod_{i=1}^{n}\Big(\frac{x_i}{x_n}\Big)^{y_i}\Big(\frac{cq^{\mathcal{N}_n}}{b}\Big)^{\mathcal{Y}_n}\\
&\times(by;q)_{\mathcal{Y}_n}q^{y_2+2y_3+\ldots+(n-1)y_n-e_2(y_1,\dots,y_n)}\Bigg\}.\end{aligned}$$

Letting $b\to 1/b$, we rewrite (13) as

$$\begin{aligned}(14)\quad &\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\dots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\prod_{r,s=1}^{n}\frac{\big(\frac{x_r}{x_s}q^{-N_s};q\big)_{y_r}}{\big(\frac{x_r}{x_s}q;q\big)_{y_r}}\\
&\times\prod_{i=1}^{n}\Big(\frac{x_i}{x_n}cx;q\Big)^{-1}_{y_i}\prod_{i=1}^{n}\Big(\frac{x_i}{x_n}\Big)^{y_i}\\
&\times\big(-cxq^{\mathcal{N}_n}\big)^{\mathcal{Y}_n}q^{y_2+2y_3+\ldots+(n-1)y_n-e_2(y_1,\dots,y_n)}q^{\binom{\mathcal{Y}_n}{2}}\Big(q^{1-\mathcal{Y}_n}\frac{b}{x};q\Big)_\infty\Bigg\}\end{aligned}$$

$$= \prod_{i=1}^{n} \frac{\left(\frac{x_i}{x_n}cy;q\right)_{N_i}}{\left(\frac{x_i}{x_n}cx;q\right)_{N_i}} \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\dots,n}} \Bigg\{ \prod_{1\leqslant r<s\leqslant n} \frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}$$

$$\times \prod_{r,s=1}^{n} \frac{\left(\frac{x_r}{x_s}q^{-N_s};q\right)_{y_r}}{\left(\frac{x_r}{x_s}q;q\right)_{y_r}} \prod_{i=1}^{n}\left(\frac{x_i}{x_n}cy;q\right)_{y_i}^{-1}\left(-cyq^{\mathcal{N}_n}\right)^{\mathcal{Y}_n}$$

$$\times q^{y_2+2y_3+\ldots+(n-1)y_n-e_2(y_1,\dots,y_n)} q^{\binom{\mathcal{Y}_n}{2}} \frac{\left(q^{1-\mathcal{Y}_n}\frac{b}{y}, q\frac{b}{x};q\right)_\infty}{\left(q\frac{b}{y};q\right)_\infty} \Bigg\}.$$

Applying the operator $W(d;e\theta)_b$ to both sides of (14) and using (11) and (12), we have

$$(15)\quad \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\dots,n}} \Bigg\{ \prod_{1\leqslant r<s\leqslant n} \frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \prod_{r,s=1}^{n} \frac{\left(\frac{x_r}{x_s}q^{-N_s};q\right)_{y_r}}{\left(\frac{x_r}{x_s}q;q\right)_{y_r}}$$

$$\times \prod_{i=1}^{n}\left(\frac{x_i}{x_n}cx;q\right)_{y_i}^{-1} \prod_{i=1}^{n}\left(\frac{x_i}{x_n}\right)^{y_i}$$

$$\times \left(cbdq^{\mathcal{N}_n}\right)^{\mathcal{Y}_n} q^{y_2+2y_3+\ldots+(n-1)y_n-e_2(y_1,\dots,y_n)} \frac{\left(\frac{x}{b},\frac{x}{de};q\right)_{\mathcal{Y}_n}}{\left(\frac{x}{e};q\right)_{\mathcal{Y}_n}} \Bigg\}$$

$$= \prod_{i=1}^{n} \frac{\left(\frac{x_i}{x_n}cy;q\right)_{N_i}}{\left(\frac{x_i}{x_n}cx;q\right)_{N_i}} \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\dots,n}} \Bigg\{ \prod_{1\leqslant r<s\leqslant n} \frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}$$

$$\times \prod_{r,s=1}^{n} \frac{\left(\frac{x_r}{x_s}q^{-N_s};q\right)_{y_r}}{\left(\frac{x_r}{x_s}q;q\right)_{y_r}} \prod_{i=1}^{n}\left(\frac{x_i}{x_n}cy;q\right)_{y_i}^{-1}\left(cbq^{\mathcal{N}_n}\right)^{\mathcal{Y}_n}$$

$$\times q^{y_2+2y_3+\ldots+(n-1)y_n-e_2(y_1,\dots,y_n)} \left(\frac{y}{b};q\right)_{\mathcal{Y}_n} {}_3\Phi_2\left(\begin{matrix} q^{-\mathcal{Y}_n}, & d, & \frac{x}{b} \\ & \frac{y}{b}, & \frac{x}{e}\end{matrix};q,q\right) \Bigg\}.$$

