Czechoslovak Mathematical Journal, 66 (141) (2016), 13–25

PACKING CONSTANT FOR CESÀRO-ORLICZ SEQUENCE SPACES

Zhen-Hua Ma, Beijing, Zhangjiakou,
Li-Ning Jiang, Qiao-Ling Xin, Beijing

(Received March 26, 2014)

Abstract. The packing constant is an important and interesting geometric parameter of Banach spaces. Inspired by the packing constant for Orlicz sequence spaces, the main purpose of this paper is calculating the Kottman constant and the packing constant of the Cesàro-Orlicz sequence spaces (ces_φ) defined by an Orlicz function φ equipped with the Luxemburg norm. In order to compute the constants, the paper gives two formulas. On the base of these formulas one can easily obtain the packing constant for the Cesàro sequence space ces_p and some other sequence spaces. Finally, a new constant $\widetilde{D}(X)$, which seems to be relevant to the packing constant, is given.

Keywords: packing constant; Cesàro sequence space; Cesàro-Orlicz sequence space

MSC 2010: 46A45, 46B20

1. Introduction

The packing constant is an important and interesting geometric parameter. It is of great importance for studying the geometric structure, isometric embedding, noncompactness and reflexivity in Banach spaces, see [1], [9], [18], [20]. From the 1950's, many mathematicians have studied this constant on different Banach spaces, see [8], [9], [18], [20].

As an extension of the Cesàro sequence space, Cesàro-Orlicz sequence spaces appeared for the first time in 1988 [12] and since then they have been studied by a number of authors [6], [7], [10], [14], [16], [19]. In this paper, we consider the packing constant for Cesàro-Orlicz spaces, and also we introduce a new constant $\widetilde{D}(X)$, which seems to be relevant to the packing constant.

The research has been supported by NSFC (Grant No. 10971011, 11371222) and Youth Fund Project of Hebei University of Architectural (Grant No. QN201411).

Let X be a Banach space, let $B(X)$ and $S(X)$ denote the unit ball and the unit sphere of X, respectively.

In a Banach space X, a sequence of balls with centers at $x_1, x_2, \dots$ and a fixed radius $r > 0$ is said to be packed into the unit ball if the following two conditions hold:

(i) $\|x_n\| \leqslant 1 - r$, $n = 1, 2, \dots$,

(ii) $\|x_n - x_m\| \geqslant 2r$, $n \neq m$, $n, m = 1, 2, \dots$

One can associate a constant to X as follows: $P(X) = \sup\{r > 0$: infinitely many disjoint balls of radius r are packed into the unit ball of $X\}$, i.e., if $P(X) \leqslant r$, then $B(X)$ contains infinitely many disjoint balls with radius r, and when $P(X) > r$, then $B(X)$ contains only finitely many such balls. If $\dim X = n < \infty$, then $r = 0$, so we only need to consider the case of $\dim X = \infty$.

In the 1950's, J. A. C. Burlak, R. A. Rankin and A. P. Robertson [1] gave the following formula of the packing constant for a Banach space X:

$$P(X) = \sup\{r > 0\colon\ \exists \{x_n\}_{n=1}^{\infty} \subset B(X),\ \|x_n\| \leqslant 1 - r,\\ \|x_n - x_m\| \geqslant 2r \text{ for } n \neq m\}.$$

Kottman [9] has shown that

$$P(X) = \frac{D(X)}{2 + D(X)},$$

where $D(X) = \sup\{\operatorname{sep}(\{x_n\})\colon \{x_n\} \subset S(X)\}$ and $\operatorname{sep}(\{x_n\}) = \inf\{\|x_n - x_m\|\colon n \neq m\}$. Also, he has given the inequality for any infinite-dimensional Banach space X:

$$\frac{1}{3} \leqslant P(X) \leqslant \frac{1}{2}.$$

In the rest of this section, we collect some definitions and basic facts related to the theory of Orlicz and Cesàro-Orlicz sequence spaces.

A map $\varphi\colon \mathbb{R} \to [0, \infty]$ is said to be an Orlicz function if φ is even, convex, left continuous on $\mathbb{R}_+$, continuous at zero, $\varphi(0) = 0$ and $\varphi(u) \to \infty$ as $u \to \infty$. If φ takes the value zero only at zero we will write $\varphi > 0$ and if φ takes only finite values we will write $\varphi < \infty$, see [2], [15], [17].

Let l^0 be the set of all real sequences $x = (x(i))_{i=1}^{\infty}$.

We define the Orlicz sequence space as follows:

$$l_{\varphi} = \{x \in l^0\colon\ \varrho_{\varphi}(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where $\varrho_{\varphi}(x) = \sum\limits_{i=1}^{\infty} \varphi(|x(i)|)$.

It is well known that l_φ equipped with the so called Luxemburg norm

$$\|x\|_\varphi = \inf\Big\{\lambda > 0,\ \varrho_\varphi\Big(\frac{x}{\lambda}\Big) \leqslant 1\Big\}$$

is a Banach space, see [2].

