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HENSTOCK-KURZWEIL INTEGRAL ON BV SETS

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Abstract. The generalized Riemann integral of Pfeffer (1991) is defined on all bounded BV subsets of $\mathbb{R}^n$, but it is additive only with respect to pairs of disjoint sets whose closures intersect in a set of σ-finite Hausdorff measure of codimension one. Imposing a stronger regularity condition on partitions of BV sets, we define a Riemann-type integral which satisfies the usual additivity condition and extends the integral of Pfeffer. The new integral is lipeomorphism-invariant and closed with respect to the formation of improper integrals. Its definition in $\mathbb{R}$ coincides with the Henstock-Kurzweil definition of the Denjoy-Perron integral.

Keywords: Henstock-Kurzweil integral; charge; BV set

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1. Introduction

More than a century ago, independently and by different means, Denjoy in [1] and Perron in [12] defined an extension of the Lebesgue integral that satisfies the fundamental theorem of calculus for each differentiable function in an interval. While the usefulness and aesthetic appeal of the Denjoy-Perron integral is undeniable, both of its definitions are complicated and resistant to a usable higher-dimensional generalization.

In the second half of the last century, Henstock in [5], [4] and Kurzweil in [6] discovered independently that the Denjoy-Perron integral can be obtained by a minor, but ingenious, modification of the classical Riemann integral. The simplicity of the

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Henstock-Kurzweil definition revitalized the surge for a multi-dimensional generalization. In spite of many attempts, it took more than twenty years before Mawhin in [11] observed that partitioning a multi-dimensional interval A to subintervals similar to A yields the divergence theorem for any differentiable vector field defined in a neighborhood of A. Several improvements of Mawhin's pioneering work followed (see [10], [9], [18], [8], [7], [17]), but all resulting integrals were defined either on intervals, or on sets not sufficiently general for applications.

The Riemann-type definition of an integral on the family of all bounded BV sets in $\mathbb{R}^n$ is due to the second author (see [16]), and we refer to it as the R-integral (Definition 2.8). For the descriptive definition of the R-integral and the detailed development of its properties we refer to [15], or in a more concise form to [14]. Note that BV sets are the most general sets for which the surface area and exterior normal can be profitably defined. In other words, BV sets form the largest family of sets for which the divergence theorem can be formulated.

The additivity property of the R-integral is limited. Improving it requires a transfinite extension akin to that of the constructive definition of the Denjoy-Perron integral. While the extended integral, i.e., the GR-integral (see [15], Section 6.3) is finitely additive in any dimension, in the real line it is still less general than the Denjoy-Perron integral (see [16], Example 6.9 and Proposition 10.8). The purpose of this paper is to show that a stricter regularity condition on the partitioning sets leads to a proper extension of the R-integral, called the R_*-integral, which is finitely additive in the usual sense (Theorem 3.14), shares the important properties of the R-integral, and coincides with the Denjoy-Perron integral in the real line (Proposition 3.6). The unrestricted Gauss-Green theorem and the area theorem for lipeomorphisms are valid for the R_*-integral (Theorems 3.19 and 3.24). The area theorem facilitates the obvious definition of the R_*-integral on Lipschitz manifolds. The R_*-integral is also closed with respect to the formation of improper integrals (Theorem 3.20), and consequently extends the GR-integral.

The main difference between the R and R_*-integrals lies in the regularity conditions placed on BV sets. The regularity applied in defining the R-integral relates diameter, perimeter and volume so that regular BV sets enjoy both the reverse isoperimetric and reverse isodiametric inequalities. If a regular BV set E of small diameter contains a generic point of the essential interior of any BV set A, then the intersection $E \cap A$ is again regular (see [15], Lemma 2.5.2), but little can be said about the difference $E - A$. An unpleasant consequence is that there are BV sets $A \subset B$ and an R-integrable function on A whose zero extension to B is not R-integrable (see [15], Section 6.1). To avoid this pathology we impose an additional condition on a regular BV set E. Specifically, we require that for any BV set A the smaller of the perimeters of $E \cap A$ and $E - A$ is controlled by the relative perimeter of A in E. Cubes in $\mathbb{R}^n$ satisfy this condition (Lemma 3.1), and its utility transpires from Lemmas 3.7 and 3.8.

2. Preliminaries

Finite and countably infinite sets are called *countable.* If A and B are sets, then $A \triangle B = (A-B) \cup (B-A)$ is their *symmetric difference.* The sets of all positive integers and all real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. When no attributes are added, functions are assumed to be *real-valued.*

The ambient space of this paper is $\mathbb{R}^n$, where $n \geqslant 1$ is a fixed integer. In $\mathbb{R}^n$ we shall use exclusively the Euclidean norm $|x|$ induced by the usual scalar product $x \cdot y$. For $x \in \mathbb{R}^m$ and $r > 0$, we denote by $U(x,r)$ and $B(x,r)$ the open and closed balls centered at x of radius r, respectively. The zero vector in $\mathbb{R}^m$ is denoted by 0, and we write $U(r)$ instead of $U(0,r)$. The diameter and closure of a set $E \subset \mathbb{R}^m$ are denoted by $d(E)$ and $\operatorname{cl} E$, respectively. By $\mathbf{1}_E$ we denote the *indicator* of a set $E \subset \mathbb{R}^m$. Equalities such as $\gamma = \gamma(n)$, $\kappa := \kappa(n), \ldots$, indicate that $\gamma, \kappa, \ldots$, are constants depending only on the dimension n.

Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathscr{L}$; however, for $E \subset \mathbb{R}^n$, we write $|E|$ instead of $\mathscr{L}(E)$. Throughout the paper,

$$\alpha(n) := \mathscr{L}(\{x \in \mathbb{R}^n : \ |x| \leqslant 1\}).$$

Unless specified otherwise, the words *measure, measurable,* and *negligible* as well as the expressions *almost all, almost everywhere*, and *absolutely continuous* always refer to Lebesgue measure $\mathscr{L}$. In $\mathbb{R}^n$ we also use the $(n-1)$-dimensional Hausdorff measure, denoted by $\mathscr{H}$.

Let $A \subset \mathbb{R}^n$ be a measurable set. We let $\operatorname{int}_* A$ and $\operatorname{ext}_* A$ be the sets of all density points of A and $\mathbb{R}^n - A$, respectively, and define

$$\partial_* A = \mathbb{R}^n - (\operatorname{int}_* A \cup \operatorname{ext}_* A) \quad \text{and} \quad \operatorname{cl}_* A = \operatorname{int}_* A \cup \partial_* A;$$

we call these sets the *essential interior, essential exterior, essential boundary*, and *essential closure* of A, respectively. We say that A is an *admissible* set if

$$\operatorname{int}_* A \subset A \subset \operatorname{cl}_* A$$

and ∂A is compact. Note that the complement $\mathbb{R}^n - A$ of an admissible set A is also admissible. The *relative perimeter* of a measurable set E in A is the number

$$P(E, \operatorname{in} A) = \mathscr{H}(\partial_* E \cap \operatorname{int}_* A).$$

The (absolute) *perimeter* of a measurable set E is the number

$$P(E) = P(E, \operatorname{in} \mathbb{R}^n) = \mathscr{H}(\partial_* E).$$

A measurable set $A \subset \mathbb{R}^n$ such that $P(A, \operatorname{in} E) < \infty$ for each bounded measurable set E is called a *locally* BV *set*; if $|A| + P(A) < \infty$, then A is called a BV *set*. Note that $\mathbb{R}^n$ is a locally BV set, and that the intersection of a BV set and a locally BV set is a BV set. If A is an admissible locally BV set, then so is $\mathbb{R}^n - A$. The families of all BV sets, all bounded BV sets, and all locally BV sets are denoted by $\mathcal{BV}$, $\mathcal{BV}_c$, and $\mathcal{BV}_{\mathrm{loc}}$, respectively. By $\mathcal{ABV}$ and $\mathcal{ABV}_{\mathrm{loc}}$ we denote, respectively, the families of all *admissible sets* in $\mathcal{BV}$ and $\mathcal{BV}_{\mathrm{loc}}$. Note that $\mathcal{ABV} \subset \mathcal{BV}_c$.

