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EXISTENCE, BLOW-UP AND EXPONENTIAL DECAY FOR A NONLINEAR LOVE EQUATION ASSOCIATED WITH DIRICHLET CONDITIONS

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Abstract. In this paper we consider a nonlinear Love equation associated with Dirichlet conditions. First, under suitable conditions, the existence of a unique local weak solution is proved. Next, a blow up result for solutions with negative initial energy is also established. Finally, a sufficient condition guaranteeing the global existence and exponential decay of weak solutions is given. The proofs are based on the linearization method, the Galerkin method associated with a priori estimates, weak convergence, compactness techniques and the construction of a suitable Lyapunov functional. To our knowledge, there has been no decay or blow up result for equations of Love waves or Love type waves before.

Keywords: nonlinear Love equation; Faedo-Galerkin method; local existence; blow up; exponential decay

MSC 2010: 35L20, 35L70, 35Q74, 37B25

1. Introduction

In this paper, we consider the following nonlinear Love equation with initial conditions and homogeneous Dirichlet boundary conditions

$$(1.1)\quad u_{tt} - u_{xx} - u_{xxtt} - \lambda_1 u_{xxt} + \lambda u_t = F(x,t,u,u_x,u_t,u_{xt}) - \frac{\partial}{\partial x}[G(x,t,u,u_x,u_t,u_{xt})] + f(x,t), \quad 0 < x < 1,\ 0 < t < T,$$

$$(1.2)\quad u(0,t) = u(1,t) = 0,$$

$$(1.3)\quad u(x,0) = \widetilde{u}_0(x), \quad u_t(x,0) = \widetilde{u}_1(x),$$

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where $\lambda > 0$, $\lambda_1 > 0$ are constants and $\widetilde{u}_0$, $\widetilde{u}_1 \in H_0^1 \cap H^2$; f, F, G are given functions satisfying conditions specified below.

When $f = F = G = 0$, $\lambda = \lambda_1 = 0$, $\Omega = (0, L)$, equation (1.1) is related to the Love equation

$$u_{tt} - \frac{E}{\varrho} u_{xx} - 2\mu^2\omega^2 u_{xxtt} = 0, \tag{1.4}$$

presented by Radochová in 1978 (see [18]). This equation, which describes the vertical oscillations of a rod, was established from Euler's variational equation of an energy functional

$$\int_0^T \mathrm{d}t \int_0^L \Big[\frac{1}{2} F\varrho(u_t^2 + \mu^2\omega^2 u_{tx}^2) - \frac{1}{2} F(E u_x^2 + \varrho\mu^2\omega^2 u_x u_{xtt})\Big]\, \mathrm{d}x. \tag{1.5}$$

The parameters in (1.5) have the following meaning: u is the displacement, L is the length of the rod, F is the area of cross-section, ω is the cross-section radius, E is the Young modulus of the material and ϱ is the mass density. By using the Fourier method, Radochová [18] obtained a classical solution of problem (1.4) associated with initial condition (1.3) and boundary conditions

$$u(0, t) = u(L, t) = 0, \tag{1.6a}$$

or

$$\begin{cases} u(0, t) = 0, \\ \varepsilon u_{xtt}(L, t) + c^2 u_x(L, t) = 0, \end{cases} \tag{1.6b}$$

where $c^2 = E/\varrho$, $\varepsilon = 2\mu^2\omega^2$. On the other hand, the asymptotic behaviour of the solution of problem (1.3), (1.4), (1.6a) or (1.6b) as $\varepsilon \to 0_+$ was also established by the method of small parameter.

Equations of Love waves or Love type waves have been studied by many authors, we refer to [4], [6], [12], [13], [14], [17] and references therein.

In [12], by combining the linearization method for the nonlinear term, the Faedo-Galerkin method and the weak compactness method, the existence of a unique weak solution of a Dirichlet problem for the nonlinear Love equation $u_{tt} - u_{xx} - u_{xxtt} = f(x, t, u, u_x, u_t, u_{xt})$ is proved.

In [19], a symmetric version of the regularized long wave equation (SRLWE)

$$\begin{cases} u_{xxt} - u_t = \varrho_x + u u_x, \\ \varrho_t + u_x = 0, \end{cases} \tag{1.7}$$

was proposed as a model for propagation of weakly nonlinear ion acoustic and space-charge waves. Obviously, eliminating ϱ from (1.7), we get

$$u_{tt} - u_{xx} - u_{xxtt} = -uu_{xt} - u_x u_t. \tag{1.8}$$

The SRLWE (1.8) is explicitly symmetric in the x and t derivatives and is very similar to the regularized long wave equation which describes shallow water waves and plasma drift waves [1], [2]. The SRLWE also arises in many other areas of mathematical physics [5], [9], [16]. We remark that equations (1.1) and (1.8) are special forms of the equation discussed in [12].

The purpose of this paper is establishing the existence, blow up and exponential decay of weak solutions for problem (1.1)–(1.3). To our knowledge, there is no decay or blow up result for equations of Love waves or Love type waves. However, the existence and exponential decay of solutions or blow up results for wave equations, with different boundary conditions, have been extensively studied by many authors, for example, we refer to [3], [10], [11], [15] and references therein. In [3], the following problem was considered:

$$\begin{cases} u_{tt} - \Delta u + g(u_t) + f(u) = 0, \quad x \in \Omega,\ t > 0, \\ u = 0,\ x \in \partial\Omega, \quad t \geqslant 0, \\ u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x), \quad x \in \Omega, \end{cases} \tag{1.9}$$

where $f(u) = -b|u|^{p-2}u$, $g(u_t) = a(1 + |u_t|^{m-2})u_t$, a, $b > 0$, m, $p > 2$, and Ω is a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Benaissa and Messaoudi showed that for suitably chosen initial data, (1.10) possesses a global weak solution, which decays exponentially even if $m > 2$. Nakao and Ono [11] extended the previous results to the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \lambda^2(x)u + g(u_t) + f(u) = 0, \quad x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x), \quad x \in \mathbb{R}^N, \end{cases} \tag{1.10}$$

where $g(u_t)$ behaves like $|u_t|^{m-2}u_t$, $f(u)$ behaves like $-|u|^{p-2}u$, and the initial data $(\widetilde{u}_0, \widetilde{u}_1)$ is small enough in $H^1(\Omega) \times L^2(\Omega)$. In [15], the existence and exponential decay for the nonlinear wave equation

$$u_{tt} - u_{xx} + Ku + \lambda u_t = a|u|^{p-2}u + f(x,t), \quad 0 < x < 1,\ t > 0, \tag{1.11}$$

with a nonlocal boundary condition, in cases $a = 1$, $a = -1$, were also established. In [10], Messaoudi established a blow up result for solutions with negative initial energy and a global existence result for arbitrary initial data of a nonlinear viscoelastic wave equation

$$(1.12)\quad u_{tt}-\Delta u+\int_0^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau+a|u_t|^{m-2}u_t=b|u|^{p-2}u,\quad x\in\Omega,\ t>0,$$

where $a,b>0$, $p>2$, $m\geqslant 1$, and Ω is a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, associated with initial and Dirichlet boundary conditions. In [8], [20], the existence, regularity, blow-up and exponential decay estimates of solutions for nonlinear wave equations associated with two-point boundary conditions were established. The proofs are based on the Galerkin method associated with a priori estimates, weak convergence, compactness techniques and the construction of a suitable Lyapunov functional. The authors in [20] proved that any weak solution with negative initial energy will blow up in finite time.

The above mentioned works lead to the study of the existence, blow-up and exponential decay estimates for a nonlinear Love equation associated with initial and Dirichlet boundary conditions (1.1)–(1.3). Our paper is organized as follows.

Section 2 is devoted to the presentation of preliminaries and an existence result via the Faedo-Galerkin method. Problem (1.1)–(1.3) here is dealt with the general case $F,G\in C^1([0,1]\times[0,T]\times\mathbb{R}^4)$.

In Sections 3, 4, 5, problem (1.1)–(1.3) is considered with $F=F(u)=a|u|^{p-2}u$, $G=G(u_x)=b|u_x|^{p-2}u_x$, $a,b\in\mathbb{R}$, $p>2$. In the case of $a>0$, $b>0$; $f(x,t)\equiv 0$, with negative initial energy, we prove that the solution of (1.1)–(1.3) blows up in finite time. In the case of $a>0$, $b<0$, it is proved that if $\|\widetilde{u}_{0x}\|^2-a\|\widetilde{u}_0\|^p_{L^p}>0$ and $f\in L^2((0,1)\times\mathbb{R}_+)$, $\|f(t)\|\leqslant C\mathrm{e}^{-\gamma_0 t}$, $\gamma_0>0$, then the energy of the solution decays exponentially as $t\to\infty$. Finally, in the case of $a<0$, $b<0$ and $\|f(t)\|$ small enough as above, we remark that problem (1.1)–(1.3) has a unique global solution with energy decaying exponentially as $t\to\infty$, without the initial data $(\widetilde{u}_0,\widetilde{u}_1)$ being small enough.

2. Existence of a weak solution

First, we put $\Omega=(0,1)$; $Q_T=\Omega\times(0,T)$, $T>0$ and denote the usual function spaces used in this paper by $C^m(\overline{\Omega})$, $W^{m,p}=W^{m,p}(\Omega)$, $L^p=W^{0,p}(\Omega)$, $H^m=W^{m,2}(\Omega)$, $1\leqslant p\leqslant\infty$, $m=0,1,\ldots$ Let $\langle\cdot,\cdot\rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$, $1\leqslant p\leqslant\infty$, the Banach space of the real functions $u\colon (0,T)\to X$ measurable such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, \mathrm{d}t\right)^{1/p} < \infty \quad \text{for } 1 \leqslant p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X \quad \text{for } p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $u_x(t)$, $u_{xx}(t)$ denote $u(x,t)$, $\partial u/\partial t(x,t)$, $\partial^2 u/\partial t^2(x,t)$, $\partial u/\partial x(x,t)$, $\partial^2 u/\partial x^2(x,t)$, respectively.

On H^1, we shall use the norm

$$\|v\|_{H^1} = (\|v\|^2 + \|v_x\|^2)^{1/2}.$$

Then the following lemma is known.

Lemma 2.1. *The imbedding* $H^1 \hookrightarrow C^0(\overline{\Omega})$ *is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \leqslant \sqrt{2}\|v\|_{H^1} \quad \forall\, v \in H^1. \tag{2.1}$$

Remark 2.1. On H_0^1, $v \mapsto \|v\|_{H^1}$ and $v \mapsto \|v_x\|$ are equivalent norms. Furthermore,

$$\|v\|_{C^0(\overline{\Omega})} \leqslant \|v_x\| \quad \text{for all } v \in H_0^1. \tag{2.2}$$

With $F \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, $F = F(x,t,y_1,\ldots,y_4)$, we put $D_1F = \partial F/\partial x$, $D_2F = \partial F/\partial t$, $D_{i+2}F = \partial F/\partial y_i$, $i = 1,\ldots,4$.

Next, we establish the local existence theorem. We need the following assumptions:

(H_1) $f \in H^1(Q_T)$, $Q_T = (0,1) \times (0,T)$;

(H_2) $F \in C^1([0,1]\times[0,T]\times\mathbb{R}^4)$, such that $F(0,t,0,y_2,0,y_4) = F(1,t,0,y_2,0,y_4) = 0$ for all $t \in [0,T]$, for all $y_2, y_4 \in \mathbb{R}$;

(H_3) $G \in C^1([0,1] \times [0,T] \times \mathbb{R}^4)$.