Letting $d \to d/e$, we rewrite (15) as

$$(16)\quad \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\dots,n}} \Bigg\{ \prod_{1\leqslant r<s\leqslant n} \frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \prod_{r,s=1}^{n} \frac{\left(\frac{x_r}{x_s}q^{-N_s};q\right)_{y_r}}{\left(\frac{x_r}{x_s}q;q\right)_{y_r}} \prod_{i=1}^{n}\left(\frac{x_i}{x_n}cx;q\right)_{y_i}^{-1}$$

$$\times \prod_{i=1}^{n}\left(\frac{x_i}{x_n}\right)^{y_i}\left(\frac{cbq^{\mathcal{N}_n}}{e}\right)^{\mathcal{Y}_n} q^{y_2+2y_3+\ldots+(n-1)y_n-e_2(y_1,\dots,y_n)}$$

$$\times \frac{(-1)^{\mathcal{Y}_n}\left(\frac{x}{b};q\right)_{\mathcal{Y}_n}}{\left(\frac{x}{e};q\right)_{\mathcal{Y}_n}} q^{-\binom{\mathcal{Y}_n}{2}}\left(\frac{dq^{1-\mathcal{Y}_n}}{x};q\right)_\infty \Bigg\}$$

$$= \prod_{i=1}^{n} \frac{\left(\frac{x_i}{x_n}cy;q\right)_{N_i}}{\left(\frac{x_i}{x_n}cx;q\right)_{N_i}} \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\dots,n}} \Bigg\{ \prod_{1\leqslant r<s\leqslant n} \frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}$$

$$\times \prod_{r,s=1}^{n} \frac{\left(\frac{x_r}{x_s} q^{-N_s}; q\right)_{y_r}}{\left(\frac{x_r}{x_s} q; q\right)_{y_r}} \prod_{i=1}^{n} \left(\frac{x_i}{x_n} cy; q\right)_{y_i}^{-1}$$
$$\times \left(cbq^{\mathcal{N}_n}\right)^{\mathcal{Y}_n} q^{y_2+\ldots+(n-1)y_n - e_2(y_1,\ldots,y_n)}$$
$$\times \left(\frac{y}{b}; q\right)_{\mathcal{Y}_n} \sum_{0\leqslant j\leqslant \mathcal{Y}_n} \frac{\left(q^{-\mathcal{Y}_n}, \frac{x}{b}; q\right)_j q^j}{(q, \frac{x}{e}, \frac{y}{b}; q)_j} \frac{\left(\frac{d}{e}, \frac{dq}{x}; q\right)_\infty}{\left(\frac{dq^j}{e}; q\right)_\infty} \Bigg\}.$$

Applying the operator $W(a_1; a_2\theta)_d$ to both sides of (16), applying (11) and (12), then letting $a_1 \to a_1/a_2$, we have

