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COMMUTATORS OF MARCINKIEWICZ INTEGRALS ON HERZ SPACES WITH VARIABLE EXPONENT

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Abstract. Let $\Omega \in L^s(\mathrm{S}^{n-1})$ for $s \geqslant 1$ be a homogeneous function of degree zero and b a BMO function. The commutator generated by the Marcinkiewicz integral μ_Ω and b is defined by

$$[b,\mu_\Omega](f)(x) = \biggl(\int_0^\infty \biggl| \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x)-b(y)] f(y)\,\mathrm{d}y \biggr|^2 \frac{\mathrm{d}t}{t^3}\biggr)^{1/2}.$$

In this paper, the author proves the $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of the Marcinkiewicz integral operator μ_Ω and its commutator $[b,\mu_\Omega]$ when $p(\cdot)$ satisfies some conditions. Moreover, the author obtains the corresponding result about μ_Ω and $[b,\mu_\Omega]$ on Herz spaces with variable exponent.

Keywords: Herz space; variable exponent; commutator; Marcinkiewicz integral

MSC 2010: 42B20, 42B35

1. Introduction

The theory of function spaces with variable exponent has been extensively studied by researchers since the work of Kováčik and Rákosník [9] appeared in 1991, see [2], [4] and the references therein. In [1], [3] and [15], the authors proved the boundedness of some integral operators on variable L^p spaces.

Given an open set $E \subset \mathbb{R}^n$ and a measurable function $p(\cdot)\colon E \to [1,\infty)$, $L^{p(\cdot)}(E)$ denotes the set of measurable functions f on E such that for some $\lambda > 0$,

$$\int_E \Bigl(\frac{|f(x)|}{\lambda}\Bigr)^{p(x)} \,\mathrm{d}x < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\Big\{\lambda > 0\colon\ \int_E \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)}\,\mathrm{d}x \leqslant 1\Big\}.$$

These spaces are referred to as variable L^p spaces, since they generalize the standard L^p spaces: if $p(x) = p$ is constant, then $L^{p(\cdot)}(E)$ is isometrically isomorphic to $L^p(E)$.

For all compact subsets $F \subset E$, the space $L^{p(\cdot)}_{\rm loc}(E)$ is defined by $L^{p(\cdot)}_{\rm loc}(E) := \{f\colon\ f \in L^{p(\cdot)}(F)\}$. Define $\mathcal{P}^0(E)$ to be the set of $p(\cdot)\colon\ E \to (0,\infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x)\colon\ x \in E\} > 0, \quad p^+ = \operatorname{ess\,sup}\{p(x)\colon\ x \in E\} < \infty.$$

Define $\mathcal{P}(E)$ to be the set of $p(\cdot)\colon\ E \to [1,\infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x)\colon\ x \in E\} > 1, \quad p^+ = \operatorname{ess\,sup}\{p(x)\colon\ x \in E\} < \infty.$$

Denote $p'(x) = p(x)/(p(x)-1)$. Let $\mathcal{B}(E)$ be the set of $p(\cdot) \in \mathcal{P}(E)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(E)$. In addition, we denote the Lebesgue measure and the characteristic function of a measurable set $A \subset \mathbb{R}^n$ by $|A|$ and χ_A, respectively. The notation $f \approx g$ means that there exist constants $C_1, C_2 > 0$ such that $C_1 g \leqslant f \leqslant C_2 g$.

In variable L^p spaces we have the following important lemmas.

Lemma 1.1 ([1]). *If* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfies*

$$|p(x) - p(y)| \leqslant \frac{C}{-\log(|x-y|)}, \quad |x-y| \leqslant \frac{1}{2} \tag{1.1}$$

and

$$|p(x) - p(y)| \leqslant \frac{C}{\log(|x|+\mathrm{e})}, \quad |y| \geqslant |x|, \tag{1.2}$$

then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, *that is the Hardy-Littlewood maximal operator* M *is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 1.2 ([9]). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,\mathrm{d}x \leqslant r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where

$$r_p = 1 + \frac{1}{p^-} - \frac{1}{p^+}.$$

This inequality is called the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 1.3 ([8]). *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leqslant C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leqslant C\Big(\frac{|S|}{|B|}\Big)^{\delta_1} \quad \text{and} \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leqslant C\Big(\frac{|S|}{|B|}\Big)^{\delta_2},$$

where δ_1, δ_2 are constants with $0 < \delta_1, \delta_2 < 1$.

Throughout this paper δ_1, δ_2 are the same as in Lemma 1.3.

