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Partial Fuzzy Metric Space and Some Fixed Point Results

Shaban Sedghi, Nabi Shobkolaei, Ishak Altun

Abstract. In this paper, we introduce the concept of partial fuzzy metric on a nonempty set X and give the topological structure and some properties of partial fuzzy metric space. Then some fixed point results are provided.

1 Introduction and preliminaries

We recall some basic definitions and results from the theory of fuzzy metric spaces, used in the sequel.

Definition 1. [5] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a *continuous t-norm* if it satisfies the following conditions:

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a * 1 = a$ for all $a \in [0,1]$,
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norms are $a*b = ab$ and $a*b = \min\{a, b\}$.

Definition 2. [1] A triple $(X, M, *)$ is called a *fuzzy metric space* (in the sense of George and Veeramani) if X is a nonempty set, $*$ is a continuous t-norm and $M : X^2 \times (0, \infty) \to [0,1]$ is a fuzzy set satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$,

1. $M(x, y, t) > 0$,
2. $M(x, y, t) = 1 \Leftrightarrow x = y$,

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3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$,
5. $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is a continuous mapping

If the fourth condition is replaced by

4′. $M(x, z, \max\{t, s\}) \geq M(x, y, t) * M(y, z, s)$,

then the space $(X, M, *)$ is said to be a *non-Archimedean fuzzy metric space*. It should be noted that any non-Archimedean fuzzy metric space is a fuzzy metric space.

The following properties of M noted in the theorem below are easy consequences of the definition.

Theorem 1. *Let $(X, M, *)$ be a fuzzy metric space.*

1. *$M(x, y, t)$ is nondecreasing with respect to t for each $x, y \in X$,*
2. *If M is non-Archimedean, then $M(x, y, t) \geq M(x, z, t) * M(z, y, t)$ for all $x, y, z \in X$ and $t > 0$.*

Example 1. Let (X, d) be an ordinary metric space and $a*b = ab$ for all $a, b \in [0, 1]$. Then the fuzzy set M on $X^2 \times (0, \infty)$ defined by

$$M(x, y, t) = \exp\left(-\frac{d(x, y)}{t}\right),$$

is a fuzzy metric on X.

Example 2. Let $a * b = ab$ for all $a, b \in [0, 1]$ and M be the fuzzy set on $\mathbb{R}^+ \times \mathbb{R}^+ \times (0, \infty)$ (where $\mathbb{R}^+ = (0, \infty)$) defined by

$$M(x, y, t) = \frac{\min\{x, y\}}{\max\{x, y\}},$$

for all $x, y \in \mathbb{R}^+$. Then $(\mathbb{R}^+, M, *)$ is a fuzzy metric space.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with centre $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the fuzzy metric M). A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$. This definition of Cauchy sequence is identical with that given by George and Veeramani.

The fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent.

The fixed point theory in fuzzy metric spaces started with the paper of Grabiec [2]. Later on, the concept of fuzzy contractive mappings, initiated by Gregori and Sapena in [3], have become of interest for many authors, see, e.g., the papers [3], [7], [8], [9], [10], [11].

In our paper we present the concept of partial fuzzy metric space and some properties of it. Then we give some fundamental fixed point theorem on complete partial fuzzy metric space.

2 Partial fuzzy metric space

In this section we introduce the concept of partial fuzzy metric space and give its properties.

Definition 3. A *partial fuzzy metric* on a nonempty set X is a function

$$P_M : X \times X \times (0, \infty) \to [0, 1]$$

such that for all $x, y, z \in X$ and $t, s > 0$

(PM1) $x = y \Leftrightarrow P_M(x, x, t) = P_M(x, y, t) = P_M(y, y, t)$,

(PM2) $P_M(x, x, t) \geq P_M(x, y, t)$,

(PM3) $P_M(x, y, t) = P_M(y, x, t)$,

(PM4) $P_M(x, y, \max\{t, s\}) * P_M(z, z, \max\{t, s\}) \geq P_M(x, z, t) * P_M(z, y, s)$.

