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CHARACTERIZATION OF THE ALTERNATING GROUPS BY THEIR ORDER AND ONE CONJUGACY CLASS LENGTH

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Abstract. Let G be a finite group, and let $N(G)$ be the set of conjugacy class sizes of G. By Thompson's conjecture, if L is a finite non-abelian simple group, G is a finite group with a trivial center, and $N(G) = N(L)$, then L and G are isomorphic. Recently, Chen et al. contributed interestingly to Thompson's conjecture under a weak condition. They only used the group order and one or two special conjugacy class sizes of simple groups and characterized successfully sporadic simple groups (see Li's PhD dissertation). In this article, we investigate validity of Thompson's conjecture under a weak condition for the alternating groups of degrees $p+1$ and $p+2$, where p is a prime number. This work implies that Thompson's conjecture holds for the alternating groups of degree $p+1$ and $p+2$.

Keywords: finite simple group; conjugacy class size; prime graph; Thompson's conjecture

MSC 2010: 20D08, 20D60, 20D06

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Denote by $\pi(G)$ the set of primes p such that G contains an element of order p. The set of conjugacy class sizes of G is denoted by $N(G)$.

We construct the *prime graph* of G, denoted by $\Gamma(G)$, as follows: the vertex set is $\pi(G)$ and two distinct vertices p and p' are joined by an edge if and only if G has an element of order pp'. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, \ldots, \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$. Some of the characterizations by prime graph can be found in [8], [16].

We can express $|G|$ as a product of integers $m_1, m_2, \ldots, m_{t(G)}$, where $\pi(m_i) = \pi_i$ for each i. The numbers m_i are called the order components of G. In particular, if m_i is odd, then we call it an odd order component of G. Write $OC(G)$ for the set $\{m_1, m_2, \ldots, m_{t(G)}\}$ of order components of G and $T(G)$ for the set of connected components of G. For related results, in [10] the authors show that $L_3(q)$, is determined up to isomorphism by order components. Similar characterizations have been found in [12], [11]. According to the classification theorem of finite simple groups and [13], [17], [9], we can list the order components of finite simple groups with disconnected prime graphs as in Tables 1–3 in [10].

In 1980, Thompson posed the following conjecture (see [15], Question 12.38).

Thompson's conjecture: If L is a finite non-abelian simple group, G is a finite group with a trivial center, and $N(G) = N(L)$, then L and G are isomorphic.

In 1994, Chen proved in his PhD dissertation [4] that Thompson's conjecture holds for all simple groups with a non-connected prime graph (also see [3], [2]). Recently, Chen and Li contributed interestingly to Thompson's conjecture under a weak condition. They only used the group order and one or two special conjugacy class sizes of simple groups and characterized successfully sporadic simple groups (see Li's PhD dissertation [14]) and simple K_3-groups (a finite simple group is called a simple K_n-group if its order is divisible by exactly n distinct primes), by which they verified Thompson's conjecture for sporadic simple groups and simple K_3-groups. Also Chen et al. in [5] verified Thompson's conjecture for the projective special linear group $L_2(p)$ by its order and one special conjugacy class size, where p is a prime. In [1], we have verified Thompson's conjecture for the alternating groups of degree p by its order and one special conjugacy class size, where p is a prime. Hence, it is an interesting topic to characterize simple groups with their orders and few conjugacy class sizes. In this paper, we characterize the alternating groups of degrees $p+1$ and $p+2$ by their order and one special conjugacy class length, where p is a prime. Our Main theorem is as follows.

Main theorem. *Let G be a group. Then*

(a) $G \cong \mathrm{Alt}_{p+1}$ *if and only if* $|G| = (p+1)!/2$ *and G has a conjugacy class of length* $(p+1)!/2p$, *where* $7 \neq p \geqslant 5$ *is a prime number. For the case* $p = 7$, $G \cong \mathrm{Alt}_8$ *or* $L_3(4)$ *if and only if* $|G| = 8!/2$ *and G has a conjugacy class of length* $2\,880$.