Theorem 2.2. *Suppose that* (H_1)–(H_3) *hold. Then problem* (1.1)–(1.3) *has a unique local solution*

$$u \in L^\infty(0,T_*;H_0^1 \cap H^2),\ u_t \in L^\infty(0,T_*;H_0^1 \cap H^2),\ u_{tt} \in L^\infty(0,T_*;H_0^1 \cap H^2), \tag{2.3}$$

for $T_* > 0$ *small enough.*

Remark 2.2. The regularity obtained by (2.3) shows that problem (1.1)–(1.3) has a unique strong solution

$$u \in C^1([0,T_*];H_0^1 \cap H^2), \quad u_{tt} \in L^\infty(0,T_*;H_0^1 \cap H^2). \tag{2.4}$$

P r o o f of Theorem 2.2. The proof is a combination of the linearization method for a nonlinear term, the Faedo-Galerkin method and the weak compactness method, and consits of two steps.

Step 1. Establish a linear recurrence sequence $\{u_m\}$ by the linearization method.

Consider $T>0$ fixed, let $M>0$, and put

$$K_M(f)=\|f\|_{H^1(Q_T)}=\Big(\|f\|^2_{L^2(Q_T)}+\Big\|\frac{\partial f}{\partial x}\Big\|^2_{L^2(Q_T)}+\Big\|\frac{\partial f}{\partial t}\Big\|^2_{L^2(Q_T)}\Big)^{1/2}, \tag{2.5}$$

$$\|F\|_{C^0(A_M)}=\sup_{(x,t,y_1,\dots,y_4)\in A_M}|F(x,t,y_1,\dots,y_4)|,$$

$$A_M=[0,1]\times[0,T]\times[-M,M]^4,$$

$$\overline{F}_M=\|F\|_{C^1(A_M)}=\|F\|_{C^0(A_M)}+\sum_{i=1}^{6}\|D_iF\|_{C^0(A_M)},$$

$$\overline{G}_M=\|G\|_{C^1(A_M)}=\|G\|_{C^0(A_M)}+\sum_{i=1}^{6}\|D_iG\|_{C^0(A_M)}.$$

For each $T_*\in(0,T]$ and $M>0$, we put

$$\begin{cases} W(M,T_*)=\{v\in L^\infty(0,T_*;H_0^1\cap H^2)\colon\ v_t\in L^\infty(0,T_*;H_0^1\cap H^2),\\ \qquad\qquad v_{tt}\in L^\infty(0,T_*;H_0^1),\\ \text{with } \|v\|_{L^\infty(0,T_*;H_0^1\cap H^2)},\ \|v_t\|_{L^\infty(0,T_*;H_0^1\cap H^2)},\ \|v_{tt}\|_{L^\infty(0,T_*;H_0^1)}\leqslant M\},\\ W_1(M,T_*)=\{v\in W(M,T_*)\colon\ v_{tt}\in L^\infty(0,T_*;H_0^1\cap H^2)\}, \end{cases} \tag{2.6}$$

where $Q_{T_*}=\Omega\times(0,T_*)$.

We establish the linear recurrence sequence $\{u_m\}$ as follows.

We choose the first term $u_0\equiv 0$, suppose that

$$u_{m-1}\in W_1(M,T_*), \tag{2.7}$$

and associate with problem (1.1)–(1.3) the following problem:

Find $u_m\in W_1(M,T_*)$ $(m\geqslant 1)$ which satisfies the linear variational problem

$$\begin{cases} \langle u_m''(t),w\rangle+\langle u_{mx}''(t)+\lambda_1u_{mx}'(t)+u_{mx}(t),w_x\rangle+\lambda\langle u_m'(t),w\rangle\\ \quad=\langle f(t),w\rangle+\langle F_m(t),w\rangle+\langle G_m(t),w_x\rangle\ \forall\,w\in H_0^1,\\ u_m(0)=\widetilde{u}_0,\ \ u_m'(0)=\widetilde{u}_1, \end{cases} \tag{2.8}$$

where

$$\begin{aligned} F_m(x,t)&=F(x,t,u_{m-1}(x,t),\nabla u_{m-1}(x,t),u_{m-1}'(x,t),\nabla u_{m-1}'(x,t))\\ &\equiv F[u_{m-1}](x,t),\\ G_m(x,t)&=G(x,t,u_{m-1}(x,t),\nabla u_{m-1}(x,t),u_{m-1}'(x,t),\nabla u_{m-1}'(x,t))\\ &\equiv G[u_{m-1}](x,t). \end{aligned} \tag{2.9}$$

Then we have the following lemma.

Lemma 2.3. *Let* (H_1)–(H_3) *hold. Then there exist positive constants* M, $T_* > 0$ *such that, for* $u_0 \equiv 0$, *there exists a recurrence sequence* $\{u_m\} \subset W_1(M, T_*)$ *defined by* (2.7)–(2.9).

P r o o f of Lemma 2.3. The proof consists of several steps.

(i) *The Faedo-Galerkin approximation* (introduced by Lions [7]). Consider a special orthonormal basis $\{w_j\}$ on H_0^1: $w_j(x) = \sqrt{2}\sin(j\pi x)$, $j \in \mathbb{N}$, formed by the eigenfunctions of the Laplacian $-\Delta = -\partial^2/\partial x^2$. Put

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \tag{2.10}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of linear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + \langle \ddot{u}_{mx}^{(k)}(t) + \lambda_1 \dot{u}_{mx}^{(k)}(t) + u_{mx}^{(k)}(t), w_{jx}\rangle + \lambda \langle \dot{u}_m^{(k)}(t), w_j\rangle \\ \quad = \langle F_m(t), w_j\rangle + \langle G_m(t), w_{jx}\rangle + \langle f(t), w_j\rangle, \quad 1 \leqslant j \leqslant k, \\ u_m^{(k)}(0) = \widetilde{u}_{0k}, \ \ \dot{u}_m^{(k)}(0) = \widetilde{u}_{1k}, \end{cases} \tag{2.11}$$

in which

$$\begin{cases} \widetilde{u}_{0k} = \displaystyle\sum_{j=1}^{k} \alpha_j^{(k)} w_j \to \widetilde{u}_0 \text{ strongly in } H_0^1 \cap H^2, \\ \widetilde{u}_{1k} = \displaystyle\sum_{j=1}^{k} \beta_j^{(k)} w_j \to \widetilde{u}_1 \text{ strongly in } H_0^1 \cap H^2. \end{cases} \tag{2.12}$$

System (2.11) can be rewritten in the form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \dfrac{\lambda_1 \overline{\lambda}_j + \lambda}{1 + \overline{\lambda}_j} \dot{c}_{mj}^{(k)}(t) + \dfrac{\overline{\lambda}_j}{1 + \overline{\lambda}_j} c_{mj}^{(k)}(t) = f_{mj}(t), \\ c_m^{(k)}(0) = \alpha_j^{(k)}, \quad \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)}, \quad 1 \leqslant j \leqslant k, \end{cases} \tag{2.13}$$

where

$$\begin{cases} f_{mj}(t) = \dfrac{1}{1 + \overline{\lambda}_j}[\langle F_m(t), w_j\rangle + \langle G_m(t), w_{jx}\rangle + \langle f(t), w_j\rangle], \\ \overline{\lambda}_j = (j\pi)^2, \quad 1 \leqslant j \leqslant k. \end{cases} \tag{2.14}$$

Note that by (2.7), it is not difficult to prove that system (2.13) has a unique solution on the interval $[0, T]$.

(ii) *A priori estimates.* Put

$$S_m^{(k)}(t) = p_m^{(k)}(t) + q_m^{(k)}(t) + r_m^{(k)}(t), \tag{2.15}$$

where

$$\begin{cases} p_m^{(k)}(t) = \|\dot{u}_m^{(k)}(t)\|^2 + \|\dot{u}_{mx}^{(k)}(t)\|^2 + \|u_{mx}^{(k)}(t)\|^2 \\ \qquad\qquad + 2\lambda_1 \int_0^t \|\dot{u}_{mx}^{(k)}(s)\|^2 \, \mathrm{d}s + 2\lambda \int_0^t \|\dot{u}_m^{(k)}(s)\|^2 \, \mathrm{d}s, \\ q_m^{(k)}(t) = \|\dot{u}_{mx}^{(k)}(t)\|^2 + \|\Delta\dot{u}_m^{(k)}(t)\|^2 + \|\Delta u_m^{(k)}(t)\|^2 \\ \qquad\qquad + 2\lambda_1 \int_0^t \|\Delta\dot{u}_m^{(k)}(s)\|^2 \, \mathrm{d}s + 2\lambda \int_0^t \|\dot{u}_{mx}^{(k)}(s)\|^2 \, \mathrm{d}s, \\ r_m^{(k)}(t) = \|\ddot{u}_m^{(k)}(t)\|^2 + \|\ddot{u}_{mx}^{(k)}(t)\|^2 + \|\dot{u}_{mx}^{(k)}(t)\|^2 + 2\lambda_1 \int_0^t \|\ddot{u}_{mx}^{(k)}(s)\|^2 \, \mathrm{d}s \\ \qquad\qquad + 2\lambda \int_0^t \|\ddot{u}_m^{(k)}(s)\|^2 \, \mathrm{d}s. \end{cases} \tag{2.16}$$

Then it follows from (2.11), (2.15), and (2.16) that

$$\begin{aligned} S_m^{(k)}(t) = S_m^{(k)}(0) &+ 2\int_0^t \langle f(s), \dot{u}_m^{(k)}(s)\rangle \, \mathrm{d}s \\ &+ 2\int_0^t \langle \nabla f(s), \dot{u}_{mx}^{(k)}(s)\rangle \, \mathrm{d}s + 2\int_0^t \langle f'(s), \ddot{u}_m^{(k)}(s)\rangle \, \mathrm{d}s \\ &+ 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s)\rangle \, \mathrm{d}s + 2\int_0^t \langle G_m(s), \dot{u}_{mx}^{(k)}(s)\rangle \, \mathrm{d}s \\ &+ 2\int_0^t \langle F_{mx}(s), \dot{u}_{mx}^{(k)}(s)\rangle \, \mathrm{d}s + 2\int_0^t \langle G_{mx}(s), \triangle\dot{u}_m^{(k)}(s)\rangle \, \mathrm{d}s \\ &+ 2\int_0^t \langle \dot{F}_m(s), \ddot{u}_m^{(k)}(s)\rangle \, \mathrm{d}s + 2\int_0^t \langle \dot{G}_m(s), \ddot{u}_{mx}^{(k)}(s)\rangle \, \mathrm{d}s \\ = S_m^{(k)}(0) &+ \sum_{j=1}^{9} I_j. \end{aligned} \tag{2.17}$$

First, we are going to estimate $\xi_m^{(k)} = \|\ddot{u}_m^{(k)}(0)\|^2 + \|\ddot{u}_{mx}^{(k)}(0)\|^2$.