(PM5) $P_M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

A partial fuzzy metric space is a 3-tuple $(X, P_M, *)$ such that X is a nonempty set and P_M is a partial fuzzy metric on X. It is clear that, if $P_M(x, y, t) = 1$, then from (PM1) and (PM2) $x = y$. But if $x = y$, $P_M(x, y, t)$ may not be 1. A basic example of a partial fuzzy metric space is the 3-tuple $(\mathbb{R}^+, P_M, *)$, where

$$P_M(x, y, t) = \frac{t}{t + \max\{x, y\}}$$

for all $t > 0$, $x, y \in \mathbb{R}^+$ and $a * b = ab$.

From (PM4) for all $x, y, z \in X$ and $t > 0$, we have:

$$P_M(x, y, t) * P_M(z, z, t) \geq P_M(x, z, t) * P_M(z, y, t).$$

Let $(X, M, *)$ and $(X, P_M, *)$ be a fuzzy metric space and partial fuzzy metric space, respectively. Then mappings $P_{M_i} : X \times X \times (0, \infty) \to [0, 1]$ $(i \in \{1, 2\})$ defined by

$$P_{M_1}(x, y, t) = M(x, y, t) * P_M(x, y, t)$$

and

$$P_{M_2}(x, y, t) = M(x, y, t) * a$$

are partial fuzzy metrics on X, where $0 < a < 1$.

Theorem 2. *The partial fuzzy metric $P_M(x,y,t)$ is nondecreasing with respect to t for each $x,y \in X$ and $t>0$, if the continuous t-norm $*$ satisfies the following condition for all $a,b,c \in [0,1]$*

$$a*b \geq a*c \Rightarrow b \geq c.$$

Proof. From (PM4) for all $x,y,z \in X$ and $t,s>0$, we have:

$$P_M(x,y,\max\{t,s\}) * P_M(z,z,\max\{t,s\}) \geq P_M(x,z,s) * P_M(z,y,t).$$

Let $t>s$, then taking $z=y$ in above inequality we have

$$P_M(x,y,t) * P_M(y,y,t) \geq P_M(x,y,s) * P_M(y,y,t),$$

hence by assume we get $P_M(x,y,t) \geq P_M(x,y,s)$. □

It is easy to see that every fuzzy metric is a partial fuzzy metric, but the converse may not be true. In the following examples, the partial fuzzy metrics fails to satisfy properties of fuzzy metric.

Example 3. Let (X,p) is a partial metric space in the sense of Matthews [6] and $P_M : X \times X \times (0,\infty) \to [0,1]$ be a mapping defined as

$$P_M(x,y,t) = \frac{t}{t+p(x,y)},$$

or

$$P_M(x,y,t) = \exp\left(-\frac{p(x,y)}{t}\right).$$

If $a*b=ab$ for all $a,b \in [0,1]$, then clearly P_M is a partial fuzzy metric, but it may not be a fuzzy metric.

Lemma 1. *Let $(X,P_M,*)$ be a partial fuzzy metric space with $a*b=ab$ for all $a,b \in [0,1]$. If we define $p : X^2 \to [0,\infty)$ by*

$$p(x,y) = \sup_{\alpha\in(0,1)} \int_{\alpha}^{1} \log_a(P_M(x,y,t))\,\mathrm{d}t,$$

then p is a partial metric on X for fixed $0<a<1$.

Proof. It is clear from the definition that $p(x,y)$ is well defined for each $x,y \in X$ and $p(x,y) \geq 0$ for all $x,y \in X$.

1. For all $t>0$

$$p(x,x)=p(x,y)=p(y,y) \Leftrightarrow P_M(x,x,t)=P_M(x,y,t)=P_M(y,y,t) \Leftrightarrow x=y.$$

2.
$$\begin{aligned} p(x,x) &= \sup_{\alpha\in(0,1)} \int_{\alpha}^{1} \log_a(P_M(x,x,t))\,\mathrm{d}t \\ &\leq \sup_{\alpha\in(0,1)} \int_{\alpha}^{1} \log_a(P_M(x,y,t))\,\mathrm{d}t \\ &= p(x,y). \end{aligned}$$

3.
$$\begin{aligned} p(x,y) &= \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a(P_M(x,y,t))\,\mathrm{d}t \\ &= \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a(P_M(y,x,t))\,\mathrm{d}t \\ &= p(y,x). \end{aligned}$$