(b) $G \cong \mathrm{Alt}_{p+2}$ *if and only if* $|G| = (p+2)!/2$ *and G has a conjugacy class of length* $(p+2)!/2p$, *where* $p \geqslant 5$ *is a prime number.*

We write $p^k \parallel m$ if $p^k \mid m$ and $p^{k+1} \nmid m$. The other notation and terminology in this paper are standard, and the reader is referred to [6] if necessary.

2. Preliminary results

For the proof of the Main theorem we need the following lemmas.

Lemma 2.1 ([7], Theorem 10.3.1). *Let G be a Frobenius group with Frobenius kernel H and Frobenius complement K. Then $|K| \equiv 1 \pmod{|H|}$.*

Lemma 2.2 ([2], Lemma 8). *Let G be a finite group with $t(G) \geqslant 2$, and N a normal subgroup of G. If N is a π_i-group for some prime graph component of G, and $\mu_1, \mu_2, \ldots, \mu_r$ are orders of some components of G but not a π_i-number then $\mu_1\mu_2 \ldots \mu_r$ is a divisor of $|N| - 1$.*

Now we state the following lemma which is proved in [13], Lemma 6, with some differences and classify the simple groups of Lie type with prime odd order component by the θ function which will be introduced later.

Lemma 2.3. *If L is a simple group of Lie type and has prime odd order component $p \geqslant 17$ and $\pi(L)$ has at most $\theta(L)$ prime numbers t, where $p + 1/2 < t < p$, then $\theta(L) \leqslant 3$.*

Throughout the proof of the above lemma, we can divide simple groups of Lie type, L, with prime odd order component $p \geqslant 17$, into the following cases.

(1) $\theta(L) = 0$ if L is isomorphic to $A_{p'-1}(q)$, $A_{p'}(q)$,
where $q - 1 \mid p' + 1$, $A_2(2)$, ${}^2A_{p'-1}(q)$, ${}^2A_{p'}(q)$,
where $q + 1 \mid p' + 1$, ${}^2A_3(2)$, $B_n(q)$,
where $n = 2^{m'}$ and q is odd, $B_{p'}(3)$, $C_n(q)$,
where $n = 2^{m'}$ or $(n, q) = (p', 3)$, $D_{p'+1}(3)$, $D_{p'}(q)$ for $q = 3, 5$,
${}^2D_n(q)$ for $(n, q) = (2^{m'}, q)$, $(p', 3)$,
where $5 \leqslant p' \neq 2^{m'} + 1$ (or $2^{m'} + 1, 3$),
where $5 \leqslant p' \neq 2^{m'} + 1$, $G_2(q)$,
where $q \equiv \varepsilon \pmod 3$ for $\varepsilon = \pm 1$, ${}^3D_4(q)$, $E_6(q)$ or ${}^2E_6(q)$;

(2) $\theta(L) = 1$ if L is isomorphic to one of the simple groups $A_1(q)$,
where $2 \mid q$, $A_2(4)$, ${}^2A_5(2)$, $C_{p'}(2)$, $D_n(2)$,
where $n = p'$ or $p' + 1$, ${}^2D_n(2)$,
where $(n, q) = (2^{m'} + 1, 2)$ (or $p' = 2^{m'} + 1, 3$),
where $m' \geqslant 2$, $E_7(2)$, $E_7(3)$, $F_4(q)$, ${}^2F_4(q)$,
where $q = 2^{2n+1} > 2$, or $G_2(q)$,
where $3 \mid q$;

(3) $\theta(L) = 2$ if L is isomorphic to the simple groups $A_1(q)$,
where $q \equiv \varepsilon \pmod 4$ for $\varepsilon = \pm 1$, ${}^2B_2(q)$,

where $q = 2^{2m'+1} > 2$, or ${}^2G_2(q)$,
where $q = 3^{2m'+1} > 3$;

(4) $\theta(L) = 3$ if L is isomorphic to the simple groups $E_8(q)$ or ${}^2E_6(2)$.