Letting $t \to 0_+$ in equation $(2.11)_1$ and multiplying the result by $\ddot{c}_{mj}^{(k)}(0)$, we get

$$\begin{aligned} &\|\ddot{u}_m^{(k)}(0)\|^2 + \|\ddot{u}_{mx}^{(k)}(0)\|^2 + \langle \lambda_1\widetilde{u}_{1kx} + \widetilde{u}_{0kx}, \ddot{u}_{mx}^{(k)}(0)\rangle + \lambda\langle\widetilde{u}_{1k}, \ddot{u}_m^{(k)}(0)\rangle \\ &\qquad = \langle F_m(0), \ddot{u}_m^{(k)}(0)\rangle + \langle G_m(0), \ddot{u}_{mx}^{(k)}(0)\rangle + \langle f(0), \ddot{u}_m^{(k)}(0)\rangle. \end{aligned} \tag{2.18}$$

This implies that

$$(2.19)\quad \begin{aligned}\xi_m^{(k)} &= \|\ddot{u}_m^{(k)}(0)\|^2 + \|\ddot{u}_{mx}^{(k)}(0)\|^2\\ &\leqslant (\lambda_1\|\widetilde{u}_{1kx}\| + \|\widetilde{u}_{0kx}\| + \|G_m(0)\|)\|\ddot{u}_{mx}^{(k)}(0)\|\\ &\quad + (\lambda\|\widetilde{u}_{1k}\| + \|F_m(0)\| + \|f(0)\|)\|\ddot{u}_m^{(k)}(0)\|\\ &\leqslant [\lambda_1\|\widetilde{u}_{1kx}\| + \|\widetilde{u}_{0kx}\| + \|G_m(0)\| + \lambda\|\widetilde{u}_{1k}\| + \|F_m(0)\| + \|f(0)\|]\sqrt{\xi_m^{(k)}}\\ &\leqslant [\lambda_1\|\widetilde{u}_{1kx}\| + \|\widetilde{u}_{0kx}\| + \|G_m(0)\| + \lambda\|\widetilde{u}_{1k}\| + \|F_m(0)\| + \|f(0)\|]^2.\end{aligned}$$

Moreover,

$$(2.20)\quad \begin{aligned}\|F_m(0)\| + \|G_m(0)\| &= \|F(\cdot,0,\widetilde{u}_0,\widetilde{u}_{0x},\widetilde{u}_1,\widetilde{u}_{1x})\|\\ &\quad + \|G(\cdot,0,\widetilde{u}_0,\widetilde{u}_{0x},\widetilde{u}_1,\widetilde{u}_{1x})\| = \text{a constant independent of } m.\end{aligned}$$

Thus,

$$(2.21)\quad \xi_m^{(k)} \leqslant \overline{X}_0 \quad \forall\, m,$$

where $\overline{X}_0$ is a constant depending only on f, $\widetilde{u}_0$, $\widetilde{u}_1$, F, G, λ, and λ_1.

By (2.12), (2.15), (2.16), and (2.21), we get

$$(2.22)\quad \begin{aligned}S_m^{(k)}(0) &= \|\widetilde{u}_{1k}\|^2 + \|\widetilde{u}_{1kx}\|^2 + \|\widetilde{u}_{0k}\|^2\\ &\quad + \|\widetilde{u}_{1kx}\|^2 + \|\Delta\widetilde{u}_{1k}\|^2 + \|\Delta\widetilde{u}_{0k}\|^2 + \|\widetilde{u}_{1kx}\|^2\\ &\quad + \xi_m^{(k)} \leqslant S_0 \quad \forall\, m,k\in\mathbb{N},\end{aligned}$$

where S_0 is a constant depending only on f, $\widetilde{u}_0$, $\widetilde{u}_1$, F, G, λ, and λ_1.

We shall estimate the terms I_j on the right hand side of (2.17) as follows.

First term I_1. By the Cauchy-Schwartz inequality, we have

$$(2.23)\quad I_1 = 2\int_0^t \langle f(s),\dot{u}_m^{(k)}(s)\rangle\,\mathrm{d}s \leqslant \|f\|^2_{L^2(Q_T)} + \int_0^t \|\dot{u}_m^{(k)}(s)\|^2\,\mathrm{d}s.$$

Similarly, for the terms I_2, I_3, we obtain

$$(2.24)\quad \begin{aligned}I_2 &= 2\int_0^t \langle \nabla f(s),\dot{u}_{mx}^{(k)}(s)\rangle\,\mathrm{d}s \leqslant \|\nabla f\|^2_{L^2(Q_T)} + \int_0^t \|\dot{u}_{mx}^{(k)}(s)\|^2\,\mathrm{d}s,\\ I_3 &= 2\int_0^t \langle f'(s),\ddot{u}_m^{(k)}(s)\rangle\,\mathrm{d}s \leqslant \|f'\|^2_{L^2(Q_T)} + \int_0^t \|\ddot{u}_m^{(k)}(s)\|^2\,\mathrm{d}s.\end{aligned}$$

Hence,

$$(2.25)\quad I_1 + I_2 + I_3 \leqslant \|f\|^2_{H^1(Q_T)} + \int_0^t S_m^{(k)}(s)\,\mathrm{d}s.$$

Fourth term $I_4 = 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s)\rangle\,\mathrm{d}s$. It is known that

$$|F_m(x,t)| \leqslant \overline{F}_M. \tag{2.26}$$

Consequently,

$$\begin{aligned} I_4 = 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s)\rangle\,\mathrm{d}s &\leqslant 2\overline{F}_M\int_0^t \|\dot{u}_m^{(k)}(s)\|\,\mathrm{d}s \\ &\leqslant T_*\overline{F}_M^2 + \int_0^t \|\dot{u}_m^{(k)}(s)\|^2\,\mathrm{d}s. \end{aligned} \tag{2.27}$$

Similarly, for the term I_5, we obtain

$$I_5 = 2\int_0^t \langle G_m(s), \dot{u}_{mx}^{(k)}(s)\rangle\,\mathrm{d}s \leqslant T_*\overline{G}_M^2 + \int_0^t \|\dot{u}_{mx}^{(k)}(s)\|^2\,\mathrm{d}s. \tag{2.28}$$

Hence,

$$I_4 + I_5 \leqslant T_*(\overline{F}_M^2 + \overline{G}_M^2) + \int_0^t p_m^{(k)}(s)\,\mathrm{d}s. \tag{2.29}$$

Sixth term $I_6 = 2\int_0^t \langle F_{mx}(s), \dot{u}_{mx}^{(k)}(s)\rangle\,\mathrm{d}s$.
It is known that

$$\begin{aligned} F_{mx}(t) = D_1F[u_{m-1}] &+ D_3F[u_{m-1}]\nabla u_{m-1} + D_4F[u_{m-1}]\Delta u_{m-1} \\ &+ D_5F[u_{m-1}]\nabla u'_{m-1} + D_6F[u_{m-1}]\Delta u'_{m-1}, \end{aligned}$$

so

$$\|F_{mx}(t)\| \leqslant (1+4M)\overline{F}_M \equiv \widetilde{F}_M. \tag{2.30}$$

Hence,

$$\begin{aligned} I_6 = 2\int_0^t \langle F_{mx}(s), \dot{u}_{mx}^{(k)}(s)\rangle\,\mathrm{d}s &\leqslant 2\int_0^t \|F_{mx}(s)\|\|\dot{u}_{mx}^{(k)}(s)\|\,\mathrm{d}s \\ \leqslant 2\widetilde{F}_M\int_0^t \|\dot{u}_{mx}^{(k)}(s)\|\,\mathrm{d}s &\leqslant T_*\widetilde{F}_M^2 + \int_0^t \|\dot{u}_{mx}^{(k)}(s)\|^2\,\mathrm{d}s. \end{aligned} \tag{2.31}$$

Similarly, for the term I_7, we find that

$$I_7 = 2\int_0^t \langle G_{mx}(s), \triangle\dot{u}_m^{(k)}(s)\rangle\,\mathrm{d}s \leqslant T_*\widetilde{G}_M^2 + \int_0^t \|\triangle\dot{u}_m^{(k)}(s)\|^2\,\mathrm{d}s, \tag{2.32}$$

with $\widetilde{G}_M = (1+4M)\overline{G}_M$. Thus

$$I_6 + I_7 \leqslant T_*(\widetilde{F}_M^2 + \widetilde{G}_M^2) + \int_0^t q_m^{(k)}(s)\,\mathrm{d}s. \tag{2.33}$$

Similarly, for the terms I_8, I_9, we obtain

$$\begin{aligned} I_8 + I_9 &= 2\int_0^t \langle \dot{F}_m(s), \ddot{u}_m^{(k)}(s)\rangle\,\mathrm{d}s + 2\int_0^t \langle \dot{G}_m(s), \ddot{u}_{mx}^{(k)}(s)\rangle\,\mathrm{d}s \\ &\leqslant T_*(\widetilde{F}_M^2 + \widetilde{G}_M^2) + \int_0^t r_m^{(k)}(s)\,\mathrm{d}s. \end{aligned} \tag{2.34}$$

Finally, (2.17), (2.22), (2.25), (2.29), (2.33), and (2.34) lead to

$$S_m^{(k)}(t) \leqslant S_0 + \|f\|_{H^1(Q_T)}^2 + T_* D_1(M) + 2\int_0^t S_m^{(k)}(s)\,\mathrm{d}s, \tag{2.35}$$

where

$$D_1(M) = [1 + 2(1+4M)^2](\overline{F}_M^2 + \overline{G}_M^2). \tag{2.36}$$

We can choose $M > 0$ sufficiently large so that

$$S_0 + \|f\|_{H^1(Q_T)}^2 \leqslant \frac{1}{2}M^2, \tag{2.37}$$

next choose $T_* \in (0, T]$ small enough so that

$$\Big(\frac{1}{2}M^2 + T_* D_1(M)\Big)\mathrm{e}^{2T_*} \leqslant M^2, \tag{2.38}$$

and

$$k_{T_*} = 2\sqrt{(\overline{F}_M^2 + \overline{G}_M^2)T_*\mathrm{e}^{T_*}} < 1. \tag{2.39}$$

It follows from (2.35), (2.37), and (2.38) that

$$S_m^{(k)}(t) \leqslant \mathrm{e}^{-2T_*}M^2 + 2\int_0^t S_m^{(k)}(s)\,\mathrm{d}s. \tag{2.40}$$

By virtue of Gronwall's Lemma, (2.40) yields

$$S_m^{(k)}(t) \leqslant \mathrm{e}^{-2T_*}M^2\mathrm{e}^{2t} \leqslant M^2 \tag{2.41}$$

for all $t \in [0, T_*]$, for all m and k. Therefore,

$$u_m^{(k)} \in W(M, T_*) \quad \forall m \text{ and } k. \tag{2.42}$$

(iii) *Limiting process.* From (2.41) we deduce the existence of a subsequence of $\{u_m^{(k)}\}$ still so denoted, such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in } L^\infty(0, T_*; H_0^1 \cap H^2) \text{ weakly*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in } L^\infty(0, T_*; H_0^1 \cap H^2) \text{ weakly*}, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in } L^\infty(0, T_*; H_0^1) \text{ weakly*}, \\ u_m \in W(M, T_*). \end{cases} \tag{2.43}$$

Passing to limit in (2.11), (2.12), we have u_m satisfying (2.8), (2.9) in $L^2(0, T_*)$.