4. Since
$$P_M(x,y,t)P_M(z,z,t) \geq P_M(x,z,t)P_M(z,y,t),$$
and $\log_a$ is decreasing, it follows that
$$\log_a(P_M(x,y,t)) + \log_a(P_M(z,z,t)) \leq \log_a(P_M(x,z,t)) + \log_a(P_M(z,y,t)),$$
hence
$$\begin{aligned} p(x,y)+p(z,z) &= \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a(P_M(x,y,t))\,\mathrm{d}t + \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a(P_M(z,z,t))\,\mathrm{d}t \\ &\leq \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a(P_M(x,z,t))\,\mathrm{d}t + \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a(P_M(z,y,t))\,\mathrm{d}t \\ &= p(x,z)+p(z,y). \end{aligned}$$

This proves that p is a partial metric on X. □

Definition 4. Let $(X, P_M, *)$ be a partial fuzzy metric space.

1. A sequence $\{x_n\}$ in a partial fuzzy metric space $(X, P_M, *)$ *converges* to x if and only if $P_M(x,x,t) = \lim_{n\to\infty} P_M(x_n,x,t)$ for every $t>0$.
2. A sequence $\{x_n\}$ in a partial fuzzy metric space $(X, P_M, *)$ is called a *Cauchy sequence* if $\lim_{n,m\to\infty} P_M(x_n,x_m,t)$ exists.
3. A partial fuzzy metric space $(X, P_M, *)$ is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges to a point $x\in X$.

Suppose that $\{x_n\}$ is a sequence in partial fuzzy metric space $(X, P_M, *)$, then we define $L(x_n) = \{x \in X : x_n \to x\}$. In the following example shows that every convergent sequence $\{x_n\}$ in a partial fuzzy metric space $(X, P_M, *)$ fails to satisfy Cauchy sequence. In particular, it shows that the limit of a convergent sequence is not unique.

Example 4. Let $X = [0,\infty)$ and $P_M(x,y,t) = \frac{t}{t+\max\{x,y\}}$, then it is clear that $(X, P_M, *)$ is a partial fuzzy metric space where $a*b = ab$ for all $a,b\in[0,1]$. Let $\{x_n\} = \{1,2,1,2,\dots\}$. Then clearly it is convergent sequence and for every $x\geq 2$ we have
$$\lim_{n\to\infty} P_M(x_n,x,t) = P_M(x,x,t),$$
therefore
$$L(x_n) = \{x\in X : x_n \to x\} = [2,\infty).$$
but $\lim_{n,m\to\infty} P_M(x_n,x_m,t)$ is not exist, that is, $\{x_n\}$ is not Cauchy sequence.

The following Lemma shows that under certain conditions the limit of a convergent sequence is unique.

Lemma 2. *Let $\{x_n\}$ be a convergent sequence in partial fuzzy metric space $(X, P_M, *)$ such that $a * b \geq a * c \Rightarrow b \geq c$ for all $a, b, c \in [0,1]$, $x_n \to x$ and $x_n \to y$. If*

$$\lim_{n\to\infty} P_M(x_n, x_n, t) = P_M(x, x, t) = P_M(y, y, t),$$

then $x = y$.

Proof. As

$$P_M(x, y, t) * P_M(x_n, x_n, t) \geq P_M(x, x_n, t) * P_M(y, x_n, t),$$

taking limit as $n \to \infty$, we have

$$P_M(x, y, t) * P_M(x, x, t) \geq P_M(x, x, t) * P_M(y, y, t).$$

By given assumptions and from (PM2), we have

$$P_M(y, y, t) \geq P_M(x, y, t) \geq P_M(y, y, t),$$

which shows that $P_M(x, y, t) = P_M(y, y, t) = P_M(x, x, t)$, therefore $x = y$. $\square$