Lemma 2.4 ([13], Lemma 1). *If $n \geqslant 6$ is a natural number, then there are at least $s(n)$ prime numbers p_i such that $n + 1/2 < p_i < n$. Here*

$$\begin{aligned} s(n) &= 6 \quad \text{for } n \geqslant 48; \\ s(n) &= 5 \quad \text{for } 42 \leqslant n \leqslant 47; \\ s(n) &= 4 \quad \text{for } 38 \leqslant n \leqslant 41; \\ s(n) &= 3 \quad \text{for } 18 \leqslant n \leqslant 37; \\ s(n) &= 2 \quad \text{for } 14 \leqslant n \leqslant 17; \\ s(n) &= 1 \quad \text{for } 6 \leqslant n \leqslant 13. \end{aligned}$$

In particular, for every natural number $n \geqslant 6$ there exists a prime p such that $n + 1/2 < p < n$, and for every natural number $n > 3$ there exists an odd prime number p such that $n - p < p < n$.

3. Proof of the Main theorem

Since the necessity of the theorem can be checked easily, we only need to prove the sufficiency.

P r o o f of Main theorem (sufficiency). By hypothesis, there exists an element x of order p in G such that $C_G(x) = \langle x \rangle$ and $C_G(x)$ is a Sylow p-subgroup of G. By the Sylow theorem, we have that $C_G(y) = \langle y \rangle$ for any element y in G of order p. So, $\{p\}$ is a prime graph component of G and $t(G) \geqslant 2$. In addition, p is the maximal prime divisor of $|G|$ and an odd order component of G. The proof of the main theorem follows from the following lemmas.

Lemma 3.1. *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group and H is a nilpotent group.*

P r o o f. Let $x \in G$ be an element of order p, then $C_G(x) = \langle x \rangle$. Set $H = O_{p'}(G)$ (the largest normal p'-subgroup of G). Then H is a nilpotent group since x acts on H fixed point freely. Let K be a normal subgroup of G such that K/H is a minimal normal subgroup of G/H. Then K/H is a direct product of copies of some simple group. Since $p \mid |K/H|$ and $p^2 \nmid |K/H|$, K/H is a simple group. Since $\langle x \rangle$ is a Sylow

p-subgroup of K, $G = N_G(\langle x \rangle)K$ by the Frattini argument and so $|G/K|$ divides $p-1$.

If $|K/H| = p$, then by Lemma 2.4 and $|G| = (p+1)!/2$ or $(p+2)!/2$, we deduce that there exists $r \in \pi(G)$ such that $p - 1/2 < r < p$. So, $r \nmid p-1$ and hence, $|G/K| \mid (p-1)$ implies that $r \in \pi(H)$.

Let R be an r-Sylow subgroup of H. Thus $|G| = (p+1)!/2$ or $(p+2)!/2$ shows that R is a cyclic subgroup of order r. On the other hand, the same reasoning as before shows that $R \rtimes P$ is a Frobenius group, where P is a p-Sylow subgroup of K, so Lemma 2.1 forces $p \mid r-1$ and hence, $p+1 < r$, which is a contradiction. $\square$

If K/H has an element of order rq where r and q are primes, then G has also such an element. Hence by definition of order components, an odd order component of G must be an odd order component of K/H. Note that $t(K/H) \geqslant 2$.

Lemma 3.2. (a) *If $t \in \pi(H)$, then $t \leqslant p + 1/2$;*

(b) *A group K/H cannot be isomorphic to a sporadic simple group. Moreover, if K/H is isomorphic to an alternating simple group, then we must have $G \cong \mathrm{Alt}_n$, where $n = p+1$ or $p+2$.*

Proof. (a) If t divides $|H|$ where $p + 1/2 < t < p$, then since H is a nilpotent subgroup of G and the order of T, which is the Sylow t-subgroup of H, is equal to t. By Lemma 2.2, we must have $p \mid t-1$, which is impossible. Thus $|H|$ is not divisible by the primes t with $p + 1/2 < t < p$.