Lemma 1.4 ([8]). *Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leqslant C.$$

In 2010, Izuki [8], [7] introduced the Herz spaces with variable exponent and proved the boundedness of some operators on these spaces. Next we recall the definition of the Herz spaces with variable exponent. Let $B_k = \{x \in \mathbb{R}^n \colon |x| \leqslant 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the sets of all positive and non-negative integers, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\widetilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\widetilde{\chi}_0 = \chi_{B_0}$.

Definition 1.1 ([8]). Let $\alpha \in \mathbb{R}$, $0 < p \leqslant \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) \colon \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{1/p}.$$

The nonhomogeneous Herz space with variable exponent $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\rm loc}(\mathbb{R}^n)\colon \ \|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{\sum_{k=0}^{\infty} 2^{k\alpha p}\|f\widetilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

Suppose that S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ $(n \geqslant 2)$ equipped with the normalized Lebesgue measure. Let $\Omega \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ for $0 < \beta \leqslant 1$ be a homogeneous function of degree zero and

$$\int_{\mathrm{S}^{n-1}} \Omega(x')\,\mathrm{d}\sigma(x') = 0, \tag{1.3}$$

where $x' = x/|x|$ for any $x \neq 0$. In 1958, Stein [13] introduced the Marcinkiewicz integral related to the Littlewood-Paley g function on $\mathbb{R}^n$ as

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2\frac{\mathrm{d}t}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)\,\mathrm{d}y.$$

It was shown that μ_Ω is of type (p,p) for $1 < p \leqslant 2$ and of weak type $(1,1)$.

Let b be a locally integrable function on $\mathbb{R}^n$; the commutator generated by the Marcinkiewicz integral μ_Ω and b is defined by

$$[b,\mu_\Omega](f)(x) = \left(\int_0^\infty \left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]f(y)\,\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{1/2}.$$

Motivated by [10], [14], we will study the boundedness for the Marcinkiewicz integral operator μ_Ω and its commutator $[b,\mu_\Omega]$ on the Herz space with variable exponent, where $\Omega \in L^s(\mathrm{S}^{n-1})$ for $s \geqslant 1$.

2. Estimate for the Marcinkiewicz integral operator

In this section we will prove the boundedness of the Marcinkiewicz integral operators μ_Ω on Herz spaces with variable exponent.

A nonnegative locally integrable function $\omega(x)$ on $\mathbb{R}^n$ is said to belong to A_p $(1 < p < \infty)$, if there is a constant $C > 0$ such that

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)\,\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\,\mathrm{d}x\right)^{p-1} \leqslant C < \infty,$$

where $p' = p/(p-1)$.

The weighted (L^p, L^p) boundedness of μ_Ω was proved by Ding, Fan and Pan [5].

Lemma 2.1 ([5]). *Suppose that* $\Omega \in L^s(\mathrm{S}^{n-1})$ $(s > 1)$ *satisfies* (1.3). *If* $\omega \in A_{p/s'}$, $s' < p < \infty$, *then there is a constant* C, *independent of* f, *such that*

$$\int_{\mathbb{R}^n} |\mu_\Omega(f)(x)|^p \omega(x)\,\mathrm{d}x \leqslant C\int_{\mathbb{R}^n} |f(x)|^p\omega(x)\,\mathrm{d}x.$$

Lemma 2.2 ([3]). *Given a family* $\mathcal{F}$ *and an open set* $E \subset \mathbb{R}^n$, *assume that for some* p_0, $0 < p_0 < \infty$ *and for every* $\omega \in A_\infty$,

$$\int_E f(x)^{p_0}\omega(x)\,\mathrm{d}x \leqslant C_0 \int_E g(x)^{p_0}\omega(x)\,\mathrm{d}x, \quad (f, g) \in \mathcal{F}.$$

Given $p(\cdot) \in \mathcal{P}^0(E)$ *such that* $p(\cdot)$ *satisfies* (1.1) *and* (1.2) *in Lemma* 1.1, *then for all* $(f, g) \in \mathcal{F}$ *such that* $f \in L^{p(\cdot)}(E)$,

$$\|f\|_{L^{p(\cdot)}(E)} \leqslant C\|g\|_{L^{p(\cdot)}(E)}.$$

Since $A_{p/s'} \subset A_\infty$, by Lemma 2.1 and Lemma 2.2 it is easy to get the $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of the Marcinkiewicz integral operators μ_Ω.

To obtain Theorem 2.1, we need the following lemmas.