Lemma 3. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in partial fuzzy metric space $(X, P_M, *)$ such that $a * b \geq a * c \Rightarrow b \geq c$ for all $a, b, c \in [0,1]$,*

$$\lim_{n\to\infty} P_M(x_n, x, t) = \lim_{n\to\infty} P_M(x_n, x_n, t) = P_M(x, x, t),$$

and

$$\lim_{n\to\infty} P_M(y_n, y, t) = \lim_{n\to\infty} P_M(y_n, y_n, t) = P_M(y, y, t),$$

then $\lim_{n\to\infty} P_M(x_n, y_n, t) = P_M(x, y, t)$. In particular, for every $z \in X$

$$\lim_{n\to\infty} P_M(x_n, z, t) = \lim_{n\to\infty} P_M(x, z, t).$$

Proof. As

$$P_M(x_n, y_n, t) * P_M(x, x, t) \geq P_M(x_n, x, t) * P_M(x, y_n, t),$$

therefore

$$\begin{aligned} P_M(x_n, y_n, t) * P_M(x, x, t) * P_M(y, y, t) &\geq P_M(x_n, x, t) * P_M(x, y_n, t) * P_M(y, y, t) \\ &\geq P_M(x_n, x, t) * P_M(x, y, t) * P_M(y, y_n, t). \end{aligned}$$

Thus

$$\begin{aligned} &\limsup_{n\to\infty} P_M(x_n, y_n, t) * P_M(x, x, t) * P_M(y, y, t) \\ &\qquad \geq \limsup_{n\to\infty} P_M(x_n, x, t) * P_M(x, y, t) * \limsup_{n\to\infty} P_M(y, y_n, t) \\ &\qquad = P_M(x, x, t) * P_M(x, y, t) * P_M(y, y, t), \end{aligned}$$

hence

$$\limsup_{n\to\infty} P_M(x_n, y_n, t) \geq P_M(x, y, t).$$

Also, as

$$P_M(x, y, t) * P_M(x_n, x_n, t) \geq P_M(x, x_n, t) * P_M(x_n, y, t),$$

therefore

$$\begin{aligned} P_M(x, y, t) * P_M(x_n, x_n, t) * P_M(y_n, y_n, t) \\ &\geq P_M(x, x_n, t) * P_M(x_n, y, t) * P_M(y_n, y_n, t) \\ &\geq P_M(x, x_n, t) * P_M(x_n, y_n, t) * P_M(y_n, y, t) \end{aligned}$$

Thus

$$\begin{aligned} P_M(x, y, t) &* P_M(x, x, t) * P_M(y, y, t) \\ &= P_M(x, y, t) * \limsup_{n\to\infty} P_M(x_n, x_n, t) * \limsup_{n\to\infty} P_M(y_n, y_n, t) \\ &\geq \limsup_{n\to\infty} P_M(x, x_n, t) * \limsup_{n\to\infty} P_M(x_n, y_n, t) * \limsup_{n\to\infty} P_M(y_n, y, t) \\ &= P_M(x, x, t) * \limsup_{n\to\infty} P_M(x_n, y_n, t) * P_M(y, y, t). \end{aligned}$$

Therefore

$$P_M(x, y, t) \geq \limsup_{n\to\infty} P_M(x_n, y_n, t).$$

That is,

$$\limsup_{n\to\infty} P_M(x_n, y_n, t) = P_M(x, y, t).$$

Similarly, we have

$$\limsup_{n\to\infty} P_M(x_n, y_n, t) = P_M(x, y, t).$$

Hence the result follows. □

Definition 5. Let $(X, P_M, *)$ be a partial fuzzy metric space. P_M is said to be upper semicontinuous on X if for every $x \in X$,

$$P_M(p, x, t) \geq \limsup_{n\to\infty} P_M(x_n, x, t),$$

whenever $\{x_n\}$ is a sequence in X which converges to a point $p \in X$.

3 Fixed point results

Let $(X, P_M, *)$ be a partial fuzzy metric space and $\emptyset \neq S \subseteq X$. Define

$$\delta_{P_M}(S, t) = \inf\{P_M(x, y, t) : x, y \in S\}$$

for all $t > 0$. For an $A_n = \{x_n, x_{n+1}, \dots\}$ in partial fuzzy metric space $(X, P_M, *)$, let $r_n(t) = \delta_{P_M}(A_n, t)$. Then $r_n(t)$ is finite for all $n \in \mathbb{N}$, $\{r_n(t)\}$ is nonincreasing, $r_n(t) \to r(t)$ for some $0 \leq r(t) \leq 1$ and also $r_n(t) \leq P_M(x_l, x_k, t)$ for all $l, k \geq n$.