Lemma 2.1. *If A and E are* BV *sets, then $P(E, \operatorname{in} A) = P(A \cap E, \operatorname{in} A)$ and*

$$P(E, \operatorname{in} A) = \frac{1}{2}[P(E \cap A) + P(A - E) - P(A)] = P(A - E, \operatorname{in} A).$$

Proof. The first equality follows from [13], Corollary 4.2.5. It shows that the remaining equalities do not change when E is replaced by $E \cap A$. Thus it suffices to prove them for $E \subset A$. In this case [13], Proposition 6.6.3 implies

$$\begin{aligned} P(E) + P(A - E) - P(A) & \\ &= 2P(E) - 2\mathscr{H}(\partial_* E \cap \partial_* A) \\ &= 2[\mathscr{H}(\partial_* E \cap \operatorname{int}_* A) + \mathscr{H}(\partial_* E \cap \partial_* A)] - 2\mathscr{H}(\partial_* E \cap \partial_* A) \\ &= 2P(E, \operatorname{in} A) \end{aligned}$$

and consequently

$$\begin{aligned} 2P(A - E, \operatorname{in} A) &= P[(A - E) \cap A] + P[A - (A - E)] - P(A) \\ &= P(A - E) + P(E \cap A) - P(A) = 2P(E, \operatorname{in} A). \end{aligned}$$

□

If A is a BV set, then for $\mathscr{H}$ almost all $x \in \partial_* A$ there exists a unique *unit exterior normal* $\nu_A(x)$ such that the *Gauss-Green formula*

$$\int_{\partial_* A} v \cdot \nu_A \, \mathrm{d}\mathscr{H} = \int_A \operatorname{div} v \, \mathrm{d}\mathscr{L}$$

for each $v \in C^1(\mathbb{R}^n; \mathbb{R}^n)$; see [13], Section 6.5.

Definition 2.2. A finitely additive function F defined on $\mathcal{BV}_c$ is called a *charge* if for each $\varepsilon > 0$ there is $\theta > 0$ such that

$$|F(B)| \leqslant \theta|B| + \varepsilon[P(B) + 1]$$

for each BV set $B \subset U(1/\varepsilon)$.

We say that a sequence $\{A_k\}$ in $\mathcal{BV}_c$ *converges* to a set A, and write $A_k \to A$, if there is a compact set $K \subset \mathbb{R}^m$ containing each A_k, $\sup P(A_k) < \infty$, and $\lim |A \triangle A_k| = 0$.

The following characterization of charges is proved in [15], Section 2.2.

Proposition 2.3. *A finitely additive function F defined on $\mathcal{BV}_c$ is a charge if and only if F satisfies the following condition: $F(A) = \lim F(A_k)$ for each sequence $\{A_k\}$ in $\mathcal{BV}_c$ converging to A.*

If F is a charge and A is a locally BV set, then

$$F \llcorner A \colon\ B \mapsto F(A \cap B) : \mathcal{BV}_c \to \mathbb{R}.$$

is also a charge. We say that F is a *charge in* A when $F = F \llcorner A$; note, however, that a charge in A is still defined on the whole of $\mathcal{BV}_c$. The linear space of all charges in a locally bounded BV set A is denoted by $CH(A)$.

Lemma 2.4. *Let F be a charge. If $F(C) \geqslant 0$ for each cube $C \subset \mathbb{R}^n$, then $F \geqslant 0$.*

The lemma is a direct consequence of [13], Proposition 6.7.3.

Lemma 2.5. *Let F be a charge and $\varepsilon > 0$. There is an absolutely continuous Radon measure μ in $\mathbb{R}^n$ such that for each* BV *set $B \subset U(1/\varepsilon)$,*

$$\bigl|F(B)\bigr| \leqslant \mu(B) + \varepsilon P(B).$$

Proof. By [2], Theorem 6.2, there are $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $v \in C(\mathbb{R}^n; \mathbb{R}^n)$ such that

$$F(B) = \int_B f \,\mathrm{d}\mathscr{L} + \int_{\partial_* B} v \cdot \nu_B \,\mathrm{d}\mathscr{H}$$

for each $B \in \mathcal{BV}_c$. As by [15], Proposition 2.1.7 and Remark 2.1.8, there is $\theta > 0$ such that

$$\int_{\partial_* B} v \cdot \nu_B \,\mathrm{d}\mathscr{H} \leqslant \theta|B| + \varepsilon P(B)$$

for every BV set $B \subset U(1/\varepsilon)$, it suffices to let $\mu = \int (f + \theta) \,\mathrm{d}\mathscr{L}$. □

R e m a r k 2.6. The word “charge” has been used in the literature to describe several distinct concepts. For instance, our notion of charge, which was introduced in [15], Section 2.1, differs from that given in [13], Section 10.1.

The *regularity* of a bounded BV set $E \subset \mathbb{R}^n$ is defined by

$$r(E) = \begin{cases} \dfrac{|E|}{d(E)P(E)} & \text{if } |E| > 0, \\ 0 & \text{if } |E| = 0. \end{cases}$$

Note that $r(E) = 1/\bigl(2n\sqrt{n}\bigr)$ when E is a cube. The *isoperimetric inequality*

$$n^n \alpha(n) |E|^{n-1} \leqslant P(E)^n \tag{2.1}$$

relates the regularity of E to the common concept of shape:

$$n^n \alpha(n) r(E)^n \leqslant \frac{|E|}{d(E)^n}. \tag{2.2}$$

The *critical boundary* of a locally BV set A is the set

$$\partial_c A = \Bigl\{ x \in \mathbb{R}^m \colon \limsup_{r \to 0} \frac{P[A, \text{in } B(x,r)]}{r^{n-1}} > 0 \Bigr\}, \tag{2.3}$$

and the sets $\text{int}_c A = \text{int}_* A - \partial_c A$ and $\text{ext}_c A = \text{ext}_* A - \partial_c A$ are called the *critical interior* and *critical exterior* of A, respectively. It is clear that

$$\partial_* A \subset \partial_c A, \quad \text{ext}_c A = \text{int}_c(\mathbb{R}^n - A), \quad \mathbb{R}^n = \text{int}_c A \cup \text{ext}_c A \cup \partial_c A,$$

and it follows from [13], Section 7.3 that $\mathscr{H}(\partial_c A - \partial_* A) = 0$.

A *gage* on a set $A \subset \mathbb{R}^n$ is a nonnegative function defined on A whose null set $\{\delta = 0\}$ is of σ-finite measure $\mathscr{H}$. A *partition* is a finite, possibly empty, collection

$$P = \{(E_1, x_1), \dots, (E_p, x_p)\},$$

where $E_1, \dots, E_p$ are disjoint bounded BV sets. The *body* of P is the union $[P] = \bigcup_{i=1}^{p} E_i$. Given $\eta > 0$ and a gage δ on a set A, we say that P is

(i) *η-regular* if $r(E_i \cup \{x_i\}) > \eta$ for $i = 1, \dots, p$;
(ii) *δ-fine* if $E_i \subset U(x_i, \delta(x_i))$ for $i = 1, \dots, p$.