Also, for an Orlicz function φ, one can define on l^0 a convex modular, see [15], [17]

$$\varrho_{\mathrm{ces}_\varphi}(x) = \sum_{i=1}^{\infty} \varphi\bigg(\frac{1}{i}\sum_{j=1}^{i} |x(j)|\bigg).$$

The space

$$\mathrm{ces}_\varphi = \{x \in l^0\colon\ \varrho_{\mathrm{ces}_\varphi}(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where φ is an Orlicz function, is called the Cesàro-Orlicz sequence space. We equip this space with the Luxemburg norm

$$\|x\|_{\mathrm{ces}_\varphi} = \inf\Big\{\lambda > 0\colon\ \varrho_{\mathrm{ces}_\varphi}\Big(\frac{x}{\lambda}\Big) \leqslant 1\Big\}.$$

Notice that if $\varphi(u) = |u|^p$, $1 \leqslant p < \infty$, the space ces_φ is nothing but the Cesàro sequence space ces_p, see [3], [4], [11] and the Luxemburg norm generated by this power function is expressed by the formula:

$$\|x\|_{\mathrm{ces}_p} = \bigg[\sum_{i=1}^{\infty}\bigg(\frac{1}{i}\sum_{j=1}^{i}|x(j)|\bigg)^p\bigg]^{1/p}.$$

We say an Orlicz function φ satisfies the Δ_2-condition at zero ($\varphi \in \Delta_2(0)$ for short) if there are $K > 0$ and $a > 0$ such that $\varphi(a) > 0$ and $\varphi(2u) \leqslant K\varphi(u)$ for all $u \in [0, a]$, see [2].

A Banach space $(X, \|\cdot\|)$, which is a subspace of l^0, is said to be a Köthe sequence space, if the following two conditions are satisfied:

(i) for any $x \in l^0$ and $y \in X$ such that $|x(i)| \leqslant |y(i)|$ for all $i \in \mathbb{N}$, we have $x \in X$ and $\|x\| \leqslant \|y\|$,

(ii) there exists $x \in X$ with $x(i) \neq 0$ for all $i \in \mathbb{N}$.

Any nontrivial Cesàro-Orlicz sequence space belongs to the class of Köthe sequence spaces, see [5].

An element x from a Köthe sequence space $(X, \|\cdot\|)$ is called order continuous if for any sequence $\{x_n\}$ in X_+ (the positive cone of X) such that $x_n(i) \leqslant |x(i)|$ for all i, $n \in \mathbb{N}$ and $x_n \to 0$ coordinatewise, we have $\|x_n\| \to 0$.

A Köthe sequence space X is said to be order continuous if any $x \in X$ is order continuous. It is easy to see that X is order continuous if and only if $\|(0, \ldots 0,$

$x(n+1), x(n+2), \ldots)\| \to 0$ as $n \to \infty$ for any $x \in X$. Any nontrivial Cesàro-Orlicz sequence space is order continuous, see [5].

We say a Köthe sequence space X has the Fatou property if for any sequence $\{x_n\}$ in X_+ and any $x \in l^0$ such that $x_n \to x$ coordinatewise and $\sup_n \|x_n\| < \infty$, we have that $x \in X$ and $\|x_n\| \to \|x\|$. It is known that for any Köthe sequence (function) space the Fatou property implies its completeness, see [13]. From [5] we know that ces_φ equipped the Luxemburg norm has the Fatou property, hence it is a Banach space.

2. Packing constant for Cesàro-Orlicz sequence spaces

Inspired by the packing constant for Orlicz sequence spaces, see [2], we will consider a number d_x for $x \in S(\mathrm{ces}_\varphi)$ in Cesàro-Orlicz sequence spaces.

Lemma 2.1. *If $\varphi \in \Delta_2(0)$, then for any $x \in S(\mathrm{ces}_\varphi)$, there exists a unique $d_x > 0$ such that*

$$\varrho_{\mathrm{ces}_\varphi}\left(\frac{x}{d_x}\right) = \frac{1}{2}.$$

P r o o f. For any fixed point $x \in S(\mathrm{ces}_\varphi)$, the function

$$f(k) = \varrho_{\mathrm{ces}_\varphi}\left(\frac{x}{k}\right), \quad k \in (0, \infty)$$

is a continuous function since φ is continuous.

Hence, by the definition of the Orlicz function φ, we obtain that $\lim_{k\to\infty} f(k) = 0$ and $\lim_{k\to 0^+} f(k) = \infty$.

Notice that for $\varphi \in \Delta_2(0)$ and $x \in S(\mathrm{ces}_\varphi)$, by Lemma 2.5 in [5], the condition $\|x\|_{\mathrm{ces}_\varphi} = 1$ implies that

$$f(1) = \varrho_{\mathrm{ces}_\varphi}(x) = 1.$$

Consequently, there exists a $k \in (0, \infty)$ such that $\varrho_{\mathrm{ces}_\varphi}(x/k) = 1/2$. Using the monotonicity of the function φ, the number k, which is determined by the choice of x, is unique, and we denote it by d_x. $\square$

In the following we define

$$d = \sup\{d_x \colon\ x \in S(\mathrm{ces}_\varphi)\},$$

and we will study the relation between this number and the packing constant of ces_φ.

Lemma 2.2. *Suppose that φ is an Orlicz function. If $\varphi \in \Delta_2(0)$, then $1 < d_x \leqslant 2$ and therefore $1 < d \leqslant 2$.*

Proof. Since $x \in S(\mathrm{ces}_\varphi)$, we have $d_x \neq 1$ by Lemma 2.5 in [5], and $d_x > 1$ since φ is nondecreasing on $[0,\infty)$.