Let $\mathcal{F}$ be the set of all continuous functions $F : [0,1]^3 \times [0,1] \to [-1,1]$ such that F is nondecreasing on $[0,1]^3$ satisfying the following condition:

- $F((u,u,u),v) \leq 0$ implies that $v \geq \gamma(u)$ where $\gamma : [0,1] \to [0,1]$ is a nondecreasing continuous function with $\gamma(s) > s$ for $s \in [0,1)$.

Example 5. Let $\gamma(s) = s^h$ for $0 < h < 1$, then the functions F defined by

$$F\big((t_1,t_2,t_3),t_4\big) = \gamma\big(\min\{t_1,t_2,t_3\}\big) - t_4$$

and

$$F\big((t_1,t_2,t_3),t_4\big) = \gamma\left(\sum_{i=1}^{3} a_i t_i\right) - t_4,$$

where $a_i \geq 0$, $\sum\limits_{i=1}^{3} a_i = 1$, belong to $\mathcal{F}$.

Now we give our main theorem.

Theorem 3. *Let $(X, P_M, *)$ be a complete bounded partial fuzzy metric space, P_M is upper semicontinuous function on X and T be a self map of X satisfying*

$$F\big(P_M(x,y,t), P_M(Tx,x,t), P_M(Tx,y,t), P_M(Tx,Ty,t)\big) \leq 0 \tag{1}$$

for all $x,y \in X$, where $F \in \mathcal{F}$. Then T has a unique fixed point p in X and T is continuous at p.

Proof. Let $x_0 \in X$ and $Tx_n = x_{n+1}$. Let $r_n(t) = \delta_{P_M}(A_n,t)$, where $A_n = \{x_n, x_{n+1}, \dots\}$. Then we know $\lim\limits_{n\to\infty} r_n(t) = r(t)$ for some $0 \leq r(t) \leq 1$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then T has a fixed point. Assume that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}$. Let $k \in \mathbb{N}$ be fixed. Taking $x = x_{n-1}$, $y = x_{n+m-1}$ in (1) where $n \geq k$ and $m \in \mathbb{N}$, we have

$$\begin{aligned} &F\begin{pmatrix} P_M(x_{n-1}, x_{n+m-1}, t), P_M(Tx_{n-1}, x_{n-1}, t), \\ P_M(Tx_{n-1}, x_{n+m-1}, t), P_M(Tx_{n-1}, Tx_{n+m-1}, t) \end{pmatrix} \\ &\qquad\qquad = F\begin{pmatrix} P_M(x_{n-1}, x_{n+m-1}, t), P_M(x_n, x_{n-1}, t), \\ P_M(x_n, x_{n+m-1}, t), P_M(x_n, x_{n+m}, t) \end{pmatrix} \leq 0 \end{aligned}$$

Thus we have

$$F\big(r_{n-1}(t), r_{n-1}(t), r_n(t), P_M(x_n, x_{n+m}, t)\big) \leq 0,$$

since F is nondecreasing on $[0,1]^3$. Also, since $r_n(t)$ is nonincreasing, we have

$$F\big(r_{k-1}(t), r_{k-1}(t), r_{k-1}(t), P_M(x_n, x_{n+m}, t)\big) \leq 0,$$

which implies that

$$P_M(x_n, x_{n+m}, t) \geq \gamma(r_{k-1}(t)).$$

Thus for all $n \geq k$, we have

$$\inf_{n\geq k}\big\{P_M(x_n, x_{n+m}, t)\big\} = r_k(t) \geq \gamma(r_{k-1}(t)).$$

Letting $k \to \infty$, we get $r(t) \geq \gamma(r(t))$. If $r(t) \neq 1$, then $r(t) \geq \gamma(r(t)) > r(t)$, which is a contradiction. Thus $r(t) = 1$ and hence $\lim_{n\to\infty} \gamma_n(t) = 1$. Thus given $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $r_n(t) > 1 - \varepsilon$. Then we have for $n \geq n_0$ and $m \in \mathbb{N}$, $P_M(x_n, x_{n+m}, t) > 1 - \varepsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists a $p \in X$ such that

$$\lim_{n\to\infty} P_M(x_n, p, t) = P_M(p, p, t).$$

Taking $x = x_n, y = p$ in (1), we have

$$\begin{aligned}F\big(&P_M(x_n, p, t), P_M(Tx_n, p, t), P_M(Tx_n, x_n, t), P_M(Tx_n, Tp, t)\big)\\ &= F\big(P_M(x_n, p, t), P_M(x_{n+1}, p, t), P_M(x_{n+1}, x_n, t), P_M(x_{n+1}, Tp, t)\big) \leq 0.\end{aligned}$$