(b) We note that if $H \neq 1$, by nilpotency of H we may assume that H is a t-group for $t \in \pi_1(G)$.

If $K/H \cong J_4$, then $p = 43$. Since $19 \in \pi(G) - \pi(\mathrm{Aut}(J_4))$, hence $19 \in \pi(H)$. By Lemma 2.2, $43 \mid 19^i - 1$ for $i = 1$ or 2, which is impossible.

If $K/H \cong M_{22}$, then $p = 11$. Since $5^2 \mid |G|$ and $5 \parallel |\mathrm{Aut}(M_{22})|$, $5 \in \pi(H)$. So by Lemma 2.2, we get a contradiction.

If $K/H \cong J_2$, then $p = 7$, but $2^4 \parallel |G|$ and $2^7 \parallel |K/H|$, which is impossible. If K/H is isomorphic to other sporadic simple groups we can obtain a contradiction similarly.

Now let K/H be isomorphic to an alternating group. By Tables 1 and 2 in [10], K/H must be isomorphic to Alt_n, where $n = p$, $p+1$ or $p+2$.

For the case $p+1$, if $K/H \cong \mathrm{Alt}_p$, then $G/H \cong \mathrm{Alt}_p$ or Sym_p. Let $G/H \cong \mathrm{Alt}_p$, then $|H| = p+1$.

If there is an odd number $t \in \pi(H)$, then $t \mid (p+1)/2$, by nilpotency of H, we may assume H is a t-group, and by Lemma 2.5, we get a contradiction.

If $p+1=2^a$, then due to nilpotency of H we may assume H is an elementary abelian 2-group. So $C_K(H)/H \trianglelefteq K/H$. Since K/H is a simple group, $H=C_K(H)$ or $K=C_K(H)$.

If $K=C_K(H)$, then 2 and p are linked in the prime graph of G, which is a contradiction. Hence $H=C_K(H)$. So $K/H=N_K(H)/C_K(H)\leqslant \mathrm{Aut}(H)$ and then $|K/H|$ divides $f(a)=\Pi_{i=0}^{a-1}(2^a-2^i)$. Since $K/H\cong \mathrm{Alt}_p$, $p-2=2^a-3 \mid f(a)$, which is impossible.

If $G/H\cong \mathrm{Sym}_p$, then $|H|=(p+1)/2$ and this contradicts Lemma 2.2.

If $K/H\cong \mathrm{Alt}_{p+1}$, then $H=1$ and $G\cong \mathrm{Alt}_{p+1}$.

If $K/H\cong \mathrm{Alt}_{p+2}$, then $|K/H|>|G|$, which is impossible.

For the case $p+2$, if $K/H\cong \mathrm{Alt}_p$, since $G/H\cong \mathrm{Alt}_p$ or Sym_p, we must have $|H|=(p+1)(p+2)$ or $(p+1)(p+2)/2$, respectively. Let t be an odd prime divisor of $p+2$. By nilpotency of H, we may assume H is a t-group. Lemma 2.2 yields $p\mid t^a-1$. Thus $t^a=p+1$, which contradicts $\gcd(p+1,p+2)=1$.

If $K/H\cong \mathrm{Alt}_{p+1}$, by the same method we must have $|H|=p+2$ when $G/H\cong \mathrm{Alt}_{p+1}$, and thus we obtain a contradiction by Lemma 2.2. Therefore, $K/H\cong \mathrm{Alt}_{p+2}$ and $G\cong \mathrm{Alt}_{p+2}$. □

Lemma 3.3. *If $t\in\pi(G/H)$ and $p+1/2<t<p$, then $t\in\pi(K/H)$.*

P r o o f. It follows from Lemma 3.2 (a), and the proof of Lemma 6 (d) in [13]. □

Lemma 3.4. *A group K/H cannot be isomorphic to a simple group of Lie type.*

P r o o f. By Lemmas 3.3 and 2.4, we must have $17\leqslant p\leqslant 37$ and $\theta(K/H)\geqslant 2$. Therefore K/H is isomorphic to one of the following simple groups:

(1) $L_2(q)$, where $q\equiv\varepsilon \pmod 4$ for $\varepsilon=\pm1$;
(2) ${}^2B_2(q)$, where $q=2^{2m'+1}>2$;
(3) ${}^2G_2(q)$, where $q=3^{2m'+1}>3$;
(4) $E_8(q)$ or ${}^2E_6(2)$.