Note that if P is η-regular and δ-fine, then each x_i is in $A - \{\delta = 0\}$. The following useful fact about partitions is established in [15], Lemma 2.6.6.

Lemma 2.7. *Let $A \in \mathcal{ABV}_{\text{loc}}$ and $\varepsilon > \eta > 0$. There is a gage δ on A with the following property: if $\{(E_1, x_1), \dots, (E_p, x_p)\}$ is an ε-regular δ-fine partition, then $\{(A \cap E_1, x_1), \dots, (A \cap E_p, x_p)\}$ is an η-regular δ-fine partition.*

The next definition originates from [16], Proposition 7.7.

Definition 2.8. A function f defined on a set $A \in \mathcal{ABV}$ is called R *integrable* if there is a charge F in A that satisfies the following condition: given $\varepsilon > 0$, we can find a gage δ on A so that

$$\sum_{i=1}^{p} \bigl| f(x_i)|A_i| - F(A_i) \bigr| < \varepsilon$$

for each ε-regular δ-fine partition $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ with $[P] \subset A$.

The charge F in Definition 2.8, which is uniquely determined by f, is called the *indefinite* R*-integral* of f, denoted by $(R)\int f$. By $\mathcal{R}(A)$ we denote the linear space of all R-integrable functions on A. The number $(R)\int_A f = F(A)$ is called the R*-integral* of f on A.

The R-integral has many useful properties, including the area theorem for local lipeomorphisms and the Gauss-Green theorem for pointwise Lipschitz vector fields; see [16], or [15], [14], where the R-integral is studied from the descriptive point of view. At the same time, the additive property of the R-integral is deficient, see [15], Proposition 5.1.8 and Section 6.1. Moreover, [16], Example 6.9 shows that if $n = 1$, then the R-integral is less general then the Denjoy-Perron integral.

3. R_*-integral

Given $\varepsilon > 0$, a set $E \in \mathcal{BV}_c$ is called ε*-isoperimetric* if for each $T \in \mathcal{BV}$,

$$\min\{P(E \cap T), P(E - T)\} \leqslant \frac{1}{\varepsilon} P(T, \mathrm{in}\, E).$$

According to Lemma 2.1, in testing for the ε-isoperimetric property of E it suffices to consider only BV sets $T \subset E$.

Lemma 3.1. *Every cube $C \subset \mathbb{R}^n$ is κ-isoperimetric for a constant $\kappa = \kappa(n)$. Making κ smaller, we assume throughout that $\kappa < 1/(2n\sqrt{n})$.*

Proof. Choose a BV set $T \subset C$ and assume $|T| \leqslant |C|/2$. Then [13], Corollary 4.2.5 and Lemma 6.7.2 show that there is $\beta = \beta(n) > 0$ such that

$$P(T) \leqslant P(\mathrm{int}\, C \cap \partial_* T) + P(\partial C \cap \partial_* T) \leqslant (1 + \beta) P(T, \mathrm{in}\, C).$$

If $|T| > |C|/2$ then $|C - T| < |C|/2$, and we obtain

$$P(C - T) = P[C \cap (C - T)] \leqslant (1 + \beta) P(C - T, \mathrm{in}\, C) = (1 + \beta) P(T, \mathrm{in}\, C)$$

by the first part of the proof and Lemma 2.1. □

Observation 3.2. Let $E \subset \mathbb{R}$ be a BV set that is ε-isoperimetric for some $\varepsilon > 0$. Then E is equivalent to a closed interval I and $\mathrm{cl}_* E = I$.

Proof. Let $E = A \cup B$ where A and B are BV sets of positive measure such that $\mathrm{cl}\, A \cap \mathrm{cl}\, B = \emptyset$, and let $T = A$. Then $P(T, \mathrm{in}\, E) = 0$ while $P(E \cap T) = P(A) > 0$ and $P(E - T) = P(B) > 0$, a contradiction. Thus E is equivalent to a closed interval I, and the equality $\mathrm{cl}_* E = I$ is obvious. □

An ε-regular partition $P = \{(E_1, x_1), \ldots, (E_p, x_p)\}$ is called *strongly ε-regular* if each E_i is ε-isoperimetric and $x_i \in \mathrm{cl}_* E_i$ for $i = 1, \ldots, p$. If each E_i is a cube, then P is strongly κ-regular, since $r(E_i) = 1/(2n\sqrt{n}) > \kappa$ for $i = 1, \ldots, p$.

Definition 3.3. Let G be a charge. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called *R_*-integrable* with respect to G if there is a charge F that satisfies the following condition: given $\varepsilon > 0$, we can find a gage δ on $\mathbb{R}^n$ so that

$$\sum_{i=1}^{p} |f(x_i)G(E_i) - F(E_i)| < \varepsilon$$

for each strongly ε-regular δ-fine partition $\{(E_1, x_1), \ldots, (E_p, x_p)\}$. The charge F is unique by the next proposition. It is called the *indefinite R_*-integral* of f with respect to G and denoted by $(R_*)\int f\, \mathrm{d}G$. If $G = \mathscr{L}$, we usually drop the reference to $\mathscr{L}$ and call F the *indefinite R_*-integral* of f.

Proposition 3.4. *If $f\colon \mathbb{R}^n \to \mathbb{R}$ is R_*-integrable with respect to a charge G, then the indefinite R_*-integral of f is unique.*

Proof. Let F_1 and F_2 be indefinite R_*-integrals of f with respect to the charge G, and let $H = |F_1 - F_2|$. Choose a cube $C \subset \mathbb{R}^m$ and $0 < \varepsilon \leqslant \kappa$, and find a gage δ corresponding to ε and both F_1 and F_2. It follows from [15], Lemma 2.6.4 that there is a strongly ε-regular δ-fine partition $P = \{(E_1, x_1), \ldots, (E_p, x_p)\}$ such that $[P] \subset C$ and $H(C - [P]) < \varepsilon$. Thus

$$\begin{aligned} H(C) &\leqslant H(K - [P]) + H([P]) < \varepsilon + \left| \sum_{i=1}^{p} [F_1(E_i) - F_2(E_i)] \right| \\ &\leqslant \varepsilon + \sum_{i=1}^{p} |F_1(E_i) - f(x_i)G(E_i)| + \sum_{i=1}^{p} |f(x_i)G(E_i) - F_2(E_i)| < 3\varepsilon. \end{aligned}$$

As ε is arbitrary, $H(C) = 0$ and the proposition follows from Lemma 2.4. □

Let G be a charge. It is clear that the set $R_*(\mathbb{R}^n, G)$ of all functions $f\colon \mathbb{R}^n \to \mathbb{R}$ that are R_*-integrable with respect to G is a linear space, and that

$$f \mapsto (R_*)\int f\,\mathrm{d}G\colon\ R_*(\mathbb{R}^n, G) \to CH(\mathbb{R}^n)$$

is a linear map. An argument similar to the proof of Proposition 3.4 shows that if $G \geqslant 0$ and $f \in R_*(\mathbb{R}^n, G)$ is nonnegative, then so is $(R_*)\int f\,\mathrm{d}G$. If $G = \mathscr{L}$, we write $R_*(\mathbb{R}^m)$ and $(R_*)\int f$ instead of $R_*(\mathbb{R}^m, \mathscr{L})$ and $(R_*)\int f\,\mathrm{d}\mathscr{L}$, respectively.