Lemma 2.3 ([11]). *If* $a > 0$, $1 \leqslant s \leqslant \infty$, $0 < d \leqslant s$ *and* $-n+(n-1)d/s < \nu < \infty$, *then*

$$\left(\int_{|y|\leqslant a|x|} |y|^\nu |\Omega(x-y)|^d\,\mathrm{d}y\right)^{1/d} \leqslant C|x|^{(\nu+n)/d}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}.$$

Lemma 2.4 ([12]). *Define a variable exponent* $\tilde{q}(\cdot)$ *by* $1/p(x) = 1/\tilde{q}(x) + 1/q$ $(x \in \mathbb{R}^n)$. *Then we have*

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}\|fg\|_{L^{q}(\mathbb{R}^n)}$$

for all measurable functions f *and* g.

Lemma 2.5 ([4]). *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfy conditions* (1.1) *and* (1.2) *in Lemma* 1.1. *Then*

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{1/p(x)} & \text{if } |Q| \leqslant 2^n \text{ and } x \in Q, \\ |Q|^{1/p(\infty)} & \text{if } |Q| \geqslant 1 \end{cases}$$

for every cube (or ball) $Q \subset \mathbb{R}^n$, *where* $p(\infty) = \lim\limits_{x\to\infty} p(x)$.

Theorem 2.1. *Suppose that* $0 < \nu \leqslant 1$, $0 < p \leqslant \infty$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfies conditions* (1.1) *and* (1.2) *in Lemma* 1.1, $\Omega \in L^s(\mathrm{S}^{n-1})$, $s > q'^-$ *and* $-n\delta_1 - \nu - n/s < \alpha < n\delta_2 - \nu - n/s$. *Then* μ_Ω *is bounded on* $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *and* $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Proof. We only prove the homogeneous case. The nonhomogeneous case can be proved in the same way. We suppose $0 < p < \infty$, since the proof of the case $p = \infty$ is easier. Let $f \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. Denote $f_j = f\chi_j$ for each $j \in \mathbb{Z}$, then we have $f(x) = \sum\limits_{j=-\infty}^{\infty} f_j(x)$. Then

$$\begin{aligned}
(2.1)\qquad \|\mu_\Omega(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|\mu_\Omega(f)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{1/p} \\
&\leqslant C\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=-\infty}^{k-2} \|\mu_\Omega(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\Big\}^{1/p} \\
&\quad + C\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k-1}^{k+1} \|\mu_\Omega(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\Big\}^{1/p} \\
&\quad + C\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k+2}^{\infty} \|\mu_\Omega(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\Big\}^{1/p} \\
&=: CI_1 + CI_2 + CI_3.
\end{aligned}$$

We first estimate I_2. By the $(L^{q(\cdot)}(\mathbb{R}^n), L^{q(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator μ_Ω we have

$$(2.2)\qquad I_2 \leqslant C\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{1/p} = C\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Now we estimate I_1. We consider

$$\begin{aligned}|\mu_\Omega(f_j)(x)| &\leqslant \left(\int_0^{|x|}\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f_j(y)\,\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{1/2}\\ &\quad+\left(\int_{|x|}^{\infty}\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f_j(y)\,\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{1/2}\\ &=: I_{11}+I_{12}.\end{aligned}$$

Note that $x\in A_k$, $y\in A_j$ and $j\leqslant k-2$. So, we know that $|x-y|\sim|x|$, and by the mean value theorem we have

$$\left|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\right|\leqslant\frac{|y|}{|x-y|^3}. \tag{2.3}$$

By (2.3), the Minkowski inequality and the generalized Hölder inequality we have

$$\begin{aligned}I_{11} &\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f_j(y)|\left(\int_{|x-y|}^{|x|}\frac{\mathrm{d}t}{t^3}\right)^{1/2}\mathrm{d}y\\ &\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f_j(y)|\left|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\right|^{1/2}\mathrm{d}y\\ &\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f_j(y)|\frac{|y|^{1/2}}{|x-y|^{3/2}}\,\mathrm{d}y\\ &\leqslant C\frac{2^{j/2}}{|x|^{n+1/2}}\int_{A_j}|\Omega(x-y)||f(y)|\,\mathrm{d}y\\ &\leqslant C2^{(j-k)/2}2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Similarly, we consider I_{12}. Noting that $|x-y|\sim|x|$, by the Minkowski inequality and the generalized Hölder inequality we have

$$\begin{aligned}I_{12} &\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f_j(y)|\left(\int_{|x|}^{\infty}\frac{\mathrm{d}t}{t^3}\right)^{1/2}\mathrm{d}y\\ &\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n}}|f_j(y)|\,\mathrm{d}y\\ &\leqslant C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