On the other hand, we have from $(2.8)_1$, $(2.43)_4$ that

$$\frac{\partial^2}{\partial x^2}(u_m'' + \lambda_1 u_m' + u_m) = u_m'' + \lambda u_m' - F_m + G_{mx} - f \in L^\infty(0, T_*; L^2). \tag{2.44}$$

Therefore,

$$u_m'' + \lambda_1 u_m' + u_m = \Psi_m \in L^\infty(0, T_*; H_0^1 \cap H^2). \tag{2.45}$$

In order to continue the proof, now we deduce from (2.45) that, if

$$u_m \in L^\infty(0, T_*; H_0^1 \cap H^2), \tag{2.46}$$

then

$$u_m', u_m'' \in L^\infty(0, T_*; H_0^1 \cap H^2). \tag{2.47}$$

Indeed, let (2.45), (2.46) hold. Then we have

$$u_m'' + \lambda_1 u_m' = \Psi_m - u_m \equiv \overline{\Psi}_m \in L^\infty(0, T_*; H_0^1 \cap H^2). \tag{2.48}$$

Integrating (2.48) gives

$$u_m' + \lambda_1 u_m = \widetilde{u}_1 + \lambda_1 \widetilde{u}_0 + \int_0^t \overline{\Psi}_m(s)\,\mathrm{d}s \equiv \widetilde{\Psi}_m \in L^\infty(0, T_*; H_0^1 \cap H^2). \tag{2.49}$$

Hence,

$$u_m' = \widetilde{\Psi}_m - \lambda_1 u_m \in L^\infty(0, T_*; H_0^1 \cap H^2). \tag{2.50}$$

It follows from (2.45) that

$$u_m'' = -\lambda_1 u_m' - u_m + \Psi_m \in L^\infty(0,T_*;H_0^1\cap H^2). \tag{2.51}$$

We will prove that (2.46) holds. We consider three cases for λ_1.

Case 1: $\lambda_1 = 2$. By (2.45), we have

$$u_m(t) = \widetilde{u}_0 \mathrm{e}^{-t} + (\widetilde{u}_0+\widetilde{u}_1)t\mathrm{e}^{-t} + \int_0^t (t-s)\mathrm{e}^{s-t}\Psi_m(s)\,\mathrm{d}s \in L^\infty(0,T_*;H_0^1\cap H^2). \tag{2.52}$$

Case 2: $\lambda_1 > 2$. Put $k_1 = \frac{1}{2}(-\lambda_1 + \sqrt{\lambda_1^2-4})$, $k_2 = \frac{1}{2}(-\lambda_1 - \sqrt{\lambda_1^2-4})$. Then (2.45) gives

$$\begin{aligned} u_m(t) &= \frac{1}{\sqrt{\lambda_1^2-4}}[(\widetilde{u}_1 - k_2\widetilde{u}_0)\mathrm{e}^{k_1 t} - (\widetilde{u}_1 - k_1\widetilde{u}_0)\mathrm{e}^{k_2 t}] \\ &\quad + \frac{1}{\sqrt{\lambda_1^2-4}}\int_0^t (\mathrm{e}^{k_1(t-s)} - \mathrm{e}^{k_2(t-s)})\Psi_m(s)\,\mathrm{d}s \in L^\infty(0,T_*;H_0^1\cap H^2). \end{aligned} \tag{2.53}$$

Case 3: $0 < \lambda_1 < 2$. Putting $\alpha = -\frac{1}{2}\lambda_1$, $\beta = \frac{1}{2}\sqrt{4-\lambda_1^2}$, (2.45) implies

$$\begin{aligned} u_m(t) &= \widetilde{u}_0\mathrm{e}^{\alpha t}\cos\beta t + \frac{1}{\beta}(\widetilde{u}_1 - \alpha\widetilde{u}_0)\mathrm{e}^{\alpha t}\sin\beta t \\ &\quad + \frac{1}{\beta}\int_0^t \mathrm{e}^{\alpha(t-s)}\sin(\beta t(t-s))\Psi_m(s)\,\mathrm{d}s \in L^\infty(0,T_*;H_0^1\cap H^2). \end{aligned} \tag{2.54}$$

Thus $u_m, u_m', u_m'' \in L^\infty(0,T_*;H_0^1\cap H^2)$, hence $u_m \in W_1(M,T_*)$ and Lemma 2.3 is proved. Hence, step 1 is complete.

Step 2. The convergence to the solution u of problem (1.1)–(1.3) *of the linear recurrence sequence* $\{u_m\}$.

We have the following lemma.

Lemma 2.4. *Let* (H_1)–(H_3) *hold. Then*

(i) *Problem* (1.1)–(1.3) *has a unique weak solution $u \in W_1(M,T_*)$, where the constants $M > 0$ and $T_* > 0$ are chosen as in Lemma* 2.3.

Furthermore,

(ii) *The linear recurrence sequence $\{u_m\}$ defined by* (2.7)–(2.9) *converges to the solution u of problem* (1.1)–(1.3) *strongly in the space*

$$W_1(T_*) = \{v \in L^\infty(0,T_*;H_0^1)\colon\ v' \in L^\infty(0,T_*;H_0^1)\}.$$

P r o o f of Lemma 2.4. We use the result obtained in Lemma 2.3 and the compact imbedding theorems to prove Lemma 2.4. It means that the existence and uniqueness of a weak solution of problem (1.1)–(1.3) is proved.

(i) *Existence.* First, we note that $W_1(T_*)$ is a Banach space with respect to the norm (see Lions [7])

$$\|v\|_{W_1(T_*)} = \|v\|_{L^\infty(0,T_*;H_0^1)} + \|v'\|_{L^\infty(0,T_*;H_0^1)}. \tag{2.55}$$

We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T_*)$. Let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\begin{cases} \langle w_m''(t), w\rangle + \langle w_{mx}''(t) + \lambda_1 w_{mx}'(t) + w_{mx}(t), w_x\rangle + \lambda\langle w_m'(t), w\rangle \\ \quad = \langle F_{m+1}(t) - F_m(t), w\rangle + \langle G_{m+1}(t) - G_m(t), w_x\rangle \quad \forall\, w \in H_0^1, \\ w_m(0) = w_m'(0) = 0. \end{cases} \tag{2.56}$$

Taking $w = w_m'$ in (2.56), after integrating in t, we get

$$\begin{aligned} Z_m(t) = 2\int_0^t &\langle F_{m+1}(s) - F_m(s), w_m'(s)\rangle \,\mathrm{d}s \\ &+ 2\int_0^t \langle G_{m+1}(s) - G_m(s), w_{mx}'(s)\rangle \,\mathrm{d}s, \end{aligned} \tag{2.57}$$

where

$$\begin{aligned} Z_m(t) = \|w_m'(t)\|^2 &+ \|w_{mx}'(t)\|^2 + \|w_{mx}(t)\|^2 \\ &+ 2\lambda_1\int_0^t \|w_{mx}'(s)\|^2 \,\mathrm{d}s + 2\lambda\int_0^t \|w_m'(s)\|^2 \,\mathrm{d}s. \end{aligned} \tag{2.58}$$

On the other hand, from (H_2), (H_3) we obtain by (2.5), (2.7), (2.9), and $(2.43)_4$ that

$$\begin{aligned} \|F_{m+1}(s) - F_m(s)\| &\leqslant 2\overline{F}_M\|w_{m-1}\|_{W_1(T_*)}, \\ \|G_{m+1}(s) - G_m(s)\| &\leqslant 2\overline{G}_M\|w_{m-1}\|_{W_1(T_*)}. \end{aligned} \tag{2.59}$$

Combining (2.57) and (2.59), we obtain

$$Z_m(t) \leqslant (\overline{F}_M^2 + \overline{G}_M^2)T_*\|w_{m-1}\|_{W_1(T_*)}^2 + \int_0^t Z_m(s)\,\mathrm{d}s. \tag{2.60}$$

Using Gronwall's Lemma, we deduce from (2.60) that

$$\|w_m\|_{W_1(T_*)} \leqslant k_{T_*}\|w_{m-1}\|_{W_1(T_*)} \quad \forall\, m \in \mathbb{N}, \tag{2.61}$$

where $0 < k_{T_*} < 1$ is defined as in (2.39). This implies

$$\|u_m - u_{m+p}\|_{W_1(T_*)} \leqslant \|u_0 - u_1\|_{W_1(T_*)}(1-k_{T_*})^{-1}k_{T_*}^m \leqslant M(1-k_{T_*})^{-1}k_{T_*}^m \quad \forall\, m,\ p \in \mathbb{N}. \tag{2.62}$$

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T_*)$. Then, there exists $u \in W_1(T_*)$ such that

$$u_m \to u \quad \text{strongly in } W_1(T_*). \tag{2.63}$$

Note that $u_m \in W_1(M, T_*)$, so there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in } L^\infty(0,T_*;H_0^1 \cap H^2) \text{ weakly*}, \\ u'_{m_j} \to u' & \text{in } L^\infty(0,T_*;H_0^1 \cap H^2) \text{ weakly*}, \\ u''_{m_j} \to u'' & \text{in } L^\infty(0,T_*;H_0^1) \text{ weakly*}, \\ u \in W(M,T_*). \end{cases} \tag{2.64}$$

Putting

$$\begin{aligned} F[u](x,t) &= F(x,t,u(x,t),\nabla u(x,t),u'(x,t),\nabla u'(x,t)), \\ G[u](x,t) &= G(x,t,u(x,t),\nabla u(x,t),u'(x,t),\nabla u'(x,t)), \end{aligned} \tag{2.65}$$

by (2.5), (2.7), (2.9) and $(2.64)_4$, we obtain

$$\begin{aligned} \|F_m(t) - F[u](t)\| &\leqslant 2\overline{F}_M\|u_{m-1} - u\|_{W_1(T_*)}, \\ \|G_m(t) - G[u](t)\| &\leqslant 2\overline{F}_M\|u_{m-1} - u\|_{W_1(T_*)}. \end{aligned} \tag{2.66}$$

Hence, (2.63) and (2.66) yield

$$\begin{aligned} &F_m \to F[u] \text{ strongly in } L^\infty(0,T_*;L^2), \\ &G_m \to G[u] \text{ strongly in } L^\infty(0,T_*;L^2). \end{aligned} \tag{2.67}$$

Finally, passing to limit in (2.8), (2.9) as $m = m_j \to \infty$, it follows from (2.63), $(2.64)_{1,3}$, and (2.67) that there exists $u \in W(M,T_*)$ satisfying the equation

$$\begin{aligned} &\langle u''(t), w\rangle + \langle u''_x(t) + \lambda_1 u'_x(t) + u_x(t), w_x\rangle + \lambda\langle u'(t), w\rangle \\ &\quad = \langle f(t), w\rangle + \langle F[u](t), w\rangle + \langle G[u](t), w_x\rangle \quad \forall\, w \in H_0^1 \end{aligned} \tag{2.68}$$

for all $w \in H_0^1$, and the initial conditions

$$u(0) = \widetilde{u}_0, \quad u'(0) = \widetilde{u}_1. \tag{2.69}$$

On the other hand, due to the assumption (H_2) we obtain from (2.64) and (2.68) that

$$\frac{\partial^2}{\partial x^2}(u'' + \lambda_1 u' + u) = u'' + \lambda u' - F[u] + \frac{\partial}{\partial x}G[u] - f \in L^\infty(0, T_*; L^2). \tag{2.70}$$

Hence,

$$u'' + \lambda_1 u' + u = \Psi \in L^\infty(0, T_*; H_0^1 \cap H^2). \tag{2.71}$$

Similarly, from (2.71) we have

$$u, u', u'' \in L^\infty(0, T_*; H_0^1 \cap H^2). \tag{2.72}$$

Consequently, $u \in W_1(M, T_*)$ and the existence follows.

