Kaveh Eftekharinasab
The Morse-Sard-Brown Theorem for Functionals on Bounded Fréchet-Finsler Manifolds

Communications in Mathematics, Vol. 23 (2015), No. 2, 101–112

Persistent URL: `http://dml.cz/dmlcz/144798`

Communications in Mathematics 23 (2015) 101–112

The Morse-Sard-Brown Theorem for Functionals on Bounded Fréchet-Finsler Manifolds

Kaveh Eftekharinasab

Abstract. In this paper we study Lipschitz-Fredholm vector fields on bounded Fréchet-Finsler manifolds. In this context we generalize the Morse-Sard-Brown theorem, asserting that if M is a connected smooth bounded Fréchet-Finsler manifold endowed with a connection $\mathcal{K}$ and if ξ is a smooth Lipschitz-Fredholm vector field on M with respect to $\mathcal{K}$ which satisfies condition (WCV), then, for any smooth functional l on M which is associated to ξ, the set of the critical values of l is of first category in $\mathbb{R}$. Therefore, the set of the regular values of l is a residual Baire subset of $\mathbb{R}$.

1 Introduction

The notion of a Fredholm vector field on a Banach manifold B with respect to a connection on B was introduced by Tromba [13]. Such vector fields arise naturally in non-linear analysis from variational problems. There are geometrical objects such as harmonic maps, geodesics and minimal surfaces which arise as the zeros of a Fredholm vector field. Therefore, it would be valuable to study the critical points of functionals which are associated to Fredholm vector fields. In [12], Tromba proved the Morse-Sard-Brown theorem for this type of functionals in the case of Banach manifolds. Such a theorem would have applications to problems in the calculus of variations in the large such as Morse theory [11] and index theory [13].

The purpose of this paper is to extend the theorem of Tromba [12, Theorem 1 (MSB)] to a new class of generalized Fréchet manifolds, the class of the so-called bounded Fréchet manifolds, which was introduced in [8]. Such spaces arise in geometry and physical field theory and have many desirable properties. For instance, the space of all smooth sections of a fibre bundle (over closed or noncompact manifolds), which is the foremost example of infinite dimensional manifolds, has the structure of a bounded Fréchet manifold, see [8, Theorem 3.34]. The idea to introduce this category of manifolds was to overcome some permanent difficulties (i.e.,

2010 MSC: 58K05, 58B20, 58B15
Key words: Fréchet manifolds, condition (CV), Finsler structures, Fredholm vector fields

problems of intrinsic nature) in the theory of Fréchet spaces. For example, the lack of a non-trivial topological group structure on the general linear group of a Fréchet space. As for the importance of bounded Fréchet manifolds, we refer to [3], [4] and [8].

Essentially, to define the index of Fredholm vector fields we need the stability of Fredholm operators under small perturbations, but this is unobtainable in the case of proper Fréchet spaces (non-normable spaces) in general, see [3]. Also, we need a subtle notion of a connection via a connection map, but (because of the aforementioned problem) such a connection can not be constructed for Fréchet manifolds in general (cf. [2]). However, in the case of bounded Fréchet manifolds under the global Lipschitz assumption on Fredholm operators, the stability of Lipschitz--Fredholm operators was established in [3]. In addition, the notion of a connection via a connection map was defined in [4]. By using these results, we introduce the notion of a Lipschitz-Fredholm vector field in Section 3. With regard to a kind of compactness assumption (condition (WCV)), which one needs to impose on vector fields, we will be interested in manifolds which admit a Finsler structure. We then define Finsler structures for bounded Fréchet manifolds in Section 4. Finally, after we explained all subsequent portions for proving the Morse-Sard-Brown theorem, we formulate the theorem in the setting of Finsler manifolds in Section 5. A key point in the proof of the theorem is Proposition 2 which in its simplest form says that a Lipschitz-Fredholm vector field ξ near *origin* locally has a representation of the form $\xi(u,v) = (u, \eta(u,v))$, where η is a smooth map. Indeed, this is a consequence of the inverse function theorem (Theorem 3). One of the most important advantage of the category of bounded Fréchet manifold is the availability of the inverse function theorem of Nash and Moser (see [8]).