So, we have

$$|\mu_\Omega(f_j)(x)|\leqslant C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.$$

Noting that $s > q'^{-}$, we denote $\tilde{q}\,'(\cdot) > 1$ and $1/q'(x) = 1/\tilde{q}\,'(x) + 1/s$. By Lemma 2.3 and Lemma 2.4 we have

$$
\begin{aligned}
(2.4)\quad \|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)} &\leqslant \|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^s(\mathbb{R}^n)}\|\chi_j(\cdot)\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\\
&\leqslant \|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^s(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C2^{-j\nu}\bigg(\int_{A_j}|\Omega(x-y)|^s|y|^{s\nu}\,\mathrm{d}y\bigg)^{1/s}\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C2^{-j\nu}2^{k(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

When $|B_j| \leqslant 2^n$ and $x_j \in B_j$, by Lemma 2.5 we have

$$
\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)} \approx |B_j|^{1/\tilde{q}'(x_j)} \approx \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s}.
$$

When $|B_j| \geqslant 1$ we have

$$
\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)} \approx |B_j|^{1/\tilde{q}'(\infty)} \approx \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s}.
$$

So, we obtain

$$
\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)} \approx \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s}. \tag{2.5}
$$

By (2.4), (2.5), Lemma 1.3 and Lemma 1.4 we have

$$
\begin{aligned}
&\|\mu_\Omega(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad\leqslant C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad\leqslant C2^{-kn}2^{-j\nu}2^{k(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad\leqslant C2^{-kn}2^{-j\nu}2^{k(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s}\\
&\qquad\times\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad\leqslant C2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\\
&\quad\leqslant C2^{(j-k)(n\delta_2-\nu-n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
I_1 &\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\bigg\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\bigg(\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\nu-n/s)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\bigg)^p\bigg\}^{1/p}\\
&= C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\bigg\{\sum_{k=-\infty}^{\infty}\bigg(\sum_{j=-\infty}^{k-2}2^{j\alpha}2^{(j-k)(n\delta_2-\nu-n/s-\alpha)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\bigg)^p\bigg\}^{1/p}.
\end{aligned}
$$

If $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $n\delta_2 - \nu - n/s - \alpha > 0$, by the Hölder inequality we have

$$
\begin{aligned}
(2.6)\qquad I_1 &\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})} \\
&\quad \times \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} 2^{j\alpha p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p/2} \right) \right. \\
&\quad \times \left. \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p'/2} \right)^{p/p'} \right\}^{1/p} \\
&\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})} \\
&\quad \times \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} 2^{j\alpha p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p/2} \right) \right\}^{1/p} \\
&= C\|\Omega\|_{L^s(\mathrm{S}^{n-1})} \\
&\quad \times \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \left(\sum_{k=j+2}^{\infty} 2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p/2} \right) \right\}^{1/p} \\
&\leqslant C\left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p} = C\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

If $0 < p \leqslant 1$, then we have

$$
\begin{aligned}
(2.7)\qquad I_1 &\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})} \left\{ \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-2} 2^{j\alpha p} 2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p} \\
&= C\|\Omega\|_{L^s(\mathrm{S}^{n-1})} \\
&\quad \times \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \left(\sum_{k=j+2}^{\infty} 2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p} \right) \right\}^{1/p} \\
&\leqslant C\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Let us now estimate I_3. Note that $x \in A_k$, $y \in A_j$ and $j \geqslant k+2$, so we have $|x-y| \sim |y|$. We consider

$$
\begin{aligned}
|\mu_\Omega(f_j)(x)| &\leqslant \left(\int_0^{|y|} \left| \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_j(y)\,\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{1/2} \\
&\quad + \left(\int_{|y|}^{\infty} \left| \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_j(y)\,\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{1/2} \\
&=: I_{31} + I_{32}.
\end{aligned}
$$

Similarly to the estimate for I_{11}, we get

$$I_{31} \leqslant C2^{(k-j)/2}2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.$$

Similarly to the estimate for I_{12}, we get

$$I_{32} \leqslant C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.$$

So, we have

$$|\mu_\Omega(f_j)(x)| \leqslant C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.$$