$$(17) \quad \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}} \Bigg\{ \prod_{1\leqslant r<s\leqslant n} \frac{1-\frac{x_r}{x_s} q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \prod_{r,s=1}^{n} \frac{\left(\frac{x_r}{x_s} q^{-N_s}; q\right)_{y_r}}{\left(\frac{x_r}{x_s} q; q\right)_{y_r}}$$
$$\times \prod_{i=1}^{n} \left(\frac{x_i}{x_n} cx; q\right)_{y_i}^{-1} \prod_{i=1}^{n} \left(\frac{x_i}{x_n}\right)^{y_i} \left(\frac{cbda_1 q^{\mathcal{N}_n}}{ea_2}\right)^{\mathcal{Y}_n}$$
$$\times q^{y_2+2y_3+\ldots+(n-1)y_n - e_2(y_1,\ldots,y_n)} \frac{\left(\frac{x}{b}; q\right)_{\mathcal{Y}_n}}{\left(\frac{x}{e}; q\right)_{\mathcal{Y}_n}} \frac{\left(\frac{x}{d}, \frac{x}{a_1}; q\right)_{\mathcal{Y}_n}}{\left(\frac{x}{a_2}\right)_{\mathcal{Y}_n}} \Bigg\}$$
$$= \prod_{i=1}^{n} \frac{\left(\frac{x_i}{x_n} cy; q\right)_{N_i}}{\left(\frac{x_i}{x_n} cx; q\right)_{N_i}} \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}} \Bigg\{ \prod_{1\leqslant r<s\leqslant n} \frac{1-\frac{x_r}{x_s} q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \prod_{r,s=1}^{n} \frac{(\frac{x_r}{x_s} q^{-N_s}; q)_{y_r}}{(\frac{x_r}{x_s} q; q)_{y_r}}$$
$$\times \prod_{i=1}^{n} \left(\frac{x_i}{x_n} cy; q\right)_{y_i}^{-1} \left(cbq^{\mathcal{N}_n}\right)^{\mathcal{Y}_n} q^{y_2+\ldots+(n-1)y_n - e_2(y_1,\ldots,y_n)} \left(\frac{y}{b}; q\right)_{\mathcal{Y}_n}$$
$$\times \sum_{0\leqslant j\leqslant \mathcal{Y}_n} \frac{\left(q^{-\mathcal{Y}_n}, \frac{x}{b}, \frac{d}{e}; q\right)_j q^j}{(q, \frac{x}{e}, \frac{y}{b}; q)_j} \, {}_3\Phi_2 \left(\begin{matrix} q^{-j}, & \frac{a_1}{a_2}, & \frac{x}{d} \\ & \frac{x}{a_2}, & \frac{e}{d} q^{1-j} \end{matrix} ; q, q \right) \Bigg\}.$$

The equation (8) follows by induction. □

Proof of Theorem 1.2. Replacing (b, c) by (bx, cx) and (by, cy) in (2), then comparing the two identities, we get

$$(18) \quad \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}} \Bigg\{ \prod_{1\leqslant r<s\leqslant n} \frac{1-\frac{x_r}{x_s} q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \prod_{i=1}^{n} \left(\frac{x_n}{x_i}\right)^{y_i} \left(\frac{cq^{N_n}}{b}\right)^{\mathcal{Y}_n}$$
$$\times \prod_{r,s=1}^{n} \frac{\left(\frac{x_r}{x_s} q^{-N_s}; q\right)_{y_r}}{\left(\frac{x_r}{x_s} q; q\right)_{y_r}} \prod_{i=1}^{n} \frac{\left(\frac{x_n}{x_i} cxq^{N_n-N_i} q^{\mathcal{Y}_n - y_i}; q\right)_{N_i}}{\left(\frac{x_n}{x_i} cxq^{N_n-N_i} q^{\mathcal{Y}_n - y_i}; q\right)_{y_i}}$$
$$\times (bx; q)_{\mathcal{Y}_n} q^{y_1+2y_2+\ldots+(n-1)y_n + e_2(y_1,\ldots,y_n)} \Bigg\}$$

$$= \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}} \Bigg\{ \prod_{1\leqslant r<s\leqslant n} \frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \prod_{i=1}^{n} \left(\frac{x_n}{x_i}\right)^{y_i} \left(\frac{cq^{N_n}}{b}\right)^{\mathcal{Y}_n}$$

$$\times \prod_{r,s=1}^{n} \frac{\left(\frac{x_r}{x_s}q^{-N_s};q\right)_{y_r}}{\left(\frac{x_r}{x_s}q;q\right)_{y_r}} \prod_{i=1}^{n} \frac{\left(\frac{x_n}{x_i}cyq^{N_n-N_i}q^{\mathcal{Y}_n-y_i};q\right)_{N_i}}{\left(\frac{x_n}{x_i}cyq^{N_n-N_i}q^{\mathcal{Y}_n-y_i};q\right)_{y_i}}$$

$$\times (by;q)_{\mathcal{Y}_n} q^{y_1+2y_2+\ldots+(n-1)y_n+e_2(y_1,\ldots,y_n)} \Bigg\}.$$

Then similarly to the proof of Theorem 1.1, we complete the proof. □

Remark 2.1. Setting $b \to 1/b$, and letting $d = e = a_1 = \ldots = a_{2t} = 0$, then setting $x = 1$, $y = 1/b$ in (8) and (9) we come back to Milne's formulas (1) and (2), respectively.