For any $x \in S(\mathrm{ces}_\varphi)$, since φ is convex and $d_x > 1$, we have

$$\frac{1}{2} = \varrho_{\mathrm{ces}_\varphi}\Big(\frac{x}{d_x}\Big) \leqslant \frac{1}{d_x}\varrho_{\mathrm{ces}_\varphi}(x) = \frac{1}{d_x},$$

which means that $d_x \leqslant 2$.

By the definition of d we get the conclusion. □

Next, we prove that for any Cesàro-Orlicz sequence space, if $\varphi \in \Delta_2(0)$, then $D(\mathrm{ces}_\varphi) = d$.

Theorem 2.1. *For any Cesàro-Orlicz sequence space:*

(1) *if $\varphi \in \Delta_2(0)$, then $D(\mathrm{ces}_\varphi) = d$, i.e., $P(\mathrm{ces}_\varphi) = d/(2+d)$,*
(2) *if $\varphi \notin \Delta_2(0)$, then $D(\mathrm{ces}_\varphi) = 2$, i.e., $P(\mathrm{ces}_\varphi) = 1/2$.*

Proof. (1) From Theorem 1 in [3] and Lemma 1 in [5] we know that if $\varphi \in \Delta_2(0)$, then for any Cesàro-Orlicz sequence space ces_φ,

$$D(\mathrm{ces}_\varphi) = \sup\Big\{\mathrm{sep}(\{x_n\})\colon\ x_n = \sum_{i=i_{n-1}+1}^{i_n} x_n(i)\, e_i \in S(\mathrm{ces}_\varphi)\Big\},$$

where $0 = i_0 < i_1 < i_2 < \ldots$

For these x_n and x_m, by Lemma 2.1, there exist $d_{x_n}, d_{x_m} \in (1,2]$ such that

$$\varrho_{\mathrm{ces}_\varphi}\Big(\frac{x_n}{d_{x_n}}\Big) = \varrho_{\mathrm{ces}_\varphi}\Big(\frac{x_m}{d_{x_m}}\Big) = \frac{1}{2}.$$

Let $\varepsilon_1 > 0$ be small enough such that $d - \varepsilon_1 \geqslant \max\{d_{x_n}, d_{x_m}\} > 0$, where $m \neq n$. By Lemma 2.3 in [5], there exists a $\delta(L,\varepsilon) > 0$ such that

$$|\varrho_{\mathrm{ces}_\varphi}(x+y) - \varrho_{\mathrm{ces}_\varphi}(y)| < \varepsilon$$

for all $x, y \in \mathrm{ces}_\varphi$ with $\varrho_{\mathrm{ces}_\varphi}(x) \leqslant L$ and $\varrho_{\mathrm{ces}_\varphi}(y) \leqslant \delta(L,\varepsilon)$.

Take $n \in \mathbb{N}$ large enough such that

$$\sum_{k=i_{m-1}+1}^{\infty} \varphi\Big(\frac{a_n}{k d_{x_m}}\Big) < \varepsilon',$$

where $\varepsilon' > 0$ is small enough and $a_n = \sum\limits_{i=1}^{i_n} |x_n(i)|$.

Since $\operatorname{supp} x_n \cap \operatorname{supp} x_m = \emptyset$, for any $m > n$ we have

$$\begin{aligned}
\varrho_{\mathrm{ces}_\varphi}\Big(\frac{x_n - x_m}{d - \varepsilon_1}\Big) &= \varrho_{\mathrm{ces}_\varphi}\Big(\frac{x_n + x_m}{d - \varepsilon_1}\Big) \\
&= \varrho_{\mathrm{ces}_\varphi}\Big(\frac{d_{x_n}}{d - \varepsilon_1}\frac{x_n}{d_{x_n}} + \frac{d_{x_m}}{d - \varepsilon_1}\frac{x_m}{d_{x_m}}\Big) \\
&= \sum_{k=1}^{i_{m-1}} \varphi\Big(\frac{d_{x_n}}{d - \varepsilon_1}\frac{1}{k}\sum_{i=1}^{k}\Big|\frac{x_n(i)}{d_{x_n}}\Big|\Big) \\
&\quad + \sum_{k=i_{m-1}+1}^{\infty} \varphi\Big(\frac{d_{x_m}}{d - \varepsilon_1}\frac{1}{k}\Big(\frac{a_n}{d_{x_m}} + \sum_{i=1}^{k}\Big|\frac{x_m(i)}{d_{x_m}}\Big|\Big)\Big) \\
&\leqslant \frac{d_{x_n}}{d - \varepsilon_1}\sum_{k=1}^{i_{m-1}} \varphi\Big(\frac{1}{k}\sum_{i=1}^{k}\Big|\frac{x_n(i)}{d_{x_n}}\Big|\Big) \\
&\quad + \frac{d_{x_m}}{d - \varepsilon_1}\sum_{k=i_{m-1}+1}^{\infty} \varphi\Big(\frac{1}{k}\Big(\frac{a_n}{d_{x_m}} + \sum_{i=1}^{k}\Big|\frac{x_m(i)}{d_{x_m}}\Big|\Big)\Big) \\
&\leqslant \frac{d_{x_n}}{d - \varepsilon_1}\varrho_{\mathrm{ces}_\varphi}\Big(\frac{x_n}{d_{x_n}}\Big) + \frac{d_{x_m}}{d - \varepsilon_1}\Big(\varrho_{\mathrm{ces}_\varphi}\Big(\frac{x_m}{d_{x_m}}\Big) + \varepsilon'\Big) \\
&= \frac{d_{x_n}}{d - \varepsilon_1}\frac{1}{2} + \frac{d_{x_m}}{d - \varepsilon_1}\Big(\frac{1}{2} + \varepsilon'\Big) \\
&\leqslant \frac{d_{x_n} + d_{x_m}}{d - \varepsilon_1}\Big(\frac{1}{2} + \varepsilon'\Big) \\
&\leqslant \frac{2d}{d - \varepsilon_1}\Big(\frac{1}{2} + \varepsilon'\Big) = \frac{d}{d - \varepsilon_1}(1 + 2\varepsilon') \\
&= \Big(1 + \frac{\varepsilon_1}{d - \varepsilon_1}\Big)(1 + 2\varepsilon') = 1 + \varepsilon'',
\end{aligned}$$