Proofs analogous to those of [16], Section 5 establish the next proposition.

Proposition 3.5. *Each R_*-integrable function is measurable, $L^1_{\mathrm{loc}}(\mathbb{R}^n) \subset R_*(\mathbb{R}^n)$ and $(R_*)\int f = \int f\,\mathrm{d}\mathscr{L}$ for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. In addition,*

$$L^1_{\mathrm{loc}}(\mathbb{R}^n) = \{f \in R_*(\mathbb{R}^n)\colon\ |f| \in R_*(\mathbb{R}^n)\}.$$

An immediate consequence of Proposition 3.4 is that neither the R_*-integrability of $f\colon\ \mathbb{R}^m \to \mathbb{R}$, nor $(R_*)\int f$, depends on the values of f on a negligible set.

Proposition 3.6. *A function f defined on $\mathbb{R}$ is R_*-integrable if and only if it is Denjoy-Perron integrable on each compact interval $A \subset \mathbb{R}$. If $F = (R_*)\int f$, then $F(A)$ equals the Denjoy-Perron integral of f on A.*

P r o o f. We prove the proposition using the Henstock-Kurzweil definition of the Denjoy-Perron integral, see [3], Definition 9.3. Let f be Denjoy-Perron integrable on each compact subinterval of $\mathbb{R}$. It follows from [3], Theorem 9.12 that the indefinite Denjoy-Perron integral is a charge, and the Henstock lemma ([3], Lemma 9.11), combined with Observation 3.2, shows that f is R_*-integrable.

Conversely, let f be R_*-integrable and $F = (R_*)\int f$. Choose a compact interval $A \subset \mathbb{R}$ and $\varepsilon > 0$. There is a gage δ on $\mathbb{R}$ such that

$$\sum_{i=1}^{p} \bigl|f(x_i)|E_i| - F(E_i)\bigr| < \varepsilon$$

for each δ-fine partition $\{(E_1, x_1), \ldots, (E_p, x_p)\}$, where every E_i is a closed interval containing x_i. Enumerate the countable set $\{\delta = 0\}$ as $\{z_1, z_2, \ldots\}$, and without loss of generality assume that $f(z_k) = 0$ for $k = 1, 2, \ldots$ There are $r_k > 0$ such that $|F(J)| < \varepsilon 2^{-k}$ for every interval $J \subset A$ with $|J| < 2r_k$. Let

$$\delta_+(x) = \begin{cases} \delta(x) & \text{if } \delta(x) > 0,\\ r_k & \text{if } x = z_k, \end{cases}$$

and select a δ_+-fine partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ so that each A_i is a closed interval containing x_i and $[P] = A$. Then P is the disjoint union of partitions Q and S such that Q is δ-fine and $S = \{(A_{i_1}, z_{k_1}), \ldots, (A_{i_s}, z_{k_s})\}$. Hence

$$\begin{aligned}\left|\sum_{i=1}^{p} f(x_i)|A_i| - F(A)\right| &\leqslant \sum_{i=1}^{p} \bigl|f(x_i)|A_i| - F(A_i)\bigr| \\ &= \sum_{(A_i,x_i)\in Q} \bigl|f(x_i)|A_i| - F(A_i)\bigr| + \sum_{j=1}^{s} |F(A_{i_j})| < 2\varepsilon.\end{aligned}$$

It follows that the Denjoy-Perron integral of f on A exists and equals $F(A)$. □

Lemma 3.7. *Let $A \in \mathcal{BV}_{\rm loc}$ and $\varepsilon > 0$. For each $x \in \operatorname{ext}_c A$, there is $\delta > 0$ such that every strongly ε-regular set E with $x \in \operatorname{cl}_* E$ and $d(E) < \delta$ satisfies*

$$P(E \cap A) \leqslant P(E - A).$$

Proof. As the case $n = 1$ is trivial, assume $n \geqslant 2$. Let $\theta = n^n \alpha(n)$ be the isoperimetric constant of (2.1). Choose $0 < \eta < \varepsilon^n \theta / 2$ and find $\delta > 0$ so that

$$|A \cap B(x,d)| \leqslant \eta d^n \quad \text{and} \quad P[A, \text{in}\, B(x,d)] \leqslant \eta d^{n-1} \tag{3.1}$$

whenever $d < \delta$. Select a strongly ε-regular set E with $x \in \operatorname{cl}_* E$ and $d = d(E) < \delta$. Then $|E| \geqslant \theta(\varepsilon d)^n$ by inequality (2.2). Seeking a contradiction, assume that

$$P(E - A) < P(E \cap A).$$

Since E is strongly ε-regular, the isoperimetric inequality (2.1) and (3.1) imply

$$\begin{aligned}\varepsilon \theta^{1/n} |E - A|^{(n-1)/n} &\leqslant \varepsilon P(E - A) \leqslant P(A, \text{in}\, E) \\ &\leqslant P[A, \text{in}\, B(x,d)] \leqslant \eta d^{n-1} \leqslant \frac{1}{2} \varepsilon^n \theta d^{n-1}\end{aligned}$$

and hence

$$|E - A| \leqslant 2^{-n/(n-1)} \theta (\varepsilon d)^n < \frac{1}{2} \theta (\varepsilon d)^n.$$

On the other hand, (3.1) shows that

$$|E - A| \geqslant |E| - |A \cap B(x,d)| \geqslant \theta(\varepsilon d)^n - \eta d^n \geqslant \frac{1}{2} \theta (\varepsilon d)^n.$$

Combining the last two inequalities yields a contradiction. □

Lemma 3.8. *Let F be a charge, and $A \in \mathcal{ABV}_{\mathrm{loc}}$. Given $\varepsilon > 0$, there is a gage δ on $\mathbb{R}^n$ such that*

$$\sum_{x_i \notin A} |F(A \cap E_i)| < \varepsilon \quad \textit{and} \quad \sum_{x_i \in A} |F(E_i - A)| < \varepsilon$$

for each strongly ε-regular δ-fine partition $\{(E_1, x_1), \ldots, (E_p, x_p)\}$.

Proof. Choose $\varepsilon > 0$ so that the compact set ∂A is contained in $U = U(1/\varepsilon^2)$. By Lemma 2.5, there is an absolutely continuous Radon measure μ in $\mathbb{R}^n$ such that

$$|F(E)| \leqslant \mu(E) + \varepsilon^2 P(E)$$

for each BV set $E \subset U$. There is a compact set $K \subset U \cap A$ such that

$$\mu(U \cap A - K) < \varepsilon.$$

Using Lemma 3.7, for each $x \in U \cap \mathrm{ext}_c A$ find $\delta_x > 0$ so that $B(x, \delta_x) \subset U$, and

$$P(E \cap A) \leqslant P(E - A) \tag{3.2}$$

for each strongly ε-regular set E with $x \in \mathrm{cl}_* E$ and $d(E) < \delta_x$. Making δ_x smaller, we may assume that $K \cap B(x, \delta_x) = \emptyset$. As A is an admissible set, $\mathrm{int}_c A \subset A$ and $A \cap \mathrm{ext}_c A = \emptyset$. Hence we can define a gage δ on $\mathbb{R}^n$ by letting

$$\delta(x) = \begin{cases} 1 & \text{if } x \in \mathrm{int}_c A, \\ 0 & \text{if } x \in \partial_c A, \\ \delta_x & \text{if } x \in U \cap \mathrm{ext}_c A, \\ \mathrm{dist}(x, \partial A) & \text{if } x \in \mathrm{ext}_c A - U. \end{cases}$$

Let $P = \{(E_1, x_1), \ldots, (E_p, x_p)\}$ be a strongly ε-regular δ-fine partition. From the definition of δ, inequality (3.2), and the strong ε-regularity of E_i, we obtain

(i) $x_i \notin \partial_c A$ for $i = 1, \ldots, p$;
(ii) $A \cap E_i \subset U \cap A - K$ when $x_i \in U \cap \mathrm{ext}_c A$;
(iii) $A \cap E_i = \emptyset$ when $x_i \in \mathrm{ext}_c A - U$;
(iv) $P(A \cap E_i) \leqslant (1/\varepsilon) P(A, \mathrm{in}\, E_i)$ when $x_i \in U \cap \mathrm{ext}_c A$.