Since one of the odd order components of K/H is equal to p, by Tables 2 and 3 in [10], we must have:

(1) $K/H\cong L_2(17)$ for $p=17$;
(2) $K/H\cong L_2(19)$, ${}^2G_2(27)$ or ${}^2E_6(2)$ for $p=19$;
(3) $K/H\cong L_2(q)$, where $p=q$ is equal to 23, 29 or 31;
(4) $K/H\cong L_2(37)$, ${}^2G_2(27)$ or ${}^2E_6(2)$ for $p=37$.

Let $p=17$. Then $K/H\cong L_2(17)$. Since $13\notin\pi(L_2(17))$, we get a contradiction by Lemma 3.3.

Let $p = 19$. If $K/H \cong L_2(19)$, then since $17 \notin \pi(L_2(19))$, we get a contradiction by Lemma 3.3. Since $37 \mid |^2G_2(27)|$, $K/H \ncong {}^2G_2(27)$. For the case $K/H \cong {}^2E_6(2)$, we obtain a contradiction by $2^{36} \mid |^2E_6(2)|$.

Let $K/H \cong L_2(q)$, where $p = q$ is equal to 23, 29, 31, or 37. If $p = 37$, then since $23 \notin \pi(L_2(37))$, we get a contradiction by Lemma 3.3. Similarly, we obtain a contradiction when $p = 23$, 29, or 31.

Let $p = 37$. If $K/H \cong L_2(37)$, then since $23 \notin \pi(L_2(37))$, we get a contradiction. Since $23 \notin \pi(^2G_2(27))$, $K/H \ncong {}^2G_2(27)$. If $K/H \cong {}^2E_6(2)$, a contradiction follows by $2^{36} \mid |^2E_6(2)|$. □

Lemma 3.5. *If $n = 8$, then $G \cong L_3(4)$, or* Alt_8.

P r o o f. We know that G has a normal series $1 \lhd H \lhd K \lhd G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, and H is a nilpotent group. By Lemma 3.2 (b), K/H cannot be isomorphic to a sporadic simple group, and if K/H is isomorphic to an alternating simple group, then we must have $G \cong \mathrm{Alt}_8$.

Let K/H be isomorphic to the simple group of Lie type $G(q)$ where $q = s^m$ and s is a prime number. We know that 7 is one of the odd order components of K/H. So $s = 2$, 3, or 7. Then the order of all Sylow t-subgroups of G are less than or equal to 64 or 81. Therefore, $K/H \cong L_2(7)$, $L_2(8)$, $L_3(4)$, $L_4(2) \cong \mathrm{Alt}_8$ or $U_3(3)$.

If $K/H \cong L_2(7)$, $L_2(8)$ or $U_3(3)$, then since $5 \in \pi(G) - \pi(\mathrm{Aut}(K/H))$, we have $5 \in \pi(H)$, which contradicts Lemma 3.2 (a).

Therefore, $K/H \cong L_3(4)$, or $L_4(2) \cong \mathrm{Alt}_8$. Since $|G| = |L_3(4)| = |\mathrm{Alt}_8|$, $G \cong L_3(4)$, or Alt_8. □

Now Lemmas 3.2 (b), 3.4, and 3.5 imply Main theorem.

Corollary 3.6. *Thompson's conjecture holds for the alternating groups of degrees $p + 1$ and $p + 2$, where p is a prime number.*

P r o o f. Let G be a group with trivial central and $N(G) = N(\mathrm{Alt}_n)$, where $n = p + 1$ or $p + 2$. Then it is proved in [3], Lemma 1.4, that $|G| = |\mathrm{Alt}_n|$. Hence, the corollary follows from Main theorem. □

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