Morse theory and index theories for Fréchet manifolds have not been developed. Nevertheless, our approach provides some essential tools (such as connection maps, covariant derivatives, Finsler structures) which would create a proper framework for these theories.

2 Preliminaries

In this section we summarize all the necessary preliminary material that we need for a self-contained presentation of the paper. We shall work in the category of smooth manifolds and bundles. We refer to [4] for the basic geometry of bounded Fréchet manifolds.

A Fréchet space (F, d) is a complete metrizable locally convex space whose topology is defined by a complete translation-invariant metric d. A metric with absolutely convex balls will be called a standard metric. Note that every Fréchet space admits a standard metric which defines its topology: If (α_n) is an arbitrary sequence of positive real numbers converging to *zero* and if (ρ_n) is any sequence of continuous seminorms defining the topology of F, then

$$d_{\alpha,\,\rho}(e,f) := \sup_{n\in\mathbb{N}} \alpha_n \frac{\rho_n(e-f)}{1+\rho_n(e-f)} \tag{1}$$

is a metric on F with the desired properties. We shall always define the topology of Fréchet spaces with this type of metrics. Let (E, g) be another Fréchet space and let $\mathcal{L}_{g,d}(E,F)$ be the set of all linear maps $L : E \to F$ such that

$$\mathrm{Lip}(L)_{g,d} \coloneqq \sup_{x \in E \setminus \{0\}} \frac{d(L(x),0)}{g(x,0)} < \infty.$$

We abbreviate $\mathcal{L}_g(E) \coloneqq \mathcal{L}_{g,g}(E,E)$ and write $\mathrm{Lip}(L)_g = \mathrm{Lip}(L)_{g,g}$ for $L \in \mathcal{L}_g(E)$. The metric $D_{g,d}$ defined by

$$D_{g,d} : \mathcal{L}_{g,d}(E,F) \times \mathcal{L}_{g,d}(E,F) \longrightarrow [0,\infty), \qquad (L,H) \mapsto \mathrm{Lip}(L-H)_{g,d}\,, \tag{2}$$

is a translation-invariant metric on $\mathcal{L}_{d,g}(E,F)$ turning it into an Abelian topological group (see [6, Remark 1.9]). The latter is not a topological vector space in general, but a locally convex vector group with absolutely convex balls. The topology on $\mathcal{L}_{d,g}(E,F)$ will always be defined by the metric $D_{g,d}$. We shall always equip the product of any finite number k of Fréchet spaces $(F_i,d_i), 1 \leq i \leq k$, with the maximum metric

$$d_{\max}\big((x_1,\dots,x_k),(y_1,\dots,y_k)\big) \coloneqq \max_{1\leq i\leq k} d_i(x_i,y_i).$$

Let E,F be Fréchet spaces, U an open subset of E and $P : U \to F$ a continuous map. Let $CL(E,F)$ be the space of all continuous linear maps from E to F topologized by the compact-open topology. We say P is differentiable at a point $p \in U$ if there exists a linear map $\mathrm{d}P(p) : E \to F$ such that

$$\mathrm{d}P(p)h = \lim_{t\to 0} \frac{P(p+th)-P(p)}{t},$$

for all $h \in E$. If P is differentiable at all points $p \in U$, if $\mathrm{d}P(p) : U \to CL(E,F)$ is continuous for all $p \in U$ and if the induced map

$$P' : U \times E \to F, \qquad (u,h) \mapsto \mathrm{d}P(u)h$$

is continuous in the product topology, then we say that P is Keller-differentiable. We define $P^{(k+1)} : U \times E^{k+1} \to F$ inductively by

$$P^{(k+1)}(u,f_1,...,f_{k+1}) = \lim_{t\to 0} \frac{P^{(k)}(u+tf_{k+1})(f_1,...,f_k) - P^{(k)}(u)(f_1,...,f_k)}{t}.$$