(ii) *Uniqueness.* Let u_1, u_2 be two weak solutions of problem (1.1)–(1.3) such that

$$u_i \in W_1(M, T_*),\ \ i = 1, 2. \tag{2.73}$$

Then $w = u_1 - u_2$ verifies

$$\begin{cases} \langle w''(t), w\rangle + \langle w_x''(t) + \lambda_1 w_x'(t) + w_x(t), w_x\rangle + \lambda\langle w'(t), w\rangle \\ \quad = \langle F[u_1](t) - F[u_2](t), w\rangle + \langle G[u_1](t) - G[u_2](t), w_x\rangle \quad \forall\, w \in H_0^1, \\ w(0) = w'(0) = 0. \end{cases} \tag{2.74}$$

Taking $v = w = u_1 - u_2$ in $(2.74)_1$ and integrating with respect to t, we obtain

$$\begin{aligned} \sigma(t) = 2a \int_0^t \langle F[u_1](s) - F[u_2](s), w'(s)\rangle \, \mathrm{d}s \\ + 2b \int_0^t \langle G[u_1](s) - G[u_2](s), w_x'(s)\rangle \, \mathrm{d}s, \end{aligned} \tag{2.75}$$

where

$$\begin{aligned} \sigma(t) = \|w'(t)\|^2 + \|w_x'(t)\|^2 + \|w_x(t)\|^2 \\ + 2\lambda_1 \int_0^t \|w_x'(s)\|^2 \, \mathrm{d}s + 2\lambda \int_0^t \|w'(s)\|^2 \, \mathrm{d}s. \end{aligned} \tag{2.76}$$

On the other hand, by (H_2), (H_3), (2.5) with $M = \max\limits_{i=1,2} \|u_i\|_{L^\infty(0,T_*;H^2\cap H_0^1)}$, we deduce from (2.76) that

$$\begin{aligned} \|F[u_1](s) - F[u_2](s)\| \leqslant 2\overline{F}_M \sqrt{\sigma(s)}, \\ \|G[u_1](s) - G[u_2](s)\| \leqslant 2\overline{G}_M \sqrt{\sigma(s)}. \end{aligned} \tag{2.77}$$

Combining (2.75) and (2.77), leads to

$$\sigma(t) = 2(\overline{F}_M + \overline{G}_M) \int_0^t \sigma(s)\,\mathrm{d}s. \tag{2.78}$$

By Gronwall's Lemma, (2.78) gives $\sigma \equiv 0$, i.e., $u_1 \equiv u_2$. Lemma 2.4 is proved completely and Theorem 2.2 follows. □

3. Blow up

In this section, problem (1.1)–(1.3) is considered with $F(x,t,u,u_x,u_t,u_{xt}) = a|u|^{p-2}u$, $G(x,t,u,u_x,u_t,u_{xt}) = b|u_x|^{p-2}u_x$, $a, b \in \mathbb{R}$, $p > 2$, as follows:

$$\begin{cases} u_{tt} - u_{xx} - u_{xxtt} - \lambda_1 u_{xxt} + \lambda u_t = a|u|^{p-2}u - b\frac{\partial}{\partial x}(|u_x|^{p-2}u_x) \\ +f(x,t),\ 0 < x < 1,\ 0 < t < T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x). \end{cases} \tag{3.1}$$

Supose that $a > 0$, $b > 0$, $p > 2$ and $f \equiv 0$. Let $u(x,t)$ be a weak solution of (3.1) satisfying

$$u \in C^1([0,T_*]; H^2 \cap H_0^1),\ u_{tt} \in L^\infty(0,T_*; H^2 \cap H_0^1). \tag{3.2}$$

We will show that the solution $u(x,t)$ of (3.1) blows up in finite time if

$$-H(0) = \frac{1}{2}\|\widetilde{u}_1\|^2 + \frac{1}{2}\|\widetilde{u}_{1x}\|^2 + \frac{1}{2}\|\widetilde{u}_{0x}\|^2 - \frac{a}{p}\|\widetilde{u}_0\|_{L^p}^p - \frac{b}{p}\|\widetilde{u}_{0x}\|_{L^p}^p < 0. \tag{3.3}$$

Theorem 3.1. *Let $H(0) > 0$. Then the solution u of problem* (3.1) *blows up in finite time.*

Proof. We denote by $E(t)$ the energy associated with the solution u, defined by

$$E(t) = \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u'_x(t)\|^2 + \frac{1}{2}\|u_x(t)\|^2 - \frac{a}{p}\|u(t)\|_{L^p}^p - \frac{b}{p}\|u_x(t)\|_{L^p}^p, \tag{3.4}$$

and we put

$$H(t) = -E(t) = \frac{a}{p}\|u(t)\|_{L^p}^p + \frac{b}{p}\|u_x(t)\|_{L^p}^p - \frac{1}{2}\|u'(t)\|^2 - \frac{1}{2}\|u'_x(t)\|^2 - \frac{1}{2}\|u_x(t)\|^2. \tag{3.5}$$

On the other hand, by multiplying $(3.1)_1$ by $u'(x,t)$ and integrating over $[0,1]$, we get

$$H'(t) = \lambda\|u'(t)\|^2 + \lambda_1\|u'_x(t)\|^2 \geqslant 0 \quad \forall\, t \in [0, T_*). \tag{3.6}$$

Hence, we can deduce from (3.6) and $H(0) > 0$ that

$$\begin{aligned} 0 < H(0) \leqslant H(t) = \frac{a}{p}\|u(t)\|^p_{L^p} + \frac{b}{p}\|u_x(t)\|^p_{L^p} \\ - \frac{1}{2}\|u'(t)\|^2 - \frac{1}{2}\|u'_x(t)\|^2 - \frac{1}{2}\|u_x(t)\|^2 \quad \forall\, t \in [0, T_*). \end{aligned} \tag{3.7}$$

Now, we define the functional

$$L(t) = H^{1-\eta}(t) + \varepsilon\psi(t), \tag{3.8}$$

where

$$\psi(t) = \langle u(t), u'(t)\rangle + \langle u_x(t), u'_x(t)\rangle + \frac{\lambda}{2}\|u(t)\|^2 + \frac{\lambda_1}{2}\|u_x(t)\|^2, \tag{3.9}$$

for ε small enough and

$$0 < \eta \leqslant \frac{p-2}{2p} < \frac{1}{2}. \tag{3.10}$$

Lemma 3.2. *There exists a constant $d_1 > 0$ such that*

$$L'(t) \geqslant d_1(H(t) + \|u'(t)\|^2 + \|u'_x(t)\|^2 + \|u_x(t)\|^2 + \|u(t)\|^p_{L^p} + \|u_x(t)\|^p_{L^p}). \tag{3.11}$$

Proof of Lemma 3.2. By multiplying $(3.1)_1$ by $u(x,t)$ and integrating over $[0,1]$, we get

$$\psi'(t) = \|u'(t)\|^2 + \|u'_x(t)\|^2 - \|u_x(t)\|^2 + a\|u(t)\|^p_{L^p} + b\|u_x(t)\|^p_{L^p}. \tag{3.12}$$

By taking a derivative of (3.8) and using (3.12), we obtain

$$\begin{aligned} L'(t) = (1-\eta)H^{-\eta}(t)H'(t) \\ + \varepsilon[\|u'(t)\|^2 + \|u'_x(t)\|^2 - \|u_x(t)\|^2 + a\|u(t)\|^p_{L^p} + b\|u_x(t)\|^p_{L^p}]. \end{aligned} \tag{3.13}$$

Since (3.7), (3.13) and due to the inequalities

$$
(3.14)\qquad \begin{cases} (1-\eta)H^{-\eta}(t)H'(t) > 0, \\ \dfrac{1}{2}\|u_x(t)\|^2 < \dfrac{a}{p}\|u(t)\|^p_{L^p} + \dfrac{b}{p}\|u_x(t)\|^p_{L^p}, \\ H(t) \leqslant \dfrac{a}{p}\|u(t)\|^p_{L^p} + \dfrac{b}{p}\|u_x(t)\|^p_{L^p}, \\ \dfrac{1}{2}\|u'(t)\|^2 + \dfrac{1}{2}\|u'_x(t)\|^2 + \dfrac{1}{2}\|u_x(t)\|^2 < \dfrac{a}{p}\|u(t)\|^p_{L^p} + \dfrac{b}{p}\|u_x(t)\|^p_{L^p}, \end{cases}
$$

we deduce that

$$
\begin{aligned}
(3.15)\qquad L'(t) &\geqslant \varepsilon[\|u'(t)\|^2 + \|u'_x(t)\|^2 - \|u_x(t)\|^2 + a\|u(t)\|^p_{L^p} + b\|u_x(t)\|^p_{L^p}] \\
&\geqslant \varepsilon\Big[\|u'(t)\|^2 + \|u'_x(t)\|^2 - \frac{2}{p}(a\|u(t)\|^p_{L^p} + b\|u_x(t)\|^p_{L^p}) \\
&\quad + a\|u(t)\|^p_{L^p} + b\|u_x(t)\|^p_{L^p}\Big] \\
&= \varepsilon\|u'(t)\|^2 + \varepsilon\|u'_x(t)\|^2 + \varepsilon\Big(1 - \frac{2}{p}\Big)(a\|u(t)\|^p_{L^p} + b\|u_x(t)\|^p_{L^p}).
\end{aligned}
$$

On the other hand, it follows from $(3.14)_{2,3}$ and the inequalities

$$
(3.16)\qquad a\|u(t)\|^p_{L^p} + b\|u_x(t)\|^p_{L^p} \geqslant pH(t),\ \ a\|u(t)\|^p_{L^p} + b\|u_x(t)\|^p_{L^p} \geqslant \frac{p}{2}\|u_x(t)\|^2
$$

that

$$
\begin{aligned}
(3.17)\qquad L'(t) &\geqslant \varepsilon\|u'(t)\|^2 + \varepsilon\|u'_x(t)\|^2 + \varepsilon\Big(1 - \frac{2}{p}\Big)(a\|u(t)\|^p_{L^p} + b\|u_x(t)\|^p_{L^p}) \\
&\geqslant \varepsilon\|u'(t)\|^2 + \varepsilon\|u'_x(t)\|^2 + \frac{\varepsilon}{3}\Big(1 - \frac{2}{p}\Big)(a\|u(t)\|^p_{L^p} + b\|u_x(t)\|^p_{L^p}) \\
&\quad + \frac{\varepsilon}{3}\Big(1 - \frac{2}{p}\Big)pH(t) + \frac{\varepsilon}{3}\Big(1 - \frac{2}{p}\Big)\frac{p}{2}\|u_x(t)\|^2 \\
&\geqslant d_1(H(t) + \|u'(t)\|^2 + \|u'_x(t)\|^2 + \|u_x(t)\|^2 + \|u(t)\|^p_{L^p} + \|u_x(t)\|^p_{L^p}),
\end{aligned}
$$

where $d_1 = \min\{\varepsilon, \frac{1}{3}\varepsilon(1 - 2/p)\}$ is a positive constant. Lemma 3.2 is proved completely. □

R e m a r k 3.1. By virtue of the formula of $L(t)$ and Lemma 3.2, we can choose ε small enough such that

$$
(3.18)\qquad L(t) \geqslant L(0) > 0 \quad \forall\, t \in [0, T_*).
$$

Now we continue to prove Theorem 3.1.