By (2.4), (2.5), Lemma 1.3 and Lemma 1.4 we have

$$\begin{aligned}
&\|\mu_\Omega(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad\leqslant C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad\leqslant C2^{-jn}2^{-j\nu}2^{k(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad\leqslant C2^{-jn}2^{-j\nu}2^{k(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s} \\
&\qquad\times\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad\leqslant C2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \\
&\quad\leqslant C2^{(k-j)(n\delta_1+\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
I_3 &\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\Bigg\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\Bigg(\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_1+\nu+n/s)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg)^p\Bigg\}^{1/p} \\
&= C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\Bigg\{\sum_{k=-\infty}^{\infty}\Bigg(\sum_{j=k+2}^{\infty}2^{j\alpha}2^{(k-j)(n\delta_1+\nu+n/s+\alpha)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg)^p\Bigg\}^{1/p}.
\end{aligned}$$

If $1<p<\infty$, take $1/p+1/p'=1$. Since $n\delta_1+\nu+n/s+\alpha>0$, by the Hölder inequality we have

$$\begin{aligned}
(2.8)\quad I_3 &\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\Bigg\{\sum_{k=-\infty}^{\infty}\Bigg(\sum_{j=k+2}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p/2}\Bigg) \\
&\qquad\times\Bigg(\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p'/2}\Bigg)^{p/p'}\Bigg\}^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\\
&\quad\times\Bigg\{\sum_{k=-\infty}^{\infty}\bigg(\sum_{j=k+2}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p/2}\bigg)\Bigg\}^{1/p}\\
&= C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\\
&\quad\times\Bigg\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\bigg(\sum_{k=-\infty}^{j-2}2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p/2}\bigg)\Bigg\}^{1/p}\\
&\leqslant C\Bigg\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg\}^{1/p}=C\|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

If $0<p\leqslant 1$, then we have

$$
\begin{aligned}
I_3 &\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\\
&\quad\times\Bigg\{\sum_{k=-\infty}^{\infty}\sum_{j=k+2}^{\infty}2^{j\alpha p}2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg\}^{1/p}\\
&= C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\\
&\quad\times\Bigg\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\bigg(\sum_{k=-\infty}^{j-2}2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p}\bigg)\Bigg\}^{1/p}\\
&\leqslant C\|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
\tag{2.9}
$$

Therefore, by (2.1), (2.2) and (2.6)–(2.9) we complete the proof of Theorem 2.1. □

3. BMO estimate for the commutator of Marcinkiewicz integral operator

Let us first recall that the space $\mathrm{BMO}(\mathbb{R}^n)$ consists of all locally integrable functions f such that

$$\|f\|_*=\sup_Q\frac{1}{|Q|}\int_Q|f(x)-f_Q|\,\mathrm{d}x<\infty,$$

where $f_Q=|Q|^{-1}\int_Q f(y)\,\mathrm{d}y$, the supremum is taken over all cubes $Q\subset\mathbb{R}^n$ with sides parallel to the coordinate axes and $|Q|$ denotes the Lebesgue measure of Q.

Let $b\in\mathrm{BMO}(\mathbb{R}^n)$. The weighted (L^p,L^p) boundedness of $[b,\mu_\Omega]$ was proved by Ding, Lu and Yabuta [6].

Lemma 3.1 ([6]). *Suppose that $\Omega \in L^s(\mathrm{S}^{n-1})$ $(s > 1)$ satisfies* (1.3). *If $b(x) \in \mathrm{BMO}(\mathbb{R}^n)$ and $\omega \in A_{p/s'}$, $s' < p < \infty$, then there is a constant C, independent of f, such that*

$$\int_{\mathbb{R}^n} |[b, \mu_\Omega](f)(x)|^p \omega(x)\,\mathrm{d}x \leqslant C \int_{\mathbb{R}^n} |f(x)|^p \omega(x)\,\mathrm{d}x.$$

By Lemma 3.1 and Lemma 2.2 it is easy to get the $(L^{p(\cdot)}(\mathbb{R}^n),\ L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $[b, \mu_\Omega]$.

Next, we will give the corresponding result about the commutator $[b, \mu_\Omega]$ on Herz spaces with variable exponent.

Theorem 3.1. *Suppose that $b \in \mathrm{BMO}(\mathbb{R}^n)$, $0 < \nu \leqslant 1$, $0 < p \leqslant \infty$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies conditions* (1.1) *and* (1.2) *in Lemma* 1.1, *$\Omega \in L^s(\mathrm{S}^{n-1})$, $s > q'^-$ and $-n\delta_1 - \nu - n/s < \alpha < n\delta_2 - \nu - n/s$. Then $[b, \mu_\Omega]$ is bounded on $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.*

In the proof of Theorem 3.1, we also need the following lemma.