3. Some special cases

Setting $t = 0$, replacing (b,d,e) by $(1/b,1/d,1/e)$, then letting $e = bdy$ in (8), we get

Corollary 3.1 ([24], Theorem 3.4)**.** *Let b, c, d, x, y and $x_1,\ldots,x_n$ be indeterminate, let N_i be nonnegative integers for $i = 1,2,\ldots,n$ with $n \geqslant 1$, and suppose that none of the denominators in* (19) *vanishes. For $e_2(y_1,\ldots,y_n)$, the second elementary symmetric function of $\{y_1,\ldots,y_n\}$, we have*

$$\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}} \Bigg\{ \prod_{1\leqslant r<s\leqslant n} \frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \prod_{r,s=1}^{n} \frac{\left(\frac{x_r}{x_s}q^{-N_s};q\right)_{y_r}}{\left(\frac{x_r}{x_s}q;q\right)_{y_r}} \tag{19}$$

$$\times \prod_{i=1}^{n} \left(\frac{x_i}{x_n}cx;q\right)_{y_i}^{-1} \prod_{i=1}^{n} \left(\frac{x_i}{x_n}\right)^{y_i} \left(cyq^{\mathcal{N}_n}\right)^{\mathcal{Y}_n}$$

$$\times q^{y_2+2y_3+\ldots+(n-1)y_n-e_2(y_1,\ldots,y_n)} \frac{(bx,dx;q)_{\mathcal{Y}_n}}{(bdxy;q)_{\mathcal{Y}_n}} \Bigg\}$$

$$= \prod_{i=1}^{n} \frac{\left(\frac{x_i}{x_n}cy;q\right)_{N_i}}{\left(\frac{x_i}{x_n}cx;q\right)_{N_i}} \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}} \Bigg\{ \prod_{1\leqslant r<s\leqslant n} \frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}$$

$$\times \prod_{r,s=1}^{n} \frac{\left(\frac{x_r}{x_s}q^{-N_s};q\right)_{y_r}}{\left(\frac{x_r}{x_s}q;q\right)_{y_r}} \prod_{i=1}^{n} \left(\frac{x_i}{x_n}cy;q\right)_{y_i}^{-1} \left(cxq^{\mathcal{N}_n}\right)^{\mathcal{Y}_n}$$

$$\times q^{y_2+2y_3+\ldots+(n-1)y_n-e_2(y_1,\ldots,y_n)} \frac{(by,dy;q)_{\mathcal{Y}_n}}{(bdxy;q)_{\mathcal{Y}_n}} \Bigg\}.$$

Setting $t = 0$, replacing (b, d, e) by $(1/b, 1/d, 1/e)$, then letting $e = bdy$ in (9), we find

Corollary 3.2 ([24], Theorem 3.16)**.** *Let b, c, d, x, y and $x_1, \ldots, x_n$ be indeterminate, let N_i be nonnegative integers for $i = 1, 2, \ldots, n$ with $n \geqslant 1$, and suppose that none of the denominators in (20) vanishes. For $e_2(y_1, \ldots, y_n)$, the second elementarily symmetric function of $\{y_1, \ldots, y_n\}$, we have*