Consequently,

$$\begin{aligned} \sum_{x_i \notin A} |F(A \cap E_i)| &= \sum_{x_i \notin \mathrm{int}_c A} |F(A \cap E_i)| = \sum_{x_i \in U \cap \mathrm{ext}_c A} |F(A \cap E_i)| \\ &\leqslant \sum_{x_i \in U \cap \mathrm{ext}_c A} [\mu(A \cap E_i) + \varepsilon^2 P(A \cap E_i)] \\ &\leqslant \mu(U \cap A - K) + \varepsilon \sum_{x_i \in U \cap \mathrm{ext}_c A} P(A, \mathrm{in}\, E_i) \leqslant \varepsilon[1 + P(A)], \end{aligned}$$

which proves the first desired inequality. Since $\mathbb{R}^n - A$ is in $\mathcal{ABV}_{\mathrm{loc}}$, the lemma follows by symmetry. □

Proposition 3.9. *Let G be a charge, $f \in R_*(\mathbb{R}^n, G)$, and $F = (R_*)\int f\,\mathrm{d}G$. If $A \in \mathcal{ABV}_{\mathrm{loc}}$, then $\mathbf{1}_A f \in R_*(\mathbb{R}^n, G)$ and $(R_*)\int \mathbf{1}_A f\,\mathrm{d}G = F \llcorner A$.*

P r o o f. Choose $\varepsilon > 0$. By the definition of R_*-integrability and Lemma 3.8, there is a gage δ on $\mathbb{R}^n$ such that

$$\sum_{i=1}^{p} |F(E_i) - f(x_i)G(E_i)| < \varepsilon,$$
$$\sum_{x_i \notin A} |F(A \cap E_i)| < \varepsilon \quad \text{and} \quad \sum_{x_i \in A} |F(E_i - A)| < \varepsilon$$

for each strongly ε-regular δ-fine partition $P = \{(E_1, x_1), \ldots, (E_p, x_p)\}$. Thus

$$\begin{aligned}
&\sum_{i=1}^{p} |(F \llcorner A)(E_i) - \mathbf{1}_A(x_i) f(x_i) G(E_i)| \\
&\qquad = \sum_{x_i \in A} |(F \llcorner A)(E_i) - f(x_i)G(E_i)| + \sum_{x_i \notin A} |(F \llcorner A)(E_i)| \\
&\qquad < \sum_{x_i \in A} |(F \llcorner A)(E_i) - F(E_i)| + \sum_{x_i \in A} |F(E_i) - f(x_i)G(E_i)| + \varepsilon \\
&\qquad < \sum_{x_i \in A} |F(E_i - A)| + 2\varepsilon < 3\varepsilon.
\end{aligned}$$

□

Corollary 3.10. *Let G be a charge, $f \in R_*(\mathbb{R}^n, G)$, and $F = (R_*)\int f\,\mathrm{d}G$. If $A, B \in \mathcal{ABV}$ and $f = 0$ on $\mathbb{R}^n - (A \cap B)$, then $F(A) = F(B)$.*

P r o o f. The assumption $f = 0$ on $\mathbb{R}^n - (A \cap B)$ implies that $f\mathbf{1}_A = f\mathbf{1}_B$. Thus, by Proposition 3.9, $F \llcorner A = F \llcorner B$ and

$$F(A) = F \llcorner A(A \cup B) = F \llcorner B(A \cup B) = F(B).$$

□

The *zero extension* of a function f defined on any set $A \subset \mathbb{R}^n$ is given by

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \in \mathbb{R}^n - A. \end{cases}$$

Definition 3.11. Let G be a charge, and $A \subset \mathbb{R}^n$. A function $f\colon A \to \mathbb{R}$ is called *R_*-integrable* with respect to G if $\overline{f} \in R_*(\mathbb{R}^n, G)$. The unique charge $F = (R_*)\int \overline{f}\,\mathrm{d}G$ is called the *indefinite R_*-integral* of f with respect to G, denoted by $(R_*)\int f\,\mathrm{d}G$. If $A \in \mathcal{ABV}$, then the number

$$(R_*)\int_A f\,\mathrm{d}G = F(A)$$

is called the *definite R_*-integral* of f over A with respect to G.

For $A \in \mathcal{ABV}_{\rm loc}$, the linear space of all functions $f\colon A \to \mathbb{R}$ that are R_*-integrable with respect to G is denoted by $R_*(A, G)$. The symbols $R_*(A)$ and $(R_*)\int_A f$ stand for $R_*(A, \mathscr{L})$ and $(R_*)\int_A f\,\mathrm{d}\mathscr{L}$, respectively.

Remark 3.12. If A is a bounded set, then we can define the definite R_*-integral of f over A with respect to G as

$$(R_*)\int_A f\,\mathrm{d}G = F(B),$$

where B is an arbitrary $\mathcal{ABV}$ set containing A. By Corollary 3.10, this definition does not depend on the choice of B. In addition, naturally defined improper integrals can determine the definite R_*-integral over unbounded subsets of $\mathbb{R}^n$ so that when applied to integration with respect to Lebesgue measure, the following is true: $(R_*)\int_A f = \int_A f$ for each measurable set A on which the Lebesgue integral $\int_A f$ exists. In this way, the definite R_*-integral extends the Lebesgue integral completely. We do not pursue the details, since only bounded sets are considered in the present paper.

Proposition 3.13. *Let G be a charge, $A \in \mathcal{ABV}_{\rm loc}$, and $f \in R_*(A, G)$. Then $f \restriction B$ belongs to $R_*(B, G)$ for each $B \in \mathcal{ABV}_{\rm loc}$ with $B \subset A$, and*

$$(R_*)\int (f \restriction B)\,\mathrm{d}G = \Big[(R_*)\int f\,\mathrm{d}G\Big] \llcorner B.$$

Proof. By our assumption $\overline{f} \in R_*(\mathbb{R}^n, G)$. Since $\overline{f \restriction B} = \mathbf{1}_B \overline{f}$ for each $B \subset A$, the proposition is a direct consequence of Proposition 3.9. $\square$

Let G be a charge, $A \in \mathcal{ABV}$, and $f \in R_*(A, G)$. If $B \in \mathcal{ABV}$ is a subset of A, we usually skip the restriction symbol $f \restriction A$, and write $f \in R_*(B, G)$ and $(R_*)\int_B f\,\mathrm{d}G$.