Lemma 3.2 ([7]). *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, let k be a positive integer and B a ball in $\mathbb{R}^n$. Then for all $b \in \mathrm{BMO}(\mathbb{R}^n)$ and all $j, i \in \mathbb{Z}$ with $j > i$,*

$$\frac{1}{C}\|b\|_*^k \leqslant \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_B)^k \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|b\|_*^k,$$
$$\|(b - b_{B_i})^k \chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C(j - i)^k \|b\|_*^k \|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $B_i = \{x \in \mathbb{R}^n \colon |x| \leqslant 2^i\}$ and $B_j = \{x \in \mathbb{R}^n \colon |x| \leqslant 2^j\}$.

Proof of Theorem 3.1. Similarly to Theorem 2.1, we only prove the homogeneous case and still suppose $0 < p < \infty$. Let $f \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, and let us write $f(x) = \sum\limits_{j=-\infty}^{\infty} f\chi_j(x) = \sum\limits_{j=-\infty}^{\infty} f_j(x)$. Then we have

$$\begin{aligned}
(3.1)\qquad \|[b, \mu_\Omega](f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|[b, \mu_\Omega](f)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \Big\}^{1/p} \\
&\leqslant C\Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \Big(\sum_{j=-\infty}^{k-2} \|[b, \mu_\Omega](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^p \Big\}^{1/p} \\
&\quad + C\Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \Big(\sum_{j=k-1}^{k+1} \|[b, \mu_\Omega](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^p \Big\}^{1/p} \\
&\quad + C\Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \Big(\sum_{j=k+2}^{\infty} \|[b, \mu_\Omega](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^p \Big\}^{1/p} \\
&=: CJ_1 + CJ_2 + CJ_3.
\end{aligned}$$

Noting that $[b,\mu]$ is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, we have

$$J_2 \leqslant C\biggl\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\biggr\}^{1/p} = C\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \tag{3.2}$$

Now we estimate J_1. We consider

$$\begin{aligned}|[b,\mu_\Omega](f_j)(x)| &\leqslant \biggl(\int_0^{|x|}\biggl|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]f_j(y)\,\mathrm{d}y\biggr|^2\frac{\mathrm{d}t}{t^3}\biggr)^{1/2}\\ &\quad+\biggl(\int_{|x|}^{\infty}\biggl|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]f_j(y)\,\mathrm{d}y\biggr|^2\frac{\mathrm{d}t}{t^3}\biggr)^{1/2}\\ &=: J_{11}+J_{12}.\end{aligned}$$

Note that $x\in A_k$, $y\in A_j$ and $j\leqslant k-2$, and we know that $|x-y|\sim|x|$. By (2.3), the Minkowski inequality and the generalized Hölder inequality we have

$$\begin{aligned}J_{11} &\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(x)-b(y)||f_j(y)|\biggl(\int_{|x-y|}^{|x|}\frac{\mathrm{d}t}{t^3}\biggr)^{1/2}\mathrm{d}y\\ &\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(x)-b(y)||f_j(y)|\biggl|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\biggr|^{1/2}\mathrm{d}y\\ &\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)||b(x)-b(y)||f_j(y)|}{|x-y|^{n-1}}\frac{|y|^{1/2}}{|x-y|^{3/2}}\,\mathrm{d}y\\ &\leqslant C\frac{2^{j/2}}{|x|^{n+1/2}}\int_{A_j}|\Omega(x-y)||b(x)-b(y)||f(y)|\,\mathrm{d}y\\ &\leqslant C2^{(j-k)/2}2^{-kn}\biggl\{|b(x)-b_{B_j}|\int_{A_j}|\Omega(x-y)||f_j(y)|\,\mathrm{d}y\\ &\quad+\int_{A_j}|\Omega(x-y)||b_{B_j}-b(y)||f_j(y)|\,\mathrm{d}y\biggr\}\\ &\leqslant C2^{(j-k)/2}2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\bigl\{|b(x)-b_{B_j}|\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\ &\quad+\|\Omega(x-\cdot)(b_{B_j}-b(\cdot))\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\bigr\}.\end{aligned}$$

Similarly, we consider J_{12}. Noting that $|x-y|\sim|x|$, by the Minkowski inequality and the generalized Hölder inequality we have

$$\begin{aligned}J_{12} &\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(x)-b(y)||f_j(y)|\biggl(\int_{|x|}^{\infty}\frac{\mathrm{d}t}{t^3}\biggr)^{1/2}\mathrm{d}y\\ &\leqslant C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n}}|b(x)-b(y)||f_j(y)|\,\mathrm{d}y\\ &\leqslant C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\bigl\{|b(x)-b_{B_j}|\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\ &\quad+\|\Omega(x-\cdot)(b_{B_j}-b(\cdot))\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\bigr\}.\end{aligned}$$