$$
\begin{aligned}
(20)\qquad & \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}} \Bigg\{ \prod_{1\leqslant r<s\leqslant n} \frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \prod_{i=1}^{n}\Big(\frac{x_n}{x_i}\Big)^{y_i}\big(cyq^{N_n}\big)^{\mathcal{Y}_n} \\
& \times \prod_{r,s=1}^{n} \frac{\big(\frac{x_r}{x_s}q^{-N_s};q\big)_{y_r}}{\big(\frac{x_r}{x_s}q;q\big)_{y_r}} \prod_{i=1}^{n} \frac{\big(\frac{x_n}{x_i}cxq^{N_n-N_i}q^{\mathcal{Y}_n-y_i};q\big)_{N_i}}{\big(\frac{x_n}{x_i}cxq^{N_n-N_i}q^{\mathcal{Y}_n-y_i};q\big)_{y_i}} \\
& \times q^{y_2+2y_3+\ldots+(n-1)y_n+e_2(y_1,\ldots,y_n)} \frac{(bx,dx;q)_{\mathcal{Y}_n}}{(bdxy;q)_{\mathcal{Y}_n}} \Bigg\} \\
= & \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}} \Bigg\{ \prod_{1\leqslant r<s\leqslant n} \frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \prod_{i=1}^{n}\Big(\frac{x_n}{x_i}\Big)^{y_i}\big(cxq^{N_n}\big)^{\mathcal{Y}_n} \\
& \times \prod_{r,s=1}^{n} \frac{\big(\frac{x_r}{x_s}q^{-N_s};q\big)_{y_r}}{\big(\frac{x_r}{x_s}q;q\big)_{y_r}} \prod_{i=1}^{n} \frac{\big(\frac{x_n}{x_i}cyq^{N_n-N_i}q^{\mathcal{Y}_n-y_i};q\big)_{N_i}}{\big(\frac{x_n}{x_i}cyq^{N_n-N_i}q^{\mathcal{Y}_n-y_i};q\big)_{y_i}} \\
& \times q^{y_2+2y_3+\ldots+(n-1)y_n+e_2(y_1,\ldots,y_n)} \frac{(by,dy;q)_{\mathcal{Y}_n}}{(bdxy;q)_{\mathcal{Y}_n}} \Bigg\}.
\end{aligned}
$$

Remark 3.1. Obviously Corollary 3.1 is a limit case of the transformation of Theorem 3.1 in [6] and Corollary 3.2 is a limit case of Theorem 3.13 in [2].

Letting $n = 1$ in (8) or (9) and then replacing (b, d, e, a_i) by $(1/b, 1/d, 1/e, 1/a_i)$, $i = 1, 2, \ldots, 2t$, we have

Corollary 3.3. *If $|e| < \min\{|x|, |y|\}$, $|a_{2j}| < 1$, $j = 1, 2, \ldots, t$, and t is a nonnegative integer, then*

$$
\begin{aligned}
(21)\qquad & {}_{t+3}\Phi_{t+2}\left(\begin{matrix} q^{-N_1},\ bx,\ dx,\ a_1x,\ \ldots,\ a_{2t-1}x \\ cx,\ ex,\ a_2x,\ \ldots,\ a_{2t}x\end{matrix};q,\frac{cea_2\ldots a_{2t}q^{N_1}}{bda_1\ldots a_{2t-1}}\right) \\
&= \frac{(cy;q)_{N_1}}{(cx;q)_{N_1}} \sum_{y_1=0}^{N_1} \frac{(q^{-N_1},by;q)_{y_1}}{(q,cy;q)_{y_1}} \Big(\frac{cq^{N_1}}{b}\Big)^{y_1} \sum_{j=0}^{y_1} \frac{\big(q^{-y_1},\frac{e}{d},bx;q\big)_j}{(q,ex,by;q)_j} q^j \\
&\times \sum_{0\leqslant j_t\leqslant\ldots\leqslant j_1\leqslant j_0=j} \prod_{i=1}^{t} \frac{\big(q^{-j_{i-1}},\frac{a_{2i}}{a_{2i-1}},a_{2i-3}x;q\big)_{j_i}}{\big(q,\frac{a_{2i-3}}{a_{2i-2}}q^{1-j_{i-1}},a_{2i}x;q\big)_{j_i}} q^{j_i},
\end{aligned}
$$

where $d = a_{-1}$, $e = a_0$.

Letting $cx = ex = a_2x = \ldots = a_{2t}x$, $a_{2i}/a_{2i-1} = q$, $x = y$, $a_{-3} = b$, $a_{-2} = c$, $i = -1, 0, 1, \ldots, t$ in (21), we find

Corollary 3.4. *If $|cx| < 1$, t is a nonnegative integer, then*

$$\sum_{k=0}^{N_1} \begin{bmatrix} N_1 \\ k \end{bmatrix} \frac{(1-bx)^{t+2}}{(1-bxq^k)^{t+2}} (-1)^k q^{\binom{k}{2}+k(t+2)} \tag{22}$$

$$= \sum_{y_1=0}^{N_1} \begin{bmatrix} N_1 \\ y \end{bmatrix}_1 \frac{1-bx}{1-bxq^{y_1}} (-1)^{y_1} q^{\binom{y_1}{2}+y_1}$$

$$\times \sum_{j=0}^{y_1} \begin{bmatrix} y_1 \\ j \end{bmatrix} \frac{(q;q)_j}{(bxq;q)_j} (-1)^j q^{\binom{j}{2}-y_1 j+j} \sum_{0 \leqslant j_t \leqslant \ldots \leqslant j_1 \leqslant j_0 = j} \prod_{i=1}^{t} \frac{1-bx}{1-bxq^{j_i}} q^{j_i}.$$