where $\varepsilon'' = 2\varepsilon' + \varepsilon_1/(d - \varepsilon_1) + 2\varepsilon_1\varepsilon'/(d - \varepsilon_1) > 0$.

Hence, by the definition of the Luxemburg norm, we know that for any $n \neq m$, $\|x_n - x_m\| \geqslant d - \varepsilon_1$, i.e., $D(\mathrm{ces}_\varphi) \geqslant d$ by the arbitrariness of $\varepsilon_1 > 0$.

Next, we will show the reverse inequality. For the above ε', let ε_2 be small enough with $\varepsilon_2 > 2\varepsilon' d$. By the same computation as above, we have

$$\begin{aligned}
\varrho_{\mathrm{ces}_\varphi}\Big(\frac{x_n - x_m}{d + \varepsilon_2}\Big) &= \varrho_{\mathrm{ces}_\varphi}\Big(\frac{x_n + x_m}{d + \varepsilon_2}\Big) \\
&= \varrho_{\mathrm{ces}_\varphi}\Big(\frac{d_{x_n}}{d + \varepsilon_2}\frac{x_n}{d_{x_n}} + \frac{d_{x_m}}{d + \varepsilon_2}\frac{x_m}{d_{x_m}}\Big) \\
&\leqslant \frac{d_{x_n}}{d + \varepsilon_2}\varrho_{\mathrm{ces}_\varphi}\Big(\frac{x_n}{d_{x_n}}\Big) + \frac{d_{x_m}}{d + \varepsilon_2}\Big(\varrho_{\mathrm{ces}_\varphi}\Big(\frac{x_m}{d_{x_m}}\Big) + \varepsilon'\Big)
\end{aligned}$$

$$\begin{aligned}
&= \frac{d_{x_n}}{d+\varepsilon_2}\frac{1}{2} + \frac{d_{x_m}}{d+\varepsilon_2}\Big(\frac{1}{2}+\varepsilon'\Big)\\
&\leqslant \frac{d_{x_n}+d_{x_m}}{d+\varepsilon_2}\Big(\frac{1}{2}+\varepsilon'\Big) \leqslant \frac{2d}{d+\varepsilon_2}\Big(\frac{1}{2}+\varepsilon'\Big)\\
&= \frac{d}{d+\varepsilon_2}(1+2\varepsilon') = \Big(1-\frac{\varepsilon_2}{d+\varepsilon_2}\Big)(1+2\varepsilon') = 1-\varepsilon'',
\end{aligned}$$

where $\varepsilon'' = \varepsilon_2(2\varepsilon'+1)/(d+\varepsilon_2) - 2\varepsilon'$ and $\varepsilon'' > 0$ since $\varepsilon_2 > 2\varepsilon' d$.

By the definition of the Luxemburg norm, for any $n \neq m$, $\|x_n - x_m\| \leqslant d + \varepsilon_2$.

By the arbitrariness of $\varepsilon_2 > 0$, we obtain $D(\mathrm{ces}_\varphi) \leqslant d$, and consequently $D(\mathrm{ces}_\varphi) = d$, i.e., $P(\mathrm{ces}_\varphi) = d/(2+d)$.

(2) From [8] and Proposition 1 in [16], we know that if $\varphi \notin \Delta_2(0)$, then ces_φ is nonreflexive and $D(\mathrm{ces}_\varphi) = 2$. □

From Lemma 2.1, one can see that for different $x \in S(\mathrm{ces}_\varphi)$ there exist unique different d_x satisfying the relation $\varrho_{\mathrm{ces}_\varphi}(x/d_x) = 1/2$. The following theorem gives a sufficient condition for d_x being constant, namely, for different $x, y \in S(\mathrm{ces}_\varphi)$, $d_x = d_y$.

Theorem 2.2. *Suppose that the Orlicz function $\varphi \in \Delta_2(0)$ and*

$$\varphi(\lambda x) = f(\lambda)\varphi(x),$$

where $\lambda \in \mathbb{R}^+$ and $f(\cdot)$ is continuous and reversible. Then for every $x \in S(\mathrm{ces}_\varphi)$,

$$D(\mathrm{ces}_\varphi) = \frac{1}{f^{-1}(\frac{1}{2})}.$$

P r o o f. First we make sure that $f^{-1}(1/2) \neq 0$. Indeed, if $f^{-1}(1/2) = 0$, then $f(0) = 1/2$ since $f(\cdot)$ is reversible.

For any $x \in S(\mathrm{ces}_\varphi)$, if

$$\varphi\Big(\frac{x}{2}\Big) = f\Big(\frac{1}{2}\Big)\varphi(x) = 0,$$

then $x = 0$ by the definition of φ, which is a contradiction with $x \in S(\mathrm{ces}_\varphi)$.