Using the inequality

$$\left(\sum_{i=1}^{5} x_i\right)^r \leqslant 5^{r-1}\sum_{i=1}^{5} x_i^r \quad \forall\, r > 1, \text{ and } x_1,\ldots,x_5 \geqslant 0, \tag{3.19}$$

we deduce from (3.8) and (3.9) that

$$\begin{aligned} L^{1/(1-\eta)}(t) &\leqslant \mathrm{Const}(H(t) + |\langle u(t), u'(t)\rangle|^{1/(1-\eta)} + |\langle u_x(t), u_x'(t)\rangle|^{1/(1-\eta)} \\ &\quad + \|u(t)\|^{2/(1-\eta)} + \|u_x(t)\|^{2/(1-\eta)}) \\ &\leqslant \mathrm{Const}\big(H(t) + \|u(t)\|^{1/(1-\eta)}\|u'(t)\|^{1/(1-\eta)} \\ &\quad + \|u_x(t)\|^{1/(1-\eta)}\|u_x'(t)\|^{1/(1-\eta)} \\ &\quad + \|u(t)\|^{2/(1-\eta)} + \|u_x(t)\|^{2/(1-\eta)}\big). \end{aligned} \tag{3.20}$$

On the other hand, using Young's inequality yields

$$\begin{aligned} \|u(t)\|^{1/(1-\eta)}\|u'(t)\|^{1/(1-\eta)} &\leqslant \frac{1-2\eta}{2(1-\eta)}\|u(t)\|^s + \frac{1}{2(1-\eta)}\|u'(t)\|^2 \\ &\leqslant \mathrm{Const}(\|u(t)\|^s + \|u'(t)\|^2) \\ &\leqslant \mathrm{Const}(\|u_x(t)\|^s + \|u'(t)\|^2), \end{aligned} \tag{3.21}$$

where $s = 2/(1-2\eta) \leqslant p$ by (3.10).

Similarly

$$\begin{aligned} \|u_x(t)\|^{1/(1-\eta)}\|u_x'(t)\|^{1/(1-\eta)} &\leqslant \frac{1-2\eta}{2(1-\eta)}\|u_x(t)\|^s + \frac{1}{2(1-\eta)}\|u_x'(t)\|^2 \\ &\leqslant \mathrm{Const}(\|u_x(t)\|^s + \|u_x'(t)\|^2). \end{aligned} \tag{3.22}$$

It follows from (3.20)–(3.22) that

$$\begin{aligned} L^{1/(1-\eta)}(t) &\leqslant \mathrm{Const}[H(t) + \|u'(t)\|^2 + \|u_x'(t)\|^2 \\ &\quad + \|u(t)\|^{2/(1-\eta)} + \|u_x(t)\|^{2/(1-\eta)} + \|u_x(t)\|^s]. \end{aligned} \tag{3.23}$$

Now, we need the following lemma.

Lemma 3.3. *Let $2 \leqslant r_1 \leqslant p$, $2 \leqslant r_2 \leqslant p$. Then we have*

$$\|v\|^{r_1} + \|v_x\|^{r_1} + \|v_x\|^{r_2} \leqslant 3(\|v_x\|^2 + \|v\|_{L^p}^p + \|v_x\|_{L^p}^p) \tag{3.24}$$

for any $v \in H_0^1$.

P r o o f of Lemma 3.3. (i) We consider two cases for $\|v\|$:

(i.1) *Case 1*: $\|v\| \leqslant 1$: By $2 \leqslant r_1 \leqslant p$, we get

$$\|v\|^{r_1} \leqslant \|v\|^2 \leqslant \|v_x\|^2 \leqslant \|v_x\|^2 + \|v\|_{L^p}^p + \|v_x\|_{L^p}^p \equiv \varrho[v]. \tag{3.25}$$

(i.2) *Case 2*: $\|v\| \geqslant 1$: By $2 \leqslant r_1 \leqslant p$, we find that

$$\|v\|^{r_1} \leqslant \|v\|^p \leqslant \|v\|_{L^p}^p \leqslant \varrho[v]. \tag{3.26}$$

Therefore,

$$\|v\|^{r_1} \leqslant \varrho[v] \quad \text{for any } v \in H_0^1. \tag{3.27}$$

(ii) We consider two cases for $\|v_x\|$:
(ii.1) *Case 1*: $\|v_x\| \leqslant 1$: By $2 \leqslant r_1 \leqslant p$, we have

$$\|v_x\|^{r_1} \leqslant \|v_x\|^2 \leqslant \varrho[v]. \tag{3.28}$$

(ii.2) *Case 2*: $\|v_x\| \geqslant 1$: By $2 \leqslant r_1 \leqslant p$, we have

$$\|v_x\|^{r_1} \leqslant \|v_x\|^p \leqslant \varrho[v]. \tag{3.29}$$

Therefore,

$$\|v_x\|^{r_1} \leqslant \varrho[v] \quad \text{for any } v \in H_0^1. \tag{3.30}$$

(iii) Similarly

$$\|v_x\|^{r_2} \leqslant \varrho[v] \quad \text{for any } v \in H_0^1. \tag{3.31}$$

Combining (3.27), (3.30), and (3.31), we get

$$\|v\|^{r_1} + \|v_x\|^{r_1} + \|v_x\|^{r_2} \leqslant 3\varrho[v] \leqslant 3(\|v_x\|^2 + \|v\|_{L^p}^p + \|v_x\|_{L^p}^p) \quad \forall\, v \in H_0^1. \tag{3.32}$$

Lemma 3.3 is proved completely. □

By (3.23) and using Lemma 3.2 with $r_1 = 2/(1-\eta)$, $r_2 = s$, we get

$$\begin{aligned} L^{1/(1-\eta)}(t) \leqslant \mathrm{Const}(&H(t) + \|u'(t)\|^2 + \|u_x'(t)\|^2 + \|u_x(t)\|^2 \\ &+ \|u(t)\|_{L^p}^p + \|u_x(t)\|_{L^p}^p) \quad \forall\, t \in [0, T_*). \end{aligned} \tag{3.33}$$

It follows from (3.11) and (3.33) that

$$L'(t) \geqslant d_2 L^{1/(1-\eta)}(t) \quad \forall\, t \in [0, T_*), \tag{3.34}$$

where d_2 is a positive constant. By integrating (3.34) over $(0,t)$, we deduce that

$$L^{\eta/(1-\eta)}(t) \geqslant \frac{1}{L^{-\eta/(1-\eta)}(0) - d_2\eta t/(1-\eta)}, \quad 0 \leqslant t < \frac{1-\eta}{d_2\eta} L^{-\eta/(1-\eta)}(0). \tag{3.35}$$

Therefore, (3.35) shows that $L(t)$ blows up in a finite time given by

$$T_* = \frac{1-\eta}{d_2\eta} L^{-\eta/(1-\eta)}(0). \tag{3.36}$$

Theorem 3.1 is proved completely. □

4. Exponential decay

Consider problem (3.1) corresponding to $a > 0$ and $b = -b_1 < 0$.

We prove that if $\|\widetilde{u}_{0x}\|^2 - a\|\widetilde{u}_0\|^p_{L^p} > 0$ and if the initial energy and the function f are small enough, then the energy of the solution decays exponentially as $t \to \infty$. For this purpose, we make the following assumption:

$$(\widetilde{\mathrm{H}}_1) \qquad f \in L^2((0,1)\times\mathbb{R}_+), \quad \|f(t)\| \leqslant C\mathrm{e}^{-\gamma_0 t}, \quad \gamma_0 > 0.$$

First, we construct the Lyapunov functional

$$L(t) = E_1(t) + \delta\psi(t), \tag{4.1}$$

where $\delta > 0$ will be chosen later and

$$\psi(t) = \langle u(t), u'(t)\rangle + \langle u_x(t), u'_x(t)\rangle + \frac{\lambda}{2}\|u(t)\|^2 + \frac{\lambda_1}{2}\|u_x(t)\|^2, \tag{4.2}$$

$$\begin{aligned} E_1(t) &= \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u'_x(t)\|^2 + \frac{1}{2}\|u_x(t)\|^2 + \frac{b_1}{p}\|u_x(t)\|^p_{L^p} - \frac{a}{p}\|u(t)\|^p_{L^p} \\ &= \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u'_x(t)\|^2 + J(t), \end{aligned} \tag{4.3}$$

$$\begin{aligned} J(t) &= \frac{1}{2}\|u_x(t)\|^2 + \frac{b_1}{p}\|u_x(t)\|^p_{L^p} - \frac{a}{p}\|u(t)\|^p_{L^p} \\ &= \Big(\frac{1}{2} - \frac{1}{p}\Big)\|u_x(t)\|^2 + \frac{b_1}{p}\|u_x(t)\|^p_{L^p} + \frac{1}{p}I(t), \end{aligned} \tag{4.4}$$

$$I(t) = I(u(t)) = \|u_x(t)\|^2 - a\|u(t)\|^p_{L^p}. \tag{4.5}$$

Then we have the following theorem.

Theorem 4.1. *Assume that* $(\widetilde{\mathrm{H}}_1)$ *holds. Let* $I(0)>0$ *and let the initial energy* $E_1(0)$ *satisfy*

$$\eta_* = a\left[\frac{2p}{p-2}\left(E_1(0)+\frac{1}{2\lambda}\int_0^\infty \|f(s)\|^2\,\mathrm{d}s\right)\right]^{(p-2)/2} < 1. \tag{4.6}$$

Then there exist positive constants C, γ *such that,*

$$\overline{E}_1(t)\leqslant C\exp(-\gamma t)\quad \forall\, t\geqslant 0, \tag{4.7}$$

where

$$\overline{E}_1(t)=\|u'(t)\|^2+\|u'_x(t)\|^2+\|u_x(t)\|^2+\|u_x(t)\|^p_{L^p}+I(t). \tag{4.8}$$

P r o o f. First, we need the following lemmas.

Lemma 4.2. *The energy functional* $E_1(t)$ *satisfies*

$$E_1'(t)\leqslant -\frac{\lambda}{2}\|u'(t)\|^2-\lambda_1\|u'_x(t)\|^2+\frac{1}{2\lambda}\|f(t)\|^2. \tag{4.9}$$

P r o o f of Lemma 4.2. Multiplying $(3.1)_1$ by $u'(x,t)$ and integrating over $[0,1]$, we get

$$E_1'(t)=-\lambda\|u'(t)\|^2-\lambda_1\|u'_x(t)\|^2+\langle f(t),u'(t)\rangle. \tag{4.10}$$

On the other hand,

$$\langle f(t),u'(t)\rangle\leqslant\frac{\lambda}{2}\|u'(t)\|^2+\frac{1}{2\lambda}\|f(t)\|^2. \tag{4.11}$$

Combining (4.10) and (4.11), it is easy to see (4.9) holds.

Lemma 4.2 is proved completely. □

Lemma 4.3. *Suppose that* $(\widetilde{\mathrm{H}}_1)$ *hold. If* $I(0) > 0$ *and*

$$\eta_* = a\left[\frac{2p}{p-2}\left(E_1(0) + \frac{1}{2\lambda}\int_0^\infty \|f(s)\|^2\,\mathrm{d}s\right)\right]^{(p-2)/2} < 1, \tag{4.12}$$

then $I(t) > 0$ *for all* $t \geqslant 0$.