Theorem 3.14. *Let G be a charge, and let $A, B \in \mathcal{ABV}$ be such that $A \cap B$ is of σ-finite measure $\mathscr{H}$. If $f\colon A \cup B \to \mathbb{R}$ satisfies $f \restriction A \in R_*(A, G)$ and $f \restriction B \in R_*(B, G)$, then $f \in R_*(A \cup B, G)$ and*

$$(R_*)\int_{A\cup B} f\,\mathrm{d}G = (R_*)\int_A f\,\mathrm{d}G + (R_*)\int_B f\,\mathrm{d}G. \tag{3.3}$$

Proof. Since integration does not depend on the values of f on sets of σ-finite measure $\mathscr{H}$, we may assume that $A \cap B = \emptyset$. By our assumptions, $\overline{f}\mathbf{1}_A$ and $\overline{f}\mathbf{1}_B$ belong to $R_*(\mathbb{R}^n, G)$, and by linearity so does $\overline{f}\mathbf{1}_{A\cup B}$, and

$$(R_*)\int \overline{f}\mathbf{1}_{A\cup B}\,\mathrm{d}G = (R_*)\int \overline{f}\mathbf{1}_A\,\mathrm{d}G + (R_*)\int \overline{f}\mathbf{1}_B\,\mathrm{d}G. \tag{3.4}$$

By Corollary 3.10,

$$\left[(R_*)\int \overline{f}\mathbf{1}_A\,\mathrm{d}G\right](A\cup B) = \left[(R_*)\int \overline{f}\mathbf{1}_A\,\mathrm{d}G\right](A) = (R_*)\int_A f\,\mathrm{d}G$$

and similarly

$$\left[(R_*)\int \overline{f}\mathbf{1}_B\,\mathrm{d}G\right](A\cup B) = \left[(R_*)\int \overline{f}\mathbf{1}_B\,\mathrm{d}G\right](B) = (R_*)\int_B f\,\mathrm{d}G.$$

Therefore, the formula (3.3) follows by evaluating (3.4) on $A \cup B$. □

Proposition 3.15. *Let G be a charge, and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be such that $f \restriction B$ belongs to $R_*(B, G)$ for each closed ball $B \subset \mathbb{R}^n$. Then $f \in R_*(\mathbb{R}^n, G)$.*

Proof. Choose $\varepsilon > 0$, and let $B_0 = \emptyset$, and for $k = 1, 2, \ldots$,

$$B_k = B(0,k), \quad g_k = \overline{f \restriction B_k}, \quad F_k = (R_*)\int g_k\,\mathrm{d}G.$$

Since $F_k = F_{k+1} \llcorner B_k$ by Proposition 3.13, letting $F(A) = F_k(A)$ for each A in $\mathcal{ABV}$ with $A \subset B_k$ defines a charge F such that $F \llcorner B_k = F_k$ for $k = 1, 2, \ldots$ There are gages δ_k in $\mathbb{R}^n$ such that

$$\sum_{i=1}^{p} |g_k(x_i)G(E_i) - F_k(E_i)| < \varepsilon 2^{-k} \tag{3.5}$$

for each strongly ε-regular δ_k-fine partition $\{(E_1, x_1), \ldots, (E_p, x_p)\}$. The formula

$$\delta(x) = \begin{cases} \min\{\delta_1(x), \operatorname{dist}(x, \partial B_1)\} & \text{if } x \in \operatorname{int} B_1, \\ \min\{\delta_k(x), \operatorname{dist}(x, \partial B_k \cup \partial B_{k-1})\} & \text{if } k \geqslant 2 \text{ and } x \in \operatorname{int} B_k - B_{k-1}, \\ 0 & \text{if } x \in \bigcup_{k=1}^{\infty} \partial B_k \end{cases}$$

defines a gage δ on $\mathbb{R}^n$. If $\{(E_1, x_1), \ldots, (E_p, x_p)\}$ is a strongly ε-regular δ-fine partition, let $I_k = \{1 \leqslant i \leqslant p\colon\ x_i \in \operatorname{int} B_k - B_{k-1}\}$. By (3.5),

$$\sum_{i=1}^{p} |f(x_i)G(E_i) - F(E_i)| = \sum_{k=1}^{\infty} \sum_{x_i \in I_k} |g_k(x_i)G(E_i) - F_k(E_i)| < \varepsilon$$

and we see that $f \in R_*(\mathbb{R}^n, G)$ and $(R_*)\int f\,\mathrm{d}G = F$. □

Proposition 3.16. *Let G be a charge, and $A \in \mathcal{ABV}_{\rm loc}$. Then*

$$R_*(A,G) = R_*(A, G \llcorner A) \quad \textit{and} \quad (R_*)\int f \,\mathrm{d}G = (R_*)\int f \,\mathrm{d}(G \llcorner A)$$

is a charge in A for each $f \in R_*(A,G)$.

Proof. Select $f\colon A \to \mathbb{R}$ and let $g = \overline{f}$. Choose $\varepsilon > 0$, and for $k = 1, 2, \ldots$, let $B_k = \{x \in \mathbb{R}^n\colon k-1 \leqslant |f(x)| < k\}$. By Lemma 3.8, there are gages δ_k, $k = 1, 2, \ldots$, on $\mathbb{R}^n$ such that

$$\sum_{x_i \in A} |G(E_i - A)| < \frac{\varepsilon}{k} 2^{-k}$$

for each strongly ε-regular δ_k-fine partition $\{(E_1, x_1), \ldots, (E_p, x_p)\}$. Define a gage δ on $\mathbb{R}^n$ by letting $\delta(x) = \delta_k(x)$ for $x \in B_k$, and select a strongly ε-regular δ-fine partition $\{(E_1, x_1), \ldots, (E_p, x_p)\}$. Now for any charge F,

$$\begin{aligned}
&\left|\sum_{i=1}^p |F(E_i) - g(x_i)(G \llcorner A)(E_i)| - \sum_{i=1}^p |F(E_i) - g(x_i) G(E_i)|\right| \\
&\qquad \leqslant \sum_{i=1}^p |g(x_i)| \bigl|G(E_i) - (G \llcorner A)(E_i)\bigr| \\
&\qquad = \sum_{i=1}^p |g(x_i)| \bigl|G(E_i - A)\bigr| \leqslant \sum_{k=1}^\infty k \sum_{x_i \in A \cap B_k} |G(E_i - A)| < \varepsilon.
\end{aligned}$$

Thus f belongs to $R_*(A,G)$ if and only if it belongs to $R_*(A, G \llcorner A)$, and the common indefinite R_*-integral F is a charge in A by Proposition 3.13. $\square$

Theorem 3.17. *Let G be a charge, $A \in \mathcal{ABV}_{\rm loc}$, and $f\colon A \to \mathbb{R}$. Then f belongs to $R_*(A,G)$ if and only if there is a charge F in A that satisfies the following condition: given $\varepsilon > 0$, we can find a gage δ on A so that*

$$\sum_{i=1}^p |f(x_i)(G \llcorner A)(E_i) - F(E_i)| < \varepsilon, \tag{3.6}$$

or equivalently

$$\sum_{i=1}^p |f(x_i) G(E_i) - F(E_i)| < \varepsilon, \tag{3.7}$$

for each strongly ε-regular δ-fine partition $\{(E_1, x_1), \ldots, (E_p, x_p)\}$. In either case

$$F = (R_*)\int f \,\mathrm{d}G.$$

P r o o f. Let F be a charge in A that satisfies condition (3.6), and let $g = \overline{f}$. Choose $\varepsilon > 0$ and find a gage δ on A corresponding to F and ε according to our assumption. By Lemma 3.8, there is a gage σ on $\mathbb{R}^n$ such that