So, we have

$$|[b,\mu_\Omega](f_j)(x)| \leqslant C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\big\{|b(x)-b_{B_j}|\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)} + \|\Omega(x-\cdot)(b_{B_j}-b(\cdot))\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\big\}.$$

Noting that $s > q'^{-}$, we denote $\tilde{q}'(\cdot) > 1$ and $1/q'(x) = 1/\tilde{q}'(x) + 1/s$. By Lemma 2.3 and Lemma 2.4 we have

$$\begin{aligned}\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)} &\leqslant \|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^s(\mathbb{R}^n)}\|\chi_j(\cdot)\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\\ &\leqslant \|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^s(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\\ &\leqslant C2^{-j\nu}\left(\int_{A_j}|\Omega(x-y)|^s|y|^{s\nu}\,\mathrm{d}y\right)^{1/s}\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\\ &\leqslant C2^{-j\nu}2^{k(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

When $|B_j| \leqslant 2^n$ and $x_j \in B_j$, by Lemma 2.5 we have

$$\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)} \approx |B_j|^{1/\tilde{q}'(x_j)} \approx \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s}.$$

When $|B_j| \geqslant 1$ we have

$$\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)} \approx |B_j|^{1/\tilde{q}'(\infty)} \approx \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s}.$$

So, we obtain $\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)} \approx \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s}$.

So, we have

$$\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \leqslant C2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}. \tag{3.3}$$

Similarly, by Lemma 3.2 we have

$$\begin{aligned}&\|\Omega(x-\cdot)(b_{B_j}-b(\cdot))\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\ &\quad\leqslant \|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^s(\mathbb{R}^n)}\|(b_{B_j}-b(\cdot))\chi_j(\cdot)\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\\ &\quad\leqslant C\|b\|_*\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^s(\mathbb{R}^n)}\\ &\quad\leqslant C\|b\|_*2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{3.4}$$

By (3.3), (3.4), Lemma 1.3, Lemma 1.4 and Lemma 3.2 we have

$$\begin{aligned}&\|[b,\mu](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\quad\leqslant C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\qquad\times\big(2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|(b(\cdot)-b_{B_j})\chi_k(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\qquad+\|b\|_*2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\big)\end{aligned}$$

$$
\begin{aligned}
&\leqslant C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\big((k-j)\|b\|_*2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad+\|b\|_*2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\big)\\
&\leqslant C(k-j)\|b\|_*\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{-kn}2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\\
&\quad\times\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C(k-j)\|b\|_*\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\\
&\leqslant C2^{(j-k)(n\delta_2-\nu-n/s)}(k-j)\|b\|_*\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
J_1&\leqslant C\bigg\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\bigg(\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\nu-n/s)}\\
&\quad\times(k-j)\|b\|_*\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\bigg)^p\bigg\}^{1/p}\\
&=C\|b\|_*\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\\
&\quad\times\bigg\{\sum_{k=-\infty}^{\infty}\bigg(\sum_{j=-\infty}^{k-2}2^{j\alpha}2^{(j-k)(n\delta_2-\nu-n/s-\alpha)}(k-j)\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\bigg)^p\bigg\}^{1/p}.
\end{aligned}
$$

If $1<p<\infty$, take $1/p+1/p'=1$. Since $n\delta_2-\nu-n/s-\alpha>0$, by the Hölder inequality we have

$$
\begin{aligned}
J_1&\leqslant C\|b\|_*\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\\
&\quad\times\bigg\{\sum_{k=-\infty}^{\infty}\bigg(\sum_{j=-\infty}^{k-2}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p/2}\bigg)\\
&\quad\times\bigg(\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p'/2}(k-j)^{p'}\bigg)^{p/p'}\bigg\}^{1/p}\\
&\leqslant C\|b\|_*\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\\
&\quad\times\bigg\{\sum_{k=-\infty}^{\infty}\bigg(\sum_{j=-\infty}^{k-2}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p/2}\bigg)\bigg\}^{1/p}\\
&=C\|b\|_*\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\\
&\quad\times\bigg\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\bigg(\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p/2}\bigg)\bigg\}^{1/p}\\
&\leqslant C\|b\|_*\bigg\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\bigg\}^{1/p}=C\|b\|_*\|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
\tag{3.5}
$$