Proof of Lemma 4.3. By the continuity of $I(t)$ and $I(0) > 0$, there exists $T_1 > 0$ such that

$$I(u(t)) \geqslant 0 \quad \forall\, t \in [0, T_1], \tag{4.13}$$

which implies

$$\begin{aligned} E_1(t) &= \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u'_x(t)\|^2 + J(t) \geqslant J(t) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u_x(t)\|^2 \\ &\quad + \frac{b_1}{p}\|u_x(t)\|^p_{L^p} + \frac{1}{p}I(t) \geqslant \left(\frac{1}{2} - \frac{1}{p}\right)\|u_x(t)\|^2. \end{aligned} \tag{4.14}$$

It follows from (4.14) that

$$\begin{aligned} \|u_x(t)\|^2 &\leqslant \frac{2p}{p-2}J(t) \leqslant \frac{2p}{p-2}E_1(t) \\ &\leqslant \frac{2p}{p-2}\left(E_1(0) + \frac{1}{2\lambda}\int_0^\infty \|f(s)\|^2\,\mathrm{d}s\right) \quad \forall\, t \in [0, T_1]. \end{aligned} \tag{4.15}$$

Hence, (4.12) and (4.15) lead to

$$\begin{aligned} a\|u(t)\|^p_{L^p} &\leqslant a\|u_x(t)\|^p = a\|u_x(t)\|^{p-2}\|u_x(t)\|^2 \\ &\leqslant a\left[\frac{2p}{p-2}\left(E_1(0) + \frac{1}{2\lambda}\int_0^\infty \|f(s)\|^2\,\mathrm{d}s\right)\right]^{(p-2)/2}\|u_x(t)\|^2 \\ &\equiv \eta_*\|u_x(t)\|^2 \quad \forall\, t \in [0, T_1]. \end{aligned} \tag{4.16}$$

Therefore, $I(t) \geqslant (1 - \eta_*)\|u_x(t)\|^2 > 0$ for all $t \in [0, T_1]$.

Now, we put $T_\infty = \sup\{T > 0\colon\ I(u(t)) > 0 \text{ for all } t \in [0, T)\}$. If $T_\infty < \infty$ then, by the continuity of $I(t)$, we have $I(T_\infty) \geqslant 0$. By the same arguments as in the above part, we can deduce that there exists $T'_\infty > T_\infty$ such that $I(t) > 0$, for all $t \in [0, T'_\infty]$. Hence, we conclude that $I(t) > 0$ for all $t \geqslant 0$.

Lemma 4.3 is proved completely. □

Lemma 4.4. *Let $I(0) > 0$ and (4.12) hold. Put*

$$\overline{E}_1(t) = \|u'(t)\|^2 + \|u'_x(t)\|^2 + \|u_x(t)\|^2 + \|u_x(t)\|^p_{L^p} + I(t). \tag{4.17}$$

Then there exist positive constants β_1, β_2 such that

$$\beta_1 \overline{E}_1(t) \leqslant L(t) \leqslant \beta_2 \overline{E}_1(t) \quad \forall\, t \geqslant 0, \tag{4.18}$$

for δ is small enough.

P r o o f of Lemma 4.4. It is easy to see that

(4.19)

$$\begin{aligned} L(t) &\leqslant \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u'_x(t)\|^2 + \Big(\frac{1}{2} - \frac{1}{p}\Big)\|u_x(t)\|^2 + \frac{b_1}{p}\|u_x(t)\|^p_{L^p} + \frac{1}{p}I(t) \\ &\quad + \frac{\delta}{2}\|u(t)\|^2 + \frac{\delta}{2}\|u'(t)\|^2 + \frac{\delta}{2}\|u_x(t)\|^2 + \frac{\delta}{2}\|u'_x(t)\|^2 + \frac{\lambda}{2}\|u(t)\|^2 + \frac{\lambda_1}{2}\|u_x(t)\|^2 \\ &\leqslant \frac{1+\delta}{2}\|u'(t)\|^2 + \frac{1+\delta}{2}\|u'_x(t)\|^2 + \Big(\frac{1}{2} - \frac{1}{p} + \delta + \frac{\lambda+\lambda_1}{2}\Big)\|u_x(t)\|^2 \\ &\quad + \frac{b_1}{p}\|u_x(t)\|^p_{L^p} + \frac{1}{p}I(t) \leqslant \beta_2 \overline{E}_1(t), \end{aligned}$$

where $\beta_2 = \max\{(1+\delta)/2, 1/2 - 1/p + \delta + (\lambda+\lambda_1)/2, b_1/p, 1/p\}$.

Similarly, we can prove that

$$\begin{aligned} L(t) &\geqslant \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u'_x(t)\|^2 + \Big(\frac{1}{2} - \frac{1}{p}\Big)\|u_x(t)\|^2 + \frac{b_1}{p}\|u_x(t)\|^p_{L^p} + \frac{1}{p}I(t) \\ &\quad - \frac{\delta}{2}\|u(t)\|^2 - \frac{\delta}{2}\|u'(t)\|^2 - \frac{\delta}{2}\|u_x(t)\|^2 - \frac{\delta}{2}\|u'_x(t)\|^2 \\ &\geqslant \frac{1-\delta}{2}\|u'(t)\|^2 + \frac{1-\delta}{2}\|u'_x(t)\|^2 + \Big(\frac{1}{2} - \frac{1}{p} - \delta\Big)\|u_x(t)\|^2 \\ &\quad + \frac{b_1}{p}\|u_x(t)\|^p_{L^p} + \frac{1}{p}I(t) \geqslant \beta_1 \overline{E}_1(t), \end{aligned} \tag{4.20}$$

where $\beta_1 = \min\{(1-\delta)/2; 1/2 - 1/p - \delta; b_1/p; 1/p\} > 0$, with $0 < \delta < 1/2 - 1/p$.

Lemma 4.4 is proved completely. □

Lemma 4.5. *Let $I(0) > 0$ and (4.12) hold. The functional $\psi(t)$ defined by (4.2) satisfies*

$$\begin{aligned} \psi'(t) &\leqslant \|u'(t)\|^2 + \|u'_x(t)\|^2 - \Big[\frac{1-\eta_*}{2} + b_1 - \frac{\varepsilon_1}{2}\Big]\|u_x(t)\|^2 \\ &\quad - b_1\|u_x(t)\|^p_{L^p} - \frac{1}{2}I(t) + \frac{1}{2\varepsilon_1}\|f(t)\|^2 \end{aligned} \tag{4.21}$$

for all $\varepsilon_1 > 0$.

P r o o f of Lemma 4.5. By multiplying $(3.1)_1$ by $u(x,t)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}\psi'(t) &= \|u'(t)\|^2 + \|u_x'(t)\|^2 - \|u_x(t)\|^2 + a\|u(t)\|_{L^p}^p \\ &\quad - b_1\|u_x(t)\|_{L^p}^p + \langle f(t), u(t)\rangle \\ &= \|u'(t)\|^2 + \|u_x'(t)\|^2 - I(t) - b_1\|u_x(t)\|_{L^p}^p + \langle f(t), u(t)\rangle.\end{aligned} \tag{4.22}$$

On the other hand,

$$\langle f(t), u(t)\rangle \leqslant \frac{\varepsilon_1}{2}\|u_x(t)\|^2 + \frac{1}{2\varepsilon_1}\|f(t)\|^2,\ I(t) \geqslant (1-\eta_*)\|u_x(t)\|^2. \tag{4.23}$$

Hence, Lemma 4.5 is proved by using some simple estimates. □

Now we continue to prove Theorem 4.1.

It follows from (4.1), (4.2), (4.9), and (4.21) that

$$\begin{aligned}L'(t) \leqslant &-\frac{\lambda}{2}\|u'(t)\|^2 - \lambda_1\|u_x'(t)\|^2 + \frac{1}{2\lambda}\|f(t)\|^2 \\ &+ \delta\|u'(t)\|^2 + \delta\|u_x'(t)\|^2 - \delta\Big[\frac{1-\eta_*}{2} + b_1 - \frac{\varepsilon_1}{2}\Big]\|u_x(t)\|^2 \\ &- \delta b_1\|u_x(t)\|_{L^p}^p - \frac{\delta}{2}I(t) + \frac{\delta}{2\varepsilon_1}\|f(t)\|^2 \\ = &-\Big(\frac{\lambda}{2} - \delta\Big)\|u'(t)\|^2 - (\lambda_1 - \delta)\|u_x'(t)\|^2 \\ &- \delta\Big[\frac{1-\eta_*}{2} + b_1 - \frac{\varepsilon_1}{2}\Big]\|u_x(t)\|^2 \\ &- \delta b_1\|u_x(t)\|_{L^p}^p - \frac{\delta}{2}I(t) + \frac{1}{2}\Big(\frac{1}{\lambda} + \frac{\delta}{\varepsilon_1}\Big)\|f(t)\|^2\end{aligned} \tag{4.24}$$

for all δ, $\varepsilon_1 > 0$, with $0 < \delta < 1/2 - 1/p$.

Let

$$0 < \varepsilon_1 < 1 - \eta_* + 2b_1. \tag{4.25}$$

Then for δ small enough, with $0 < \delta < \min\{\lambda/2, \lambda_1, 1/2 - 1/p\}$, we deduce from (4.18) and (4.24) that there exists a constant $\gamma > 0$ such that

$$L'(t) \leqslant -\gamma(t) + C\mathrm{e}^{-2\gamma_0 t} \quad \forall\, t \geqslant 0. \tag{4.26}$$

Combining (4.18) and (4.26), we get (4.7). Theorem 4.1 is proved completely. □

5. A remark

Consider problem (3.1) corresponding to $a = -a_1 < 0$ and $b = -b_1 < 0$:

$$(5.1)\qquad \begin{cases} u_{tt} - u_{xx} - u_{xxtt} - \lambda_1 u_{xxt} + \lambda u_t + a_1|u|^{p-2}u - b_1\dfrac{\partial}{\partial x}(|u_x|^{p-2}u_x) \\ \qquad = f(x,t), \quad 0 < x < 1,\ 0 < t < T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \widetilde{u}_0(x), \quad u_t(x,0) = \widetilde{u}_1(x). \end{cases}$$

With suitable conditions on f, we remark that problem (5.1) has a unique global solution $u(t)$ with energy decaying exponentially as $t \to \infty$, without the initial data $(\widetilde{u}_0, \widetilde{u}_1)$ being small enough.

Theorem 5.1. *Suppose that $f \in H^1(Q_T)$. Then problem* (5.1) *has a unique solution*

$$(5.2)\qquad u \in C^1([0,T_*]; H_0^1 \cap H^2), \quad u_{tt} \in L^\infty(0,T_*; H_0^1 \cap H^2),$$

for $T_* > 0$ small enough.

This is a special case of Theorem 2.2.

Theorem 5.2. *Assume that* $(\widetilde{\mathrm{H}}_1)$ *holds. Then there exist positive constants C, γ such that*

$$(5.3)\quad \|u'(t)\|^2 + \|u_x'(t)\|^2 + \|u_x(t)\|^2 + \|u(t)\|^p_{L^p} + \|u_x(t)\|^p_{L^p} \leqslant C\exp(-\gamma t) \quad \forall\, t \geqslant 0.$$

P r o o f. First, we construct the Lyapunov functional

$$(5.4)\qquad L_1(t) = \widetilde{E}_1(t) + \delta\psi(t),$$

where $\delta > 0$ will be chosen later and

$$(5.5)\ \widetilde{E}_1(t) = \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u_x'(t)\|^2 + \frac{1}{2}\|u_x(t)\|^2 + \frac{a_1}{p}\|u(t)\|^p_{L^p} + \frac{b_1}{p}\|u_x(t)\|^p_{L^p},$$

$$(5.6)\quad \psi(t) = \langle u'(t), u(t)\rangle + \langle u_x'(t), u_x(t)\rangle + \frac{\lambda}{2}\|u(t)\|^2 + \frac{\lambda_1}{2}\|u_x(t)\|^2.$$

Next, we need the following lemmas.