By Theorem 2.1, for any $x \in S(\mathrm{ces}_\varphi)$, let

$$\begin{aligned}
\varrho_{\mathrm{ces}_\varphi}\Big(\frac{x}{d_x}\Big) &= \sum_{i=1}^{\infty}\varphi\Big(\frac{1}{i}\sum_{j=1}^{i}\Big|\frac{x(j)}{d_x}\Big|\Big)\\
&= \sum_{i=1}^{\infty}\varphi\Big(\frac{1}{d_x}\frac{1}{i}\sum_{j=1}^{i}|x(j)|\Big)\\
&= f\Big(\frac{1}{d_x}\Big)\sum_{i=1}^{\infty}\varphi\Big(\frac{1}{i}\sum_{j=1}^{i}|x(j)|\Big) = f\Big(\frac{1}{d_x}\Big) = \frac{1}{2}.
\end{aligned}$$

Thus $d_x = 1/f^{-1}(1/2)$ and $d = d_x$ by the arbitrariness of $x \in S(\mathrm{ces}_\varphi)$, which means $D(\mathrm{ces}_\varphi) = 1/f^{-1}(1/2)$. □

Now for some Cesàro-Orlicz sequence spaces, one can easily compute the packing constant by Theorem 2.2. Let us see first an example.

Example 2.1. If φ is the Lebesgue N-function: $\varphi(u) = |u|^p/p$, $1 < p < \infty$, then

$$D(X) = 2^{1/p}.$$

Proof. Clearly $\varphi(u) \in \Delta_2(0)$, and

$$\varphi(\lambda u) = \frac{|\lambda u|^p}{p} = |\lambda|^p \frac{|u|^p}{p} = |\lambda|^p \varphi(u),$$

so $f(\lambda) = |\lambda|^p$.

Hence, we have

$$D(X) = \frac{1}{f^{-1}(\frac{1}{2})} = 2^{1/p}.$$

□

Next, we will give the second formula which can be used to compute the packing constant. In the following, we denote by $C(\mathrm{ces}_\varphi)$ the number

$$\sup\{\inf \varrho_{\mathrm{ces}_\varphi}(x_n - x_m),\ m \neq n\},$$

where $x_n = \sum\limits_{i=i_{n-1}+1}^{i_n} \in S(\mathrm{ces}_\varphi)$, and $0 = i_0 < i_1 < i_2 < \ldots$

Theorem 2.3. *If $\varphi \in \Delta_2(0)$, then $C(\mathrm{ces}_\varphi) = 2$.*

Proof. Using the convexity of φ, we have the following inequality

$$\varphi(a + b) \geqslant \varphi(a) + \varphi(b) \quad \text{for all } a, b \in \mathbb{R}_+.$$

Since $\operatorname{supp} x_n \cap \operatorname{supp} x_m = \emptyset$ $(n \neq m)$, we have

$$\varrho_{\mathrm{ces}_\varphi}(x_n - x_m) = \varrho_{\mathrm{ces}_\varphi}(x_n + x_m) \geqslant \varrho_{\mathrm{ces}_\varphi}(x_n) + \varrho_{\mathrm{ces}_\varphi}(x_m) = 2.$$

On the other hand, by Lemma 2.3 in [5], for a fixed $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left|\sum_{n=1}^{\infty} \varphi\left(\frac{1}{n}\sum_{i=1}^{n} |x(i) + y(i)|\right) - \sum_{n=1}^{\infty} \varphi\left(\frac{1}{n}\sum_{i=1}^{n} |x(i)|\right)\right| < \varepsilon,$$

whenever $\sum\limits_{n=1}^{\infty} \varphi\Big(n^{-1} \sum\limits_{i=1}^{n} |x(i)|\Big) \leqslant 1$ and $\sum\limits_{n=1}^{\infty} \varphi\Big(n^{-1} \sum\limits_{i=1}^{n} |y(i)|\Big) < \delta$.

Take $n_1 \in \mathbb{N}$ large enough such that $\sum\limits_{k=i_{n_1}+1}^{\infty} \varphi(a_1/k) < \varepsilon$. Hence for any $m > n$,

$$\begin{aligned}
\varrho_{\mathrm{ces}_\varphi}(x_m - x_1) &= \varrho_{\mathrm{ces}_\varphi}(x_m + x_1) \\
&= \sum_{k=1}^{i_{m-1}} \varphi\Big(\frac{1}{k}\sum_{i=1}^{k}|x_1(i)|\Big) + \sum_{k=i_{m-1}+1}^{\infty} \varphi\Big(\frac{1}{k}\Big(a_1 + \sum_{i=1}^{k}|x_m(i)|\Big)\Big) \\
&< \sum_{k=1}^{\infty} \varphi\Big(\frac{1}{k}\sum_{i=1}^{k}|x_1(i)|\Big) + \sum_{k=i_{m-1}+1}^{\infty} \varphi\Big(\frac{1}{k}\sum_{i=1}^{k}|x_m(i)|\Big) + \varepsilon \\
&= \varrho_{\mathrm{ces}_\varphi}(x_1) + \varrho_{\mathrm{ces}_\varphi}(x_m) + \varepsilon = 2 + \varepsilon.
\end{aligned}$$

Repeating this process, we have $\inf \varrho_{\mathrm{ces}_\varphi}(x_n - x_m) \leqslant 2 + \varepsilon$, $m \neq n$.