Hence, we have

$$\begin{aligned}\limsup_{n\to\infty} &F\big(P_M(x_n, p, t), P_M(x_{n+1}, p, t), P_M(x_{n+1}, x_n, t), P_M(x_{n+1}, Tp, t)\big)\\ &= F\big(P_M(p, p, t), P_M(p, p, t), 1, \limsup_{n\to\infty} P_M(x_{n+1}, Tp, t)\big) \leq 0.\end{aligned}$$

Since

$$\begin{aligned}F\big(&P_M(p, p, t), P_M(p, p, t), P_M(p, p, t), \limsup_{n\to\infty} P_M(x_{n+1}, Tp, t)\big)\\ &\leq F\big(P_M(p, p, t), P_M(p, p, t), 1, \limsup_{n\to\infty} P_M(x_{n+1}, Tp, t)\big) \leq 0,\end{aligned}$$

which implies

$$P_M(p, Tp, t) \geq \limsup_{n\to\infty} P_M(x_{n+1}, Tp, t) \geq \gamma(P_M(p, p, t)).$$

On the other hand, we have

$$P_M(p, p, t) \geq P_M(p, Tp, t) \geq \gamma(P_M(p, p, t)).$$

Hence $P_M(p, p, t) = 1$. Also, since

$$P_M(p, Tp, t) \geq \gamma(P_M(p, p, t)) = \gamma(1) = 1,$$

this implies that $P_M(p, Tp, t) = 1$, therefore, we get $Tp = p$.

For the uniqueness, let p and w be fixed points of T. Taking $x = p, y = w$ in (1), we have

$$\begin{aligned}F\big(&P_M(p, w, t), P_M(Tp, p, t), P_M(Tp, w, t), P_M(Tp, Tw, t)\big)\\ &= F\big(P_M(p, w, t), P_M(p, p, t), P_M(p, w, t), P_M(p, w, t)\big) \leq 0.\end{aligned}$$

Since F is nondecreasing on $[0,1]^3$, we have

$$F\big(P_M(p,w,t),P_M(p,w,t),P_M(p,w,t),P_M(p,w,t)\big)\le 0,$$

which implies

$$P_M(p,w,t)\ge\gamma(P_M(p,w,t))>P_M(p,w,t)$$

which is a contradiction. Thus we have $P_M(p,w,t)=1$, therefore, $p=w$. Now, we show that T is continuous at p. Let $\{y_n\}$ be a sequence in X and $\lim_{n\to\infty} y_n=p$. Taking $x=p, y=y_n$ in (1), we have

$$\begin{aligned}F\big(&P_M(p,y_n,t),P_M(Tp,p,t),P_M(Tp,y_n,t),P_M(Tp,Ty_n,t)\big)\\&=F\big(P_M(p,y_n,t),P_M(p,p,t),P_M(p,y_n,t),P_M(p,Ty_n,t)\big)\le 0,\end{aligned}$$

hence

$$\begin{aligned}F\big(&P_M(p,p,t),P_M(p,p,t),P_M(p,p,t),\limsup_{n\to\infty}P_M(p,Ty_n,t)\big)\\&=F\begin{pmatrix}\limsup_{n\to\infty}P_M(p,y_n,t),\limsup_{n\to\infty}P_M(p,p,t),\\ \limsup_{n\to\infty}P_M(p,y_n,t),\limsup_{n\to\infty}P_M(p,Ty_n,t)\end{pmatrix}\le 0,\end{aligned}$$