Setting $bx = q$ in the above identity, then letting $q \to 1$, we have

Corollary 3.5. *If t is a nonnegative integer, then*

$$\sum_{k=0}^{N_1} \frac{\binom{N_1}{k}(-1)^k}{(k+1)^{t+2}} = \sum_{y_1=0}^{N_1} \frac{\binom{N_1}{y_1}(-1)^{y_1}}{y_1+1} \sum_{j=0}^{y_1} \frac{\binom{y_1}{j}(-1)^j}{j+1} \sum_{0 \leqslant j_t \leqslant \ldots \leqslant j_1 \leqslant j_0 = j} \prod_{i=1}^{t} \frac{1}{j_i+1}. \tag{23}$$

Letting $cx = ex = a_2x = \ldots = a_{2t}x$, $a_{2i}/a_{2i-1} = q$, $qx = y$ in (21), we get

Corollary 3.6. *If $|cx| < 1$, t is a nonnegative integer, then*

$$\sum_{k=0}^{N_1} \begin{bmatrix} N_1 \\ k \end{bmatrix} \frac{(1-bx)^{t+2}}{(1-bxq^k)^{t+2}} (-1)^k q^{\binom{k}{2}+k(t+2)} \tag{24}$$

$$= \frac{1-bxq^{N_1+1}}{(1-bxq)} \sum_{y_1=0}^{N_1} \begin{bmatrix} N_1 \\ y \end{bmatrix}_1 \frac{(1-bxq)(-1)^{y_1}}{1-bxq^{y_1+1}}$$

$$\times q^{\binom{y_1}{2}+y_1} \sum_{j=0}^{y_1} \begin{bmatrix} y_1 \\ j \end{bmatrix} \frac{(q;q)_j(1-bx)(-1)^j q^{\binom{j}{2}+j-y_1 j}}{(bxq;q)_j(1-bxq^j)}$$

$$\times \sum_{0 \leqslant j_t \leqslant \ldots \leqslant j_1 \leqslant j_0 = j} \prod_{i=1}^{t} \frac{1-bx}{1-bxq^{j_i}} q^{j_i}.$$

Setting $bx = q$ in the above identity, then letting $q \to 1$, we have

Corollary 3.7. *If t is a nonnegative integer, then*

$$\tag{25} \frac{1}{N_1+2}\sum_{k=0}^{N_1}\frac{\binom{N_1}{k}(-1)^k}{(k+1)^{t+2}} = \sum_{y_1=0}^{N_1}\frac{\binom{N_1}{y_1}(-1)^{y_1}}{y_1+2}\sum_{j=0}^{y_1}\frac{\binom{y_1}{j}(-1)^j}{(j+1)^2}\sum_{0\leqslant j_t\leqslant\ldots\leqslant j_1\leqslant j_0=j}\prod_{i=1}^{t}\frac{1}{j_i+1}.$$

Setting $x \to a_{2t-1}$ in (8), we get

Corollary 3.8. *Let b, c, d, e, y and $x_1, \ldots, x_n, a_1, a_2, \ldots, a_{2t}$ be indeterminate, let N_i be nonnegative integers for $i = 1, 2, \ldots, n$ with $n \geqslant 1$, and suppose that none of the denominators in (26) vanishes. For $|e| < \min\{|x|, |y|\}$, $|a_{2j}| < 1$, $j = 1, 2, \ldots, t$, $e_2(y_1, \ldots, y_n)$ being the second elementary symmetric function of $\{y_1, \ldots, y_n\}$, we have*