By the arbitrariness of $\varepsilon > 0$, we obtain $\sup\{\inf \varrho_{\mathrm{ces}_\varphi}(x_n - x_m) \colon m \neq n\} = 2$, and consequently $C(\mathrm{ces}_\varphi) = 2$. □

Remark 2.1. By Theorem 2.3, using the relation between norm and modular, one can calculate $D(X)$ by $C(X)$.

Let us compute the number $D(\mathrm{ces}_p)$ again.

Example 2.2. As we have seen, if $\varphi(u) = |u|^p$, then the space ces_φ is the Cesàro sequence space ces_p. For any $x \in \mathrm{ces}_p$, because $\|x\|^p_{\mathrm{ces}_p} = \|x\|^p_{\mathrm{ces}_\varphi} = \varrho_{\mathrm{ces}_\varphi}(x)$, by Theorem 2.3, we get $D(\mathrm{ces}_\varphi) = C^{1/p} = 2^{1/p}$.

3. A new constant on BK sequence spaces

We say a nonzero space X is a real sequence space, if X is a linear space and $X \subset l^0$. If X is a sequence space, for any $k \in \mathbb{Z}^+$, we call the mapping $p_k \colon X \to \mathbb{R}$ a coordinate mapping, if

$$p_k(x) = x_k, \quad x = \{x_k\} \in X.$$

We say a sequence space X has the K property if any p_k is continuous. We call a sequence space X a BK sequence space if it is a Banach space with the K property.

We list some classic sequence spaces with the corresponding norms:

(1) $c_0 = \big\{x \in l^0 \colon \lim\limits_{k\to\infty} x_k = 0\big\}$, $\|x\| = \sup\limits_k |x_k|$.

(2) $l_p = \Big\{x \in l^0 \colon \sum\limits_{k=1}^{\infty} |x_k|^p < \infty,\ 1 \leqslant p < \infty\Big\}$, $\|x\|_p = \Big(\sum\limits_{k=1}^{\infty} |x_k|^p\Big)^{1/p}$.

(3) $l^\infty = \{x \in l^0 \colon \sup\limits_k |x_k| < \infty\}$, $\|x\| = \sup\limits_k |x_k|$.

(4) $c = \left\{x \in l^0 \colon \lim\limits_{k\to\infty} x_k < \infty\right\}$, $\|x\| = \sup\limits_k |x_k|$.

(5) $\mathrm{ces}_p = \left\{x \in l^0 \colon \left[\sum\limits_{n=1}^{\infty} \left(n^{-1} \sum\limits_{k=1}^{n} |x_k|\right)^p\right]^{1/p} < \infty,\ 1 < p < \infty\right\}$,

$\|x\|^p = \sum\limits_{n=1}^{\infty} (\sigma x(n))^p$, where $\sigma x(n) = n^{-1} \sum\limits_{k=1}^{n} |x_k|$.

All spaces above are BK sequence spaces, see [21]. Especially, the real separable Hilbert sequence space is a BK space.

In a BK sequence space X we denote $e_n = (0, 0, \ldots, \overset{n\text{th}}{1}, 0, \ldots)$. In general, $\|e_n\| \neq 1$, but then $\|s_n\| = \|e_n/\|e_n\|\| = 1$, i.e., $s_n \in S(X)$. We will consider a geometric constant which seems to be related to the packing constant. In the following, we denote by $\widetilde{D}(X)$ the number

$$\sup\{\mathrm{sep}(\{s_n\})\}, \quad n = 1, 2, \ldots,$$

where $\mathrm{sep}(\{s_n\}) = \inf\{\|s_n - s_m\| \colon n \neq m\}$.

Clearly $\widetilde{D}(X) \leqslant D(X)$. In the following, we will compute $\widetilde{D}(X)$ on the above mentioned BK sequence spaces.

Example 3.1. For any $p \in (1, \infty)$ there holds the formula

$$\widetilde{D}(\mathrm{ces}_p) = D(\mathrm{ces}_p) = 2^{1/p}.$$

Proof. For $s_n = e_n/\|e_n\|$, $n = 1, 2, \ldots$ and $e_n = (0, 0, \ldots, \overset{n\text{th}}{1}, 0, \ldots)$, certainly $s_n \in S(\mathrm{ces}_p)$.

We suppose without loss of generality that $n > m$ when $m = 1$. Then we have

$$\begin{aligned}
\|s_n - s_1\|^p &= \sum_{m=1}^{n-1} \left(\frac{1}{m} \frac{1}{\|e_1\|}\right)^p + \sum_{m=n}^{\infty} \left[\frac{1}{m}\left(\frac{1}{\|e_1\|} + \frac{1}{\|e_n\|}\right)\right]^p \\
&= \left(\frac{1}{\|e_1\|}\right)^p \left(1 + \frac{1}{2^p} + \ldots + \frac{1}{(n-1)^p}\right) \\
&\quad + \left(\frac{1}{\|e_1\|} + \frac{1}{\|e_n\|}\right)^p \left(\frac{1}{n^p} + \frac{1}{(n+1)^p} + \ldots\right) \\
&= \frac{1}{\|e_1\|^p}(\|e_1\|^p - \|e_n\|^p) + \left(\frac{1}{\|e_1\|} + \frac{1}{\|e_n\|}\right)^p \|e_n\|^p \\
&= \frac{\|e_1\|^p}{\|e_1\|^p} - \frac{\|e_n\|^p}{\|e_1\|^p} + \left[\left(\frac{1}{\|e_1\|} + \frac{1}{\|e_n\|}\right)\|e_n\|\right]^p \\
&= 1 - \left(\frac{\|e_n\|}{\|e_1\|}\right)^p + \left(\frac{\|e_n\|}{\|e_1\|} + 1\right)^p.
\end{aligned}$$

Define $t = \|e_n\|/\|e_1\|$. Then $t \in (0, 1)$ since $0 < \|e_n\| < \|e_1\|$.