If $0 < p \leqslant 1$, then we have

$$(3.6)\qquad J_1 \leqslant C\|b\|_*\|\Omega\|_{L^s(\mathrm{S}^{n-1})} \times \Big\{\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2} 2^{j\alpha p}2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p}(k-j)^p\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{1/p}$$
$$= C\|b\|_*\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\Big\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \times \Big(\sum_{k=j+2}^{\infty} 2^{(j-k)(n\delta_2-\nu-n/s-\alpha)p}(k-j)^p\Big)\Big\}^{1/p}$$
$$\leqslant C\|b\|_*\|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Let us now estimate J_3. Note that $x \in A_k$, $y \in A_j$ and $j \geqslant k+2$, so we have $|x-y| \sim |y|$. We consider

$$|[b,\mu_\Omega](f_j)(x)| \leqslant \Big(\int_0^{|y|}\Big|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]f_j(y)\,\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{1/2} + \Big(\int_{|y|}^{\infty}\Big|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]f_j(y)\,\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{1/2}$$
$$=: J_{31} + J_{32}.$$

Similarly to the estimate for J_{11}, we get

$$J_{31} \leqslant C2^{(k-j)/2}2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\big\{|b(x)-b_{B_j}|\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)} + \|\Omega(x-\cdot)(b_{B_j}-b(\cdot))\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\big\}.$$

Similarly to the estimate for J_{12}, we get

$$J_{32} \leqslant C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\big\{|b(x)-b_{B_j}|\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)} + \|\Omega(x-\cdot)(b_{B_j}-b(\cdot))\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\big\}.$$

So, we have

$$|[b,\mu_\Omega](f_j)(x)| \leqslant C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\big\{|b(x)-b_{B_j}|\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)} + \|\Omega(x-\cdot)(b_{B_j}-b(\cdot))\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\big\}.$$

By (3.3), (3.4), Lemma 1.3, Lemma 1.4 and Lemma 3.2 we have

$$\|[b,\mu_\Omega](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \times \big(2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|(b(\cdot)-b_{B_j})\chi_k(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^n)} + \|b\|_*2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\big)$$

$$
\begin{aligned}
&\leqslant C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\Big((j-k)\|b\|_*2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad+\|b\|_*2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)\\
&\leqslant C(j-k)\|b\|_*\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{-jn}2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\\
&\quad\times\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C(j-k)\|b\|_*\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-j)(\nu+n/s)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\\
&\leqslant C2^{(k-j)(n\delta_1+\nu+n/s)}(j-k)\|b\|_*\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
J_3&\leqslant C\|b\|_*\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\\
&\quad\times\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_1+\nu+n/s)}(j-k)\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\Big\}^{1/p}\\
&=C\|b\|_*\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\\
&\quad\times\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=k+2}^{\infty}2^{j\alpha}2^{(k-j)(n\delta_1+\nu+n/s+\alpha)}(j-k)\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\Big\}^{1/p}.
\end{aligned}
$$

If $1<p<\infty$, take $1/p+1/p'=1$. Since $n\delta_1+\nu+n/s+\alpha>0$, by the Hölder inequality we have

$$
\begin{aligned}
(3.7)\qquad J_3&\leqslant C\|b\|_*\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\\
&\quad\times\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=k+2}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p/2}\Big)\\
&\quad\times\Big(\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p'/2}(j-k)^{p'}\Big)^{p/p'}\Big\}^{1/p}\\
&\leqslant C\|b\|_*\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\\
&\quad\times\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=k+2}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p/2}\Big)\Big\}^{1/p}\\
&=C\|b\|_*\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\\
&\quad\times\Big\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big(\sum_{k=-\infty}^{j-2}2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p/2}\Big)\Big\}^{1/p}\\
&\leqslant C\|b\|_*\Big\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{1/p}=C\|b\|_*\|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

If $0 < p \leqslant 1$, then we have

$$
\begin{aligned}
(3.8)\quad J_3 &\leqslant C\|b\|_*\|\Omega\|_{L^s(\mathrm{S}^{n-1})} \\
&\quad\times\Bigg\{\sum_{k=-\infty}^{\infty}\sum_{j=k+2}^{\infty} 2^{j\alpha p}2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p}(j-k)^p\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg\}^{1/p} \\
&= C\|b\|_*\|\Omega\|_{L^s(\mathrm{S}^{n-1})} \\
&\quad\times\Bigg\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Bigg(\sum_{k=-\infty}^{j-2} 2^{(k-j)(n\delta_1+\nu+n/s+\alpha)p}(j-k)^p\Bigg)\Bigg\}^{1/p} \\
&\leqslant C\|b\|_*\|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Therefore, by (3.1), (3.2) and (3.5)–(3.8) we complete the proof of Theorem 3.1. □

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