$$\tag{26} \begin{aligned} 1 = \prod_{i=1}^{n}\frac{\left(\frac{x_i}{x_n}cy;q\right)_{N_i}}{\left(\frac{x_i}{x_n}ca_{2t-1};q\right)_{N_i}}\sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\prod_{r,s=1}^{n}\frac{\left(\frac{x_r}{x_s}q^{-N_s};q\right)_{y_r}}{\left(\frac{x_r}{x_s}q;q\right)_{y_r}} \\ \times\left(cbq^{\mathcal{N}_n}\right)^{\mathcal{Y}_n}\prod_{i=1}^{n}\left(\frac{x_i}{x_n}cy;q\right)_{y_i}^{-1}q^{y_2+2y_3+\ldots+(n-1)y_n-e_2(y_1,\ldots,y_n)}\left(\frac{y}{b};q\right)_{\mathcal{Y}_n} \\ \times\sum_{0\leqslant j\leqslant\mathcal{Y}_n}\frac{\left(q^{-\mathcal{Y}_n},\frac{a_{2t-1}}{b},\frac{d}{e};q\right)_j q^j}{\left(q,\frac{a_{2t-1}}{e},\frac{y}{b};q\right)_j} \\ \times\sum_{0\leqslant j_t\leqslant\ldots\leqslant j_0}\prod_{i=1}^{t}\frac{\left(q^{-j_{i-1}},\frac{a_{2i-1}}{a_{2i}},\frac{a_{2t-1}}{a_{2i-3}};q\right)_{j_i}q^{j_i}}{\left(q,\frac{a_{2t-1}}{a_{2i}},\frac{a_{2i-2}}{a_{2i-3}}q^{1-j_{i-1}};q\right)_{j_i}}\Bigg\}, \end{aligned}$$

where $a_{-1} = d$, $a_0 = e$, $j_0 = j$, and t is a nonnegative integer.

Setting $x \to a_{2t-1}$ in (9), we find

Corollary 3.9. *Let b, c, d, e, y and $x_1, \ldots, x_n$, $a_1, \ldots, a_{2t}$ be indeterminate, let N_i be nonnegative integers for $i = 1, 2, \ldots, n$ with $n \geqslant 1$, and suppose that none of the denominators in (27) vanishes. For $|e| < \min\{|x|, |y|\}$, $|a_{2j}| < 1$, $j = 1, 2, \ldots, t$, $e_2(y_1, \ldots, y_n)$ being the second elementary symmetric function of $\{y_1, \ldots, y_n\}$, we* *have*

$$(27)\quad \prod_{i=1}^{n}\Big(\frac{x_n}{x_i}ca_{2t-1}q^{N_n-N_i};q\Big)_{N_i}$$

$$= \sum_{\substack{0\leqslant y_i\leqslant N_i\\ i=1,2,\ldots,n}}\Bigg\{\prod_{1\leqslant r<s\leqslant n}\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\prod_{i=1}^{n}\Big(\frac{x_n}{x_i}\Big)^{y_i}\big(bcq^{N_n}\big)^{\mathcal{Y}_n}\prod_{r,s=1}^{n}\frac{\big(\frac{x_r}{x_s}q^{-N_s};q\big)_{y_r}}{\big(\frac{x_r}{x_s}q;q\big)_{y_r}}$$

$$\times\prod_{i=1}^{n}\frac{\big(\frac{x_n}{x_i}cyq^{N_n-N_i}q^{\mathcal{Y}_n-y_i};q\big)_{N_i}}{\big(\frac{x_n}{x_i}cyq^{N_n-N_i}q^{\mathcal{Y}_n-y_i};q\big)_{y_i}}q^{y_2+2y_3+\ldots+(n-1)y_n+e_2(y_1,\ldots,y_n)}\Big(\frac{y}{b};q\Big)_{\mathcal{Y}_n}$$

$$\times\sum_{0\leqslant j\leqslant\mathcal{Y}_n}\frac{\big(q^{-\mathcal{Y}_n},\frac{a_{2t-1}}{b},\frac{d}{e};q\big)_j q^j}{\big(q,\frac{a_{2t-1}}{e},\frac{y}{b};q\big)_j}$$

$$\times\sum_{0\leqslant j_t\leqslant\ldots\leqslant j_0=j}\prod_{i=1}^{t}\frac{\big(q^{-j_{i-1}},\frac{a_{2i-1}}{a_{2i}},\frac{a_{2t-1}}{a_{2i-3}};q\big)_{j_i}q^{j_i}}{\big(q,\frac{a_{2t-1}}{a_{2i}},\frac{a_{2i-2}}{a_{2i-3}}q^{1-j_{i-1}};q\big)_{j_i}}\Bigg\}.$$

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