Next, for the continuous function $f(t) = 1 - t^p + (1+t)^p$, $p > 1$, we have

$$f'(t) = -p\,t^{p-1} + p(1+t)^{p-1} = p[(1+t)^{p-1} - t^{p-1}] \geqslant 0,$$

so, $f(t) \geqslant f(0) = 2$ and $\|s_n - s_1\|^p \geqslant 2$, so certainly $\inf \|s_n - s_1\| \geqslant 2^{1/p}$. On the other hand, note that $\|e_n\| \to 0$, $n \to \infty$, and for any $\varepsilon > 0$,

$$\|s_n - s_1\|^p = 1 - \left(\frac{\|e_n\|}{\|e_1\|}\right)^p + \left(\frac{\|e_n\|}{\|e_1\|} + 1\right)^p < 1 + \left(1 + \frac{\|e_n\|}{\|e_1\|}\right)^p \to 1 + (1+\varepsilon)^p, \quad n \to \infty,$$

i.e., $\inf \|s_n - s_1\|^p < (2+\varepsilon)^{1/p}$.

By the arbitrariness of $\varepsilon > 0$, we obtain $\sup\{\inf \|s_n - s_1\| \colon n \neq 1\} = 2^{1/p}$.

When $m = 2$,

$$\begin{aligned}
\|s_n - s_2\|^p &= \sum_{m=2}^{n-1} \left(\frac{1}{m}\frac{1}{\|e_2\|}\right)^p + \sum_{m=n}^{\infty} \left[\frac{1}{m}\left(\frac{1}{\|e_2\|} + \frac{1}{\|e_n\|}\right)\right]^p \\
&= \left(\frac{1}{\|e_2\|}\right)^p \left(\frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{(n-1)^p}\right) \\
&\quad + \left(\frac{1}{\|e_2\|} + \frac{1}{\|e_n\|}\right)^p \left(\frac{1}{n^p} + \frac{1}{(n+1)^p} + \ldots\right) \\
&= \frac{1}{\|e_2\|^p}(\|e_2\|^p - \|e_n\|^p) + \left(\frac{1}{\|e_2\|} + \frac{1}{\|e_n\|}\right)^p \|e_n\|^p \\
&= \frac{\|e_2\|^p}{\|e_2\|^p} - \frac{\|e_n\|^p}{\|e_2\|^p} + \left[\left(\frac{1}{\|e_2\|} + \frac{1}{\|e_n\|}\right)\|e_n\|\right]^p \\
&= 1 - \left(\frac{\|e_n\|}{\|e_2\|}\right)^p + \left(\frac{\|e_n\|}{\|e_2\|} + 1\right)^p.
\end{aligned}$$

Similarly, one can obtain that $\sup\{\inf \|s_n - s_2\| \colon n \neq 1\} = 2^{1/p}$. Repeating the above process, for any $m \geqslant 3$, $m \neq n$, we get $\sup\{\inf\{\|s_n - s_m\| \colon n \neq m\} = 2^{1/p}$. So, $\widetilde{D}(\mathrm{ces}_p) = D(\mathrm{ces}_p) = 2^{1/p}$, which completes the proof. □

Example 3.2. For any $p \in (1, \infty)$ there holds the formula $\widetilde{D}(l_p) = D(l_p) = 2^{1/p}$.

Proof. For l_p with $p \in (1, \infty)$, we have $\|e_n\| = 1$, so $s_n = e_n$. Since for any $n \neq m$, $\sup\{\inf\{\|s_n - s_m\| \colon n \neq m\} = \|s_n - s_m\| = 2^{1/p}$, we have $\widetilde{D}(l_p) = D(l_p) = 2^{1/p}$. □

Example 3.3. For any separable Hilbert space H, there holds the formula

$$\widetilde{D}(H) = D(H) = \sqrt{2}.$$

Proof. Since the sequence $e_n = (0, 0, \ldots, \overset{n\text{th}}{1}, 0, \ldots)$, $n \in \mathbb{N}$, is an orthonormal basis in H, $\sup\{\inf \|s_n - s_m\| \colon n \neq m\} = \|s_2 - s_1\| = \sqrt{2}$. Therefore $\widetilde{D}(H) = D(H) = \sqrt{2}$. □

Remark 3.1. From Examples 3.1–3.3 one can see that $\widetilde{D}(X) = D(X)$ for these BK sequence spaces. If this equality held for any BK sequence space, then we could easily compute the packing constant. Unfortunately, for some BK sequence spaces this equality does not hold, see the following examples.

Example 3.4. $D(c_0) = 2$ and $\widetilde{D}(c_0) = 1$.