$$\sum_{x_i \notin A} |F(A \cap E_i)| < \varepsilon$$

for each strongly ε-regular σ-fine partition $\{(E_1, x_1), \dots, (E_p, x_p)\}$. The formula

$$\varDelta(x) = \begin{cases} \min\{\delta(x), \sigma(x)\} & \text{if } x \in A, \\ \sigma(x) & \text{if } x \in \mathbb{R}^n - A \end{cases}$$

defines a gage $\varDelta$ on $\mathbb{R}^n$, and if $\{(E_1, x_1), \dots, (E_p, x_p)\}$ is a $\varDelta$-fine partition, then $\{(E_i, x_i)\colon\ x_i \in A\}$ is a δ-fine partition. Since $F = F \llcorner A$,

$$\begin{aligned}\sum_{i=1}^{p} |F(E_i) - g(x_i)(G \llcorner A)(E_i)| &= \sum_{x_i \in A} |F(E_i) - g(x_i)(G \llcorner A)(E_i)| + \sum_{x_i \notin A}^{p} |F(E_i)| \\ &< \varepsilon + \sum_{x_i \notin A}^{p} |F(A \cap E_i)| < 2\varepsilon\end{aligned}$$

and we see that $g \in R_*(\mathbb{R}^n, G \llcorner A)$. By definition f belongs to $R_*(A, G \llcorner A)$, and hence to $R_*(A, G)$ according to Proposition 3.16. In addition,

$$F = (R_*)\int f \, \mathrm{d}(G \llcorner A) = (R_*)\int f \, \mathrm{d}G.$$

Conversely, let $f \in R_*(A, G)$. Then there is a gage δ on $\mathbb{R}^n$ such that

$$\sum_{i=1}^{p} |f(x_i)G(E_i) - F(E_i)| = \sum_{i=1}^{p} |g(x_i)G(E_i) - F(E_i)| < \varepsilon$$

for each strongly ε-regular $(\delta \restriction A)$-fine partition $P = \{(E_1, x_1), \dots, (E_p, x_p)\}$, since such P is also δ-fine. Thus F satisfies condition (3.7), and we infer that conditions (3.6) and (3.7) are equivalent. □

Corollary 3.18. *Let $A \in \mathcal{ABV}$. Then $R(A) \subset R_*(A)$ and for each $f \in R(A)$,*

$$(R_*)\int_A f = (R)\int_A f.$$

Proof. Let $F = (R)\int f$, and choose $\varepsilon > \eta > 0$. There is a gage δ on A such that

$$\sum_{i=1}^{p} \bigl| f(x_i)|A_i| - F(A_i) \bigr| < \eta$$

for each η-regular δ-fine partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ with $[P] \subset A$. Making δ smaller, we may assume that if $\{(E_1, x_1), \ldots, (E_p, x_p)\}$ is an ε-regular δ-fine partition, then $\{(A \cap E, x_i) \colon i = 1, \ldots, p\}$ is an η-regular δ-fine partition, see [15], Lemma 2.6.6. Thus if $\{(E_1, x_1), \ldots, (E_p, x_p)\}$ is a strongly ε-regular δ-fine partition, then

$$\sum_{i=1}^{p} \bigl| f(x_i)|A \cap E_i| - F(E_i) \bigr| = \sum_{i=1}^{p} \bigl| f(x_i)|A \cap E_i| - F(A \cap E_i) \bigr| < \eta < \varepsilon$$

and the corollary follows from Theorem 3.17. □

The following divergence theorem is an immediate consequence of Corollary 2.13 and [16], Theorem 5.19.

Theorem 3.19 (Gauss-Green). *Let $A \in \mathcal{ABV}$, let v be a continuous vector field defined on $\operatorname{cl} A$, and let $S \subset A$ be a set of σ-finite measure $\mathscr{H}$. If v is pointwise Lipschitz in $A - S$, then $\operatorname{div} v$ belongs to $R_*(A)$ and*

$$(R_*)\int_A \operatorname{div} v \, \mathrm{d}\mathscr{L} = \int_{\partial_* A} v \cdot \nu_A \, \mathrm{d}\mathscr{H}.$$

Theorem 3.20. *Let G be a charge, and let f be a function defined on $A \in \mathcal{ABV}$. Suppose there are a charge F in A and a sequence $\{A_k\}$ in $\mathcal{ABV}$ such that*

$$A_k \subset A, \quad f \in R_*(A_k, G), \quad F \llcorner A_k = (R_*)\int (f \restriction A_k) \, \mathrm{d}G$$

for $k = 1, 2, \ldots$ If $A_k \to A$, then $f \in R_*(A, G)$ and

$$(R_*)\int_A f \, \mathrm{d}G = F(A).$$

Proof. It follows from [15], Corollary 6.2.7 that the set $C = A - \bigcup_{k=1}^{\infty} A_k$ has σ-finite measure $\mathscr{H}$. Choose $\varepsilon > 0$, and find gages δ_k on $\mathbb{R}^n$ so that

$$(3.8) \quad \sum_{x_i \in A_k} |f(x_i)G(E_i) - F(A_k \cap E_i)| < \varepsilon 2^{-k} \quad \text{and} \quad \sum_{x_i \in A_k} |F(E_i - A_k)| < \varepsilon 2^{-k}$$

for each strongly ε-regular δ_k-fine partition $\{(E_1,x_1),\dots,(E_p,x_p)\}$; see Theorem 3.17 and Lemma 3.8. For each $x\in A-C$ let $k_x=\min\{k\in\mathbb{N}\colon x\in A_k\}$, and define a gage δ on A by letting $\delta(x)=\delta_{k_x}(x)$ if $x\in A-C$, and $\delta(x)=0$ if $x\in C$. Given a δ-fine partition $\{(E_1,x_1),\dots,(E_p,x_p)\}$, let $I_k=\{i\colon k_{x_i}=k\}$ and observe that $\{1,\dots,p\}=\bigcup_{k=1}^{q} I_k$, where $q=\max_i k_{x_i}$ and some I_k may be empty. Now

$$\begin{aligned}\sum_{i=1}^{p}|f(x_i)G(E_i)-F(E_i)| &= \sum_{k=1}^{q}\sum_{i\in I_k}|f(x_i)G(E_i)-F(E_i)|,\\ \sum_{i\in I_k}|f(x_i)G(E_i)-F(E_i)| &= \sum_{x_i\in A_k}|f(x_i)\,G(E_i)-F(A_k\cap E_i)|\\ &\quad+\sum_{x_i\in A_k}|F(E_i-A_k)|<\varepsilon 2^{-k+1},\end{aligned}$$

where the inequality follows from (3.8). Consequently,

$$\sum_{i=1}^{p}|f(x_i)G(E_i)-F(E_i)|<2\varepsilon$$

and the theorem follows from Theorem 3.17. □

R e m a r k 3.21. Corollary 3.18 and Theorem 3.20 show that the R_*-integral extends the GR-*integral* defined in [15], Section 6.3. The extension is proper, since the R-integral and GR-integral coincide in dimension one; see [16], Corollary 9.12, where the GR-integral is called the *continuous integral*.