If P is Keller-differentiable, $\mathrm{d}P(p) \in \mathcal{L}_{d,g}(E,F)$ for all $p \in U$, and the induced map $\mathrm{d}P(p) : U \to \mathcal{L}_{d,g}(E,F)$ is continuous, then P is called b-differentiable. We say P is MC^0 and write $P^0 = P$ if it is continuous. We say P is an MC^1 and write $P^{(1)} = P'$ if it is b-differentiable. Let $\mathcal{L}_{d,g}(E,F)_0$ be the connected component of $\mathcal{L}_{d,g}(E,F)$ containing the zero map. If P is b-differentiable and if $V \subseteq U$ is a connected open neighbourhood of $x_0 \in U$, then $P'(V)$ is connected and hence contained in the connected component $P'(x_0)+\mathcal{L}_{d,g}(E,F)_0$ of $P'(x_0)$ in $\mathcal{L}_{d,g}(E,F)$. Thus,

$$P'|_V - P'(x_0) : V \to \mathcal{L}_{d,g}(E,F)_0$$

is again a map between subsets of Fréchet spaces. This makes possible a recursive definition: If P is MC^1 and V can be chosen for each $x_0 \in U$ such that

$$P'|_V - P'(x_0) : V \to \mathcal{L}_{d,g}(E, F)_0$$

is MC^{k-1}, then P is called an MC^k-map. We make a piecewise definition of $P^{(k)}$ by $P^{(k)}|_V := (P'|_V - P'(x_0))^{(k-1)}$ for x_0 and V as before. The map P is MC^∞ if it is MC^k for all $k \in \mathbb{N}_0$. We shall denote the derivative of P at p by $\mathrm{D}\, P(p)$.

A bounded Fréchet manifold is a second countable Hausdorff space with an atlas of coordinate charts taking their values in Fréchet spaces such that the coordinate transition functions are all MC^∞-maps.

3 Lipschitz-Fredholm vector fields

Throughout the paper we assume that (F, d) is a Fréchet space and M is a bounded Fréchet manifold modelled on F. Let $(U_\alpha, \varphi_\alpha)_{\alpha\in\mathcal{A}}$ be a compatible atlas for M. The latter gives rise to a trivializing atlas $(\pi_M^{-1}(U_\alpha), \psi_\alpha)_{\alpha\in\mathcal{A}}$ on the tangent bundle $\pi_M : TM \to M$, with

$$\psi_\alpha : \pi_M^{-1}(U_\alpha) \to \varphi_\alpha(U_\alpha) \times F, \qquad j_p^1(f) \mapsto \big(\varphi_\alpha(p), (\varphi_\alpha \circ f)'(0)\big),$$

where $j_p^1(f)$ stands for the 1-jet of an MC^∞-mapping $f : \mathbb{R} \to M$ that sends *zero* to $p \in M$. Let N be another bounded Fréchet manifold and $h : M \to N$ an MC^k-map. The tangent map $Th : TM \to TN$ is defined by $Th(j_p^1(f)) = j_{h(p)}^1(h \circ f)$. Let $\Pi_{TM} : T(TM) \to TM$ be an ordinary tangent bundle over TM with the corresponding trivializing atlas $\big(\Pi_{TM}^{-1}(\pi_M^{-1}(U_\alpha)), \widetilde{\psi}_\alpha\big)_{\alpha\in\mathcal{A}}$. A connection map on the tangent bundle TM (possible also for general vector bundles) was defined in [4]. It is a smooth bundle morphism

$$\mathcal{K} : T(TM) \to TM$$

such that the maps $\tau_\alpha : \varphi_\alpha(U_\alpha) \times F \to \mathcal{L}_d(F)$ defined by the local forms

$$\mathcal{K}_\alpha := \psi_\alpha \circ \mathcal{K} \circ (\widetilde{\psi}_\alpha)^{-1} : \varphi_\alpha(U_\alpha) \times F \times F \times F \to \varphi_\alpha(U_\alpha) \times F, \quad \alpha \in \mathcal{A} \tag{3}$$