which implies

$$\limsup_{n\to\infty}P_M(p,Ty_n,t))\ge\gamma(P_M(p,p,t))=\gamma(1)=1.$$

Thus,

$$\limsup_{n\to\infty}P_M(p,Ty_n,t)=1.$$

Similarly, taking limit inf, we have

$$\limsup_{n\to\infty}P_M(p,Ty_n,t)=1.$$

Therefore, $\limsup_{n\to\infty}P_M(Ty_n,p,t)=1$, this implies that

$$\limsup_{n\to\infty}P_M(Ty_n,Tp,t)=1=P_M(p,p,t)=P_M(Tp,Tp,t).$$

Thus $\lim_{n\to\infty}Ty_n=p=Tp$. Hence T is continuous at p. □

Corollary 1. *Let $(X,P_M,*)$ be a complete bounded partial fuzzy metric space, $m\in\mathbb{N}$ and T be a self map of X satisfying for all $x,y\in X$,*

$$F\big(P_M(x,y,t),P_M(T^mx,x,t),P_M(T^mx,y,t),P_M(T^mx,T^my,t)\big)\le 0$$

where $F\in\mathcal{F}$. Then T has a unique fixed point p in X and T^m is continuous at p.

Proof. From Theorem 3, T^m has a unique fixed point p in X and T^m is continuous at p. Since $Tp=TT^mp=T^mTp$, Tp is also a fixed point of T^m, By the uniqueness it follows $Tp=p$. □

In Theorem 3, if we take $F((t_1,t_2,t_3),t_4) = \gamma(\min\{t_1,t_2,t_3\}) - t_4$ then we have the next result.

Corollary 2. *Let $(X, P_M, *)$ be a complete bounded partial fuzzy metric space and T be a self map of X satisfying for all $x, y \in X$,*

$$P_M(Tx,Ty,t) \geq \gamma\big(\min\big\{P_M(x,y,t), P_M(Tx,x,t), P_M(Tx,y,t)\big\}\big).$$

Then T has a unique fixed point p in X and T is continuous at p.

Example 6. Let $X = \mathbb{R}^+$. Define $P_M : X^2 \times [0,\infty) \to [0,1]$ by

$$P_M(x,y,t) = \exp\bigg(-\frac{\max\{x,y\}}{t}\bigg)$$

for all $x, y \in X$ and $t > 0$. Then $(X, P_M, *)$ is a complete partial fuzzy metric space where $a * b = ab$. Define map $T : X \to X$ by $Tx = \frac{x}{2}$ for $x \in X$ and let $\gamma : [0,1] \to [0,1]$ defined by $\gamma(s) = s^{\frac{1}{2}}$. It is easy to see that

$$\begin{aligned} P_M(Tx,Ty,t) &= \exp\bigg(-\frac{\max\{\frac{x}{2},\frac{y}{2}\}}{t}\bigg) \\ &= \sqrt{\exp\bigg(-\frac{\max\{x,y\}}{t}\bigg)} \\ &= \sqrt{P_M(x,y,t)} \\ &\geq \sqrt{\min\big\{P_M(x,y,t), P_M(Tx,x,t), P_M(Tx,y,t)\big\}}. \end{aligned}$$

Thus T satisfy all the hypotheses of Corollary 2 and hence T has a unique fixed point.

Corollary 3. *Let $(X, P_M, *)$ be a complete bounded partial fuzzy metric space, $m \in \mathbb{N}$ and T be a self map of X satisfying for all $x, y \in X$,*

$$P_M(T^m x, T^m y, t) \geq \gamma\big(\min\big\{P_M(x,y,t), P_M(T^m x,x,t), P_M(T^m x,y,t)\big\}\big).$$

Then T has a unique fixed point p in X and and T^m is continuous at p.

Corollary 4. *Let $(X, P_M, *)$ be a complete bounded partial fuzzy metric space and T be a self map of X satisfying for all $x, y \in X$,*

$$P_M(Tx,Ty,t) \geq \sqrt{a_1 P_M(x,y,t) + a_2 P_M(Tx,x,t) + a_3 P_M(Tx,y,t)},$$

such that for every $a_i \geq 0$, $\sum\limits_{i=1}^{3} a_i = 1$. Then T has a unique fixed point p in X and T is continuous at p.

Corollary 5. *Let $(X, M, *)$ be a complete bounded fuzzy metric space and T be a self map of X satisfying for all $x, y \in X$ the*

$$F\big(M(x,y,t), M(Tx,x,t), M(Tx,y,t), M(Tx,Ty,t)\big) \leq 0$$

where $F \in \mathcal{F}$. Then T has a unique fixed point p in X and T is continuous at p.

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