Proposition 3.22. *Let G be a charge, $h\in R_*(\mathbb{R}^n,G)$, and $H=(R_*)\int h\,\mathrm{d}G$. If $A\in\mathcal{ABV}_{\mathrm{loc}}$, then $f\colon A\to\mathbb{R}$ belongs to $R_*(A,H)$ if and only if fh belongs to $R_*(A,G)$, in which case*

$$(R_*)\int fh\,\mathrm{d}G=(R_*)\int f\,\mathrm{d}H.$$

P r o o f. Choose $\varepsilon>0$, and find gages δ_k, $k=1,2,\dots$, on $\mathbb{R}^n$ so that

$$\sum_{i=1}^{p}|h(x_i)G(E_i)-H(E_i)|<\frac{\varepsilon}{k}2^{-k}$$

for each strongly ε-regular δ_k-fine partition $\{(E_1,x_1),\dots,(E_p,x_p)\}$. Let

$$A_k=\{x\in A\colon k-1\leqslant|f(x)|<k\},\quad k=1,2,\dots,$$

and define gage δ on A by letting $\delta(x) = \delta_k(x)$ when $x \in A_k$. Let $f \in R_*(A, H)$ and $F = (R_*)\int f \,\mathrm{d}H$. Theorem 3.17 shows that making δ smaller, we may assume

$$\sum_{i=1}^{p} |f(x_i)H(E_i) - F(E_i)| < \varepsilon$$

for each strongly ε-regular δ-fine partition $P = \{(E_1, x_1), \ldots, (E_p, x_p)\}$. For such P,

$$\begin{aligned}\sum_{i=1}^{p} |f(x_i)h(x_i)G(E_i) - F(E_i)| &\leqslant \sum_{k=1}^{\infty} \sum_{x_i \in A_k} |f(x_i)| \bigl|h(x_i)G(E_i) - H(E_i)\bigr| \\ &\quad + \sum_{i=1}^{p} |f(x_i)H(E_i) - F(E_i)| < \varepsilon \sum_{k=1}^{\infty} 2^{-k} + \varepsilon = 2\varepsilon.\end{aligned}$$

Thus $fh \in R_*(A, G)$ and $(R_*)\int fh \,\mathrm{d}G = (R_*)\int f \,\mathrm{d}H$ according to Theorem 3.17. Proving the converse is similar. □

Lemma 3.23. *Let $\Omega \subset \mathbb{R}^m$ be an open set, and let $\varphi\colon \Omega \to \mathbb{R}^n$ be a lipeomorphism. There is a constant $\gamma \geqslant 1$, depending only on φ, such that given $\varepsilon > 0$ and a gage δ on $\varphi(\Omega)$, the following is true: $\delta \circ \varphi$ is a gage on Ω, and if*

$$P = \{(E_1, x_1), \ldots, (E_p, x_p)\}$$

is a strongly ε-regular $(\delta \circ \varphi)$-fine partition with $\mathrm{cl}\,[P] \subset \Omega$, then

$$\varphi(P) = \{(\varphi(E_1), \varphi(x_1)), \ldots, (\varphi(E_p), \varphi(x_p))\}$$

is a strongly (ε/γ)-regular $(\gamma\delta)$-fine partition with $[\varphi(P)] \subset \varphi(\Omega)$.

Proof. Let $\psi = \varphi^{-1}$ and $c = \max\{1, \mathrm{Lip}\,\varphi, \mathrm{Lip}\,\psi\}$. Clearly $\delta \circ \varphi$ is a gage on Ω, and $\varphi(P)$ is $(c\delta)$-fine partition with $[\varphi(P)] \subset \varphi(\Omega)$. By a direct calculation, $\varphi(P)$ is $(c^{-2m}\varepsilon)$-regular. As $\mathrm{cl}\,[P]$ is a compact subset of Ω, there is an open BV set U with $\mathrm{cl}\,[P] \subset U$ and $\mathrm{cl}\,U \subset \Omega$. Thus we may assume from the onset that Ω and $\varphi(\Omega)$ are BV sets. In view of this, we can verify the isoperimetric property of $A_i = \varphi(E_i)$ by considering only a BV set $T \subset \varphi(\Omega)$. Letting $S = \psi(T)$, we calculate

$$\begin{aligned}\min\{P(A_i \cap T), P(A_i - T)\} &= \min\{P[\varphi(E_i \cap S)], P[\varphi(E_i - S)]\} \\ &\leqslant c^{m-1} \min\{P(E_i \cap S), P(E_i - S)\} \leqslant c^{m-1}\varepsilon^{-1} P(S, \mathrm{in}\, E_i) \\ &= c^{m-1}\varepsilon^{-1} \mathscr{H}(E_i \cap \partial_* S) = c^{m-1}\varepsilon^{-1} \mathscr{H}[\psi(A_i \cap \partial_* T)] \\ &\leqslant c^{2(m-1)}\varepsilon^{-1} \mathscr{H}(A_i \cap \partial_* T) = c^{2(m-1)}\varepsilon^{-1} P(T, \mathrm{in}\, A_i).\end{aligned}$$

Now it suffices to let $\gamma = c^{-2m}$. □

Theorem 3.24. *Let $\Omega \subset \mathbb{R}^n$ be an open nonempty set, and let $\varphi\colon \Omega \to \mathbb{R}^n$ be a lipeomorphism. If $A \in \mathcal{ABV}$, $\operatorname{cl} A \subset \Omega$, and $f\colon \varphi(A) \to \mathbb{R}$, then $f \in R_*[\varphi(A)]$ if and only if $(f \circ \varphi)|\det \varphi|$ belongs to $R_*(A)$, in which case*

$$(R_*)\int_{\varphi(A)} f = (R_*)\int_A (f \circ \varphi)|\det \varphi|. \tag{3.9}$$

Proof. Assume $f \in R_*[\varphi(A)]$. As $\operatorname{cl} A$ is a compact set, there is an open set $U \in \mathcal{ABV}$ such that $\operatorname{cl} A \subset U \subset \Omega$. Note that for any charge G, the set function $E \mapsto G(\varphi(E \cap U))\colon \mathcal{BV}_c \to \mathbb{R}$ is a charge as well. Define charges H and F in U by

$$H(E) = |\varphi(E \cap U)| \quad \text{and} \quad F(E) = (R_*)\int_{\varphi(E\cap U)} f, \quad \text{where } E \in \mathcal{BV}_c.$$

By Proposition 3.5 and the area theorem for the Lebesgue integral,

$$H(E) = |\varphi(E \cap U)| = (R_*)\int_E |\det \varphi| \restriction U$$

for each $E \in \mathcal{ABV}$. Choose $\varepsilon > 0$, and let $\gamma \geqslant 1$ be the constant associated with φ according to Lemma 3.23. By Theorem 3.17, there is a gage δ on $\varphi(A)$ such that

$$\sum_{i=1}^{p} \left| f(y_i)|A_i| - (R_*)\int_{A_i} f \right| < \varepsilon \tag{3.10}$$

for each strongly (ε/γ)-regular $(\gamma\delta)$-fine partition $\{(A_1, y_1), \dots, (A_p, y_p)\}$. Now select a strongly ε-regular $(\delta \circ \varphi)$-fine partition $P = \{(E_1, x_1), \dots, (E_p, x_p)\}$. Making δ smaller, we may assume that $\operatorname{cl}[P] \subset U$. Lemma 3.23 and (3.10) imply

$$\sum_{i=1}^{p} |f[\varphi(x_i)]H(E_i) - F(E_i)| = \sum_{i=1}^{p} \left| f[\varphi(x_i)]|\varphi(E_i)| - (R_*)\int_{\varphi(E_i)} f \right| < \varepsilon.$$

Theorem 3.20 shows that $f \circ \varphi \in R_*(A, H)$ and $(R_*)\int_A f \circ \varphi \, \mathrm{d}H = (R_*)\int_{\varphi(A)} f$, and equality (3.9) follows from Proposition 3.22. The converse is obtained from symmetry. □

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