Lemma 5.3. *The energy functional $\widetilde{E}_1(t)$ satisfies*

$$\widetilde{E}_1'(t) \leqslant -\frac{\lambda}{2}\|u'(t)\|^2 - \lambda_1\|u_x'(t)\|^2 + \frac{1}{2\lambda}\|f(t)\|^2. \tag{5.7}$$

P r o o f of Lemma 5.3. Multiplying $(5.1)_1$ by $u'(x,t)$ and integrating over $[0,1]$, we get

$$\widetilde{E}_1'(t) = -\lambda\|u'(t)\|^2 - \lambda_1\|u_x'(t)\|^2 + \langle f(t), u'(t)\rangle. \tag{5.8}$$

We have

$$\langle f(t), u'(t)\rangle \leqslant \frac{\lambda}{2}\|u'(t)\|^2 + \frac{1}{2\lambda}\|f(t)\|^2. \tag{5.9}$$

Combining (5.8) and (5.9) gives (5.7). Lemma 5.3 is proved completely. □

By (5.7), we obtain

$$\widetilde{E}_1'(t) \leqslant -\frac{\lambda}{2}\|u'(t)\|^2 - \lambda_1\|u_x'(t)\|^2 + \frac{1}{2\lambda}\|f(t)\|^2 \leqslant \frac{1}{2\lambda}\|f(t)\|^2. \tag{5.10}$$

Integrating with respect to t, we get

$$\widetilde{E}_1(t) \leqslant \widetilde{E}_1(0) + \frac{1}{2\lambda}\int_0^\infty \|f(t)\|^2\,\mathrm{d}t = E_* \quad \forall\, t \geqslant 0. \tag{5.11}$$

Putting

$$\widetilde{E}_*(t) = \|u'(t)\|^2 + \|u_x'(t)\|^2 + \|u_x(t)\|^2 + \|u(t)\|_{L^p}^p + \|u_x(t)\|_{L^p}^p, \tag{5.12}$$

we have the following lemma.

Lemma 5.4. *There exist positive constants $\overline{\beta}_1$ and $\overline{\beta}_2$ such that*

$$\overline{\beta}_1\widetilde{E}_*(t) \leqslant L_1(t) \leqslant \overline{\beta}_2\widetilde{E}_*(t) \quad \forall\, t \geqslant 0, \tag{5.13}$$

for δ small enough.

P r o o f of Lemma 5.4. It is clear that

$$\begin{aligned} L_1(t) = {} & \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u_x'(t)\|^2 + \frac{1}{2}\|u_x(t)\|^2 + \frac{a_1}{p}\|u(t)\|_{L^p}^p + \frac{b_1}{p}\|u_x(t)\|_{L^p}^p \\ & + \delta\langle u'(t), u(t)\rangle + \delta\langle u_x'(t), u_x(t)\rangle + \frac{\delta\lambda}{2}\|u(t)\|^2 + \frac{\delta\lambda_1}{2}\|u_x(t)\|^2. \end{aligned} \tag{5.14}$$

From the inequalities

$$
(5.15)\qquad \begin{cases} \delta\langle u'(t),u(t)\rangle \leqslant \delta\|u'(t)\|\|u_x(t)\| \leqslant \dfrac{1}{2}\delta\|u'(t)\|^2 + \dfrac{1}{2}\delta\|u_x(t)\|^2, \\ \delta\langle u'_x(t),u_x(t)\rangle \leqslant \delta\|u'_x(t)\|\|u_x(t)\| \leqslant \dfrac{1}{2}\delta\|u'_x(t)\|^2 + \dfrac{1}{2}\delta\|u_x(t)\|^2, \\ \dfrac{\delta\lambda}{2}\|u(t)\|^2 \leqslant \dfrac{\delta\lambda}{2}\|u_x(t)\|^2, \end{cases}
$$

we deduce that

$$
\begin{aligned}
(5.16)\qquad L_1(t) &\geqslant \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u'_x(t)\|^2 + \frac{1}{2}\|u_x(t)\|^2 + \frac{a_1}{p}\|u(t)\|^p_{L^p} + \frac{b_1}{p}\|u_x(t)\|^p_{L^p} \\
&\quad + \delta\langle u'(t),u(t)\rangle + \delta\langle u'_x(t),u_x(t)\rangle \\
&\geqslant \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u'_x(t)\|^2 + \frac{1}{2}\|u_x(t)\|^2 + \frac{a_1}{p}\|u(t)\|^p_{L^p} + \frac{b_1}{p}\|u_x(t)\|^p_{L^p} \\
&\quad - \frac{1}{2}\delta\|u'(t)\|^2 - \frac{1}{2}\delta\|u_x(t)\|^2 - \frac{1}{2}\delta\|u'_x(t)\|^2 - \frac{1}{2}\delta\|u_x(t)\|^2 \\
&= \frac{1-\delta}{2}\|u'(t)\|^2 + \frac{1-\delta}{2}\|u'_x(t)\|^2 + \frac{a_1}{p}\|u(t)\|^p_{L^p} \\
&\quad + \Big(\frac{1-2\delta}{2} + \frac{b_1}{p}\Big)\|u_x(t)\|^p_{L^p} \\
&\geqslant \overline{\beta}_1\widetilde{E}_*(t),
\end{aligned}
$$

where we choose $\overline{\beta}_1 = \min\{(1-2\delta)/2, a_1/p\}$, δ small enough, $0<\delta<\frac{1}{2}$.

Similarly, we can prove that

$$
\begin{aligned}
(5.17)\qquad L_1(t) &\leqslant \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u'_x(t)\|^2 + \frac{1}{2}\|u_x(t)\|^2 + \frac{a_1}{p}\|u(t)\|^p_{L^p} + \frac{b_1}{p}\|u_x(t)\|^p_{L^p} \\
&\quad + \frac{1}{2}\delta\|u'(t)\|^2 + \frac{1}{2}\delta\|u_x(t)\|^2 + \frac{1}{2}\delta\|u'_x(t)\|^2 + \frac{1}{2}\delta\|u_x(t)\|^2 \\
&\quad + \frac{\delta\lambda}{2}\|u_x(t)\|^2 + \frac{\delta\lambda_1}{2}\|u_x(t)\|^2 \\
&= \frac{1+\delta}{2}\|u'(t)\|^2 + \frac{1+\delta}{2}\|u'_x(t)\|^2 + \frac{1}{2}[1+\delta(2+\lambda+\lambda_1)]\|u_x(t)\|^2 \\
&\quad + \frac{a_1}{p}\|u(t)\|^p_{L^p} + \frac{b_1}{p}\|u_x(t)\|^p_{L^p} \\
&\leqslant \frac{1+\delta(2+\lambda+\lambda_1)}{2}\widetilde{E}_*(t) = \overline{\beta}_2\widetilde{E}_*(t),
\end{aligned}
$$

where $\overline{\beta}_2 = \max\{(1+\delta(2+\lambda+\lambda_1))/2, a_1/p, b_1/p\}$.

Lemma 5.4 is proved completely. □

Lemma 5.5. *The functional* $\psi(t)$ *defined by* (5.6) *satisfies*

$$\psi'(t) \leqslant \|u'(t)\|^2+\|u'_x(t)\|^2-\frac{1}{2}\|u_x(t)\|^2-a_1\|u(t)\|^p_{L^p}-b_1\|u_x(t)\|^p_{L^p}+\frac{1}{2}\|f(t)\|^2. \tag{5.18}$$

P r o o f of Lemma 5.5. Multiplying $(5.1)_1$ by $u(x,t)$ and integrating over $[0,1]$, we obtain

$$\psi'(t) = \|u'(t)\|^2+\|u'_x(t)\|^2-\|u_x(t)\|^2-a_1\|u(t)\|^p_{L^p}-b_1\|u_x(t)\|^p_{L^p}+\langle f(t),u(t)\rangle. \tag{5.19}$$

Note that

$$\langle f(t),u(t)\rangle \leqslant \|f(t)\|\|u_x(t)\| \leqslant \frac{1}{2}\|u_x(t)\|^2+\frac{1}{2}\|f(t)\|^2. \tag{5.20}$$

Combining (5.19) and (5.20) leads to (5.18). Lemma 5.5 is proved completely. □

Now we continue to prove Theorem 5.2.

It follows from (5.4), (5.7), and (5.18) that

$$\begin{aligned} L_1'(t) \leqslant & -\frac{\lambda}{2}\|u'(t)\|^2-\lambda_1\|u'_x(t)\|^2+\frac{1}{2\lambda}\|f(t)\|^2 \\ & +\delta\|u'(t)\|^2+\delta\|u'_x(t)\|^2-\frac{\delta}{2}\|u_x(t)\|^2 \\ & -\delta a_1\|u(t)\|^p_{L^p}-\delta b_1\|u_x(t)\|^p_{L^p}+\frac{\delta}{2}\|f(t)\|^2 \\ = & -\Big(\frac{\lambda}{2}-\delta\Big)\|u'(t)\|^2-(\lambda_1-\delta)\|u'_x(t)\|^2 \\ & -\frac{\delta}{2}\|u_x(t)\|^2-\delta a_1\|u(t)\|^p_{L^p}-\delta b_1\|u_x(t)\|^p_{L^p}+\frac{1}{2}\Big(\frac{1}{\lambda}+\delta\Big)\|f(t)\|^2 \\ \leqslant & -\Big(\frac{\lambda}{2}-\delta\Big)\|u'(t)\|^2-(\lambda_1-\delta)\|u'_x(t)\|^2 \\ & -\frac{\delta}{2}\|u_x(t)\|^2-\delta a_1\|u(t)\|^p_{L^p}-\delta b_1\|u_x(t)\|^p_{L^p}+C_1\mathrm{e}^{-2\gamma_0 t}. \end{aligned} \tag{5.21}$$

Choosing $0<\delta<\min\{1/2,\lambda/2,\lambda_1\}$, we deduce from (5.21) that

$$\begin{aligned} L_1'(t) \leqslant & -\beta_*[\|u'(t)\|^2+\|u'_x(t)\|^2+\|u_x(t)\|^2+\|u(t)\|^p_{L^p}+\|u_x(t)\|^p_{L^p}] \\ & +C_1\mathrm{e}^{-2\gamma_0 t} \\ = & -\beta_*\widetilde{E}_*(t)+C_1\mathrm{e}^{-2\gamma_0 t} \\ \leqslant & -\frac{\beta_*}{\overline{\beta}_2}L_1(t)+C_1\mathrm{e}^{-2\gamma_0 t} \leqslant -\gamma L_1(t)+C_1\mathrm{e}^{-2\gamma_0 t}, \end{aligned} \tag{5.22}$$

where $\beta_*=\min\{\lambda/2-\delta,\lambda_1-\delta,\delta/2,\delta a_1,\delta b_1\}$, $0<\gamma<\min\{\beta_*/\overline{\beta}_2,2\gamma_0\}$.

Combining (5.12), (5.13), and (5.22), we get (5.3). Theorem 5.2 is proved completely. □

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