Proof. Since c_0 is nonreflexive, from [8] we know that $P(c_0) = 1/2$, i.e., $D(c_0) = 2$. Because $\|e_n\| = 1$ implies that $s_n = e_n$, we have for any $n \neq m$,

$$\sup\{\inf \|s_n - s_m\| \colon\ n \neq m\} = \|s_n - s_m\| = 1.$$

Consequently, $\widetilde{D}(c_0) = 1 \neq D(c_0)$. □

Example 3.5. We also have the equalities $D(l^\infty) = 2$ and $\widetilde{D}(l^\infty) = 1$.

Proof. Similarly to the computations in Example 3.4, we have

$$\sup\{\inf \|s_n - s_m\| \colon\ n \neq m\} = \|s_n - s_m\| = 1.$$

Then $\widetilde{D}(l^\infty) = 1 \neq D(l^\infty)$, because of the equality $D(l^\infty) = 2$, which follows from [8] since l^∞ is not reflexive. □

Remark 3.2. It is clear that BK sequence spaces considered in Examples 3.1 − 3.3 are all reflexive spaces with Schauder basis, while the space considered in Example 3.4 is not reflexive and the space in Example 3.5 has no Schauder basis. So it is natural to ask the following question.

Question 3.1. We know that $\widetilde{D}(X) \leqslant D(X)$ for every BK sequence space. Is it true that $\widetilde{D}(X) = D(X)$ for any reflexive BK sequence space with Schauder basis? The problem is under consideration now.

References

[1] *J. A. C. Burlak, R. A. Rankin, A. P. Robertson*: The packing of spheres in the space l_p. Proc. Glasg. Math. Assoc. *4* (1958), 22–25.

[2] *S. Chen*: Geometry of Orlicz Spaces. With a preface by Julian Musielak, Dissertationes Math. (Rozprawy Mat.) *356* (1996), 204.

[3] *Y. Cui, H. Hudzik*: Packing constant for cesaro sequence spaces. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *47* (2001), 2695–2702.

[4] *Y. Cui, H. Hudzik*: On the banach-saks and weak banach-saks properties of some banach sequence spaces. Acta Sci. Math. *65* (1999), 179–187.

[5] *Y. Cui, H. Hudzik, N. Petrot, S. Suantai, A. Szymaszkiewicz*: Basic topological and geometric properties of cesàro-orlicz spaces. Proc. Indian Acad. Sci., Math. Sci. *115* (2005), 461–476.

[6] *P. Foralewski, H. Hudzik, A. Szymaszkiewicz*: Some remarks on cesàro-orlicz sequence spaces. Math. Inequal. Appl. *13* (2010), 363–386.
[7] *P. Foralewski, H. Hudzik, A. Szymaszkiewicz*: Local rotundity structure of cesàro-orlicz sequence spaces. J. Math. Anal. Appl. *345* (2008), 410–419.
[8] *H. Hudzik*: Every nonreflexive banach lattice has the packing constant equal to 1/2. Collect. Math. *44* (1993), 129–134.
[9] *C. A. Kottman*: Packing and reflexivity in banach spaces. Trans. Am. Math. Soc. *150* (1970), 565–576.
[10] *D. Kubiak*: A note on cesàro-orlicz sequence spaces. J. Math. Anal. Appl. *349* (2009), 291–296.
[11] *P. Y. Lee*: Cesàro sequence spaces. Math. Chron. *13* (1984), 29–45.
[12] *S. K. Lim, P. Y. Lee*: An orlicz extension of cesàro sequence spaces. Ann. Soc. Math. Pol., Ser. I, Commentat. Math. *28* (1988), 117–128.
[13] *W. A. J. Luxemburg*: Banach Function Spaces. Thesis, Technische Hogeschool te Delft, 1955.
[14] *Z. Ma, Y. Cui*: Some important geometric properties in cesàro-orlicz sequence spaces. Adv. Math., Beijing *42* (2013), 348–354.
[15] *L. Maligranda*: Orlicz Spaces and Interpolation. Seminars in Mathematics 5, Univ. Estadual de Campinas, Dep. de Matemática, Campinas, 1989.
[16] *L. Maligranda, N. Petrot, S. Suantai*: On the james constant and B-convexity of cesàro and cesàro-orlicz sequences spaces. J. Math. Anal. Appl. *326* (2007), 312–331.
[17] *J. Musielak*: Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics 1034, Springer, Berlin, 1983.
[18] *R. A. Rankin*: On packings of spheres in hilbert space. Proc. Glasg. Math. Assoc. *2* (1955), 145–146.
[19] *S. Saejung*: Another look at cesàro sequence spaces. J. Math. Anal. Appl. *366* (2010), 530–537.
[20] *J. R. L. Webb, W. Zhao*: On connections between set and ball measures of noncompactness. Bull. Lond. Math. Soc. *22* (1990), 471–477.
[21] *C. X. Wu, P. Lin, Q. Y. Piao, P. Y. Lee*: Sequence Space and Its Application. Harbin Institute of Technology Press, 2001. (In Chinese.)

Authors' addresses: Zhen-Hua Ma, School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, P. R. China, and Department of Mathematics and Physics, Hebei University of Architecture, No. 13, Chaoyang West Street, Zhangjiakou, Hebei, 075024, P. R. China, e-mail: `mazhenghua_1981@163.com`; Li-Ning Jiang, Qiao-Ling Xin, School of Mathematics and Statistics, Beijing Institute of Technology, Higher Education Park, Liangxiang, Fangshan District, Beijing 100081, P. R. China, e-mail: `jianglining@bit.edu.cn`, `xinqiaoling0923@163.com`.