of $\mathcal{K}$ by the rule

$$\mathcal{K}_\alpha(f, g, h, k) = (f, k + \tau_\alpha(f, g) \cdot h),$$

are smooth. A connection on M is a connection map on its tangent bundle $\pi_M : TM \to M$. A connection $\mathcal{K}$ is linear if and only if it is linear on the fibres of the tangent map. Locally $T\pi$ is the map $U_\alpha \times F \times F \times F \to U_\alpha \times F$ defined by $T\pi(f, \xi, h, \gamma) = (f, h)$, hence locally its fibres are the spaces $\{f\} \times F \times \{h\} \times F$. Therefore, $\mathcal{K}$ is linear on these fibres if and only if the maps $(g, k) \mapsto k + \tau_\alpha(f, g)h$ are linear, and this means that the mappings τ_α need to be linear with respect to their second variables.

A linear connection $\mathcal{K}$ is determined by the family $(\Gamma_\alpha)_{\alpha\in\mathcal{A}}$ of its Christoffel symbols consisting of smooth mappings

$$\Gamma_\alpha : \varphi_\alpha(U_\alpha) \to \mathcal{L}(F \times F; F), \quad p \mapsto \Gamma_\alpha(p)$$

defined by $\Gamma_\alpha(p)(g, h) = \tau_\alpha(p, g)h$.

Remark 1. If $\varphi : U \subset M \to F$ is a local coordinate chart for M, then a vector field ξ on M induces a vector field ξ on F called the local representative of ξ by the formula $\xi(x) = T\varphi \cdot \xi(\varphi^{-1}(x))$. Here and in what follows we use ξ itself to denote this local representation.

In the following we adopt Elliason's definition of a covariant derivative [5].

Definition 1. Let $\pi_M : TM \to M$ be the tangent bundle of M. Let N be a bounded Fréchet manifold modelled on F, $\lambda : N \to M$ a Fréchet vector bundle with fibre F, and K_λ a connection map on TN. If $\xi : M \to N$ is a smooth section of λ, we define the covariant derivative of ξ at $p \in M$ to be the bundle map $\nabla\xi : TM \to N$ given by

$$\nabla\xi(p) = K_\lambda \circ T_p\xi, \quad T_p\xi = T\xi|_{T_pM}.$$

In a local coordinate chart (U, Φ) we have

$$\nabla\xi(x) \cdot y = \mathrm{D}\,\xi(x) \cdot y + \Gamma_\Phi(x) \cdot (y, \xi(x)),$$

where Γ_Φ is the Christoffel symbol for K_λ with respect to the chart (U, Φ).

The covariant derivative $\nabla\xi(p)$ is a linear map from the tangent space T_pM to $F_p := \lambda^{-1}(p)$. This is because it is the combination of the tangent map $T_p\xi$ that maps T_pM linearly into $T_{\xi(p)}N$ with K_λ which is a linear map from $T_{\xi(p)}N$ to F_p.

Definition 2. ([3], Definition 3.2) Let (F, d) and (E, g) be Fréchet spaces. A map φ in $\mathcal{L}_{g,d}(E, F)$ is called a Lipschitz-Fredholm operator if it satisfies the following conditions:

1. The image of φ is closed.
2. The dimension of the kernel of φ is finite.
3. The co-dimension of the image of φ is finite.

We denote by $\mathcal{LF}(E, F)$ the set of all Lipschitz-Fredholm operators from E into F. For $\varphi \in \mathcal{LF}(E, F)$ we define the index of φ as follows:

$$\operatorname{Ind}\varphi = \dim\ker\varphi - \operatorname{codim}\operatorname{Img}\varphi.$$

Theorem 1. *([3], Theorem 3.2) $\mathcal{LF}(E, F)$ is open in $\mathcal{L}_{g,d}(E, F)$ with respect to the topology defined by the metric* (2). *Furthermore, the function $T \to \operatorname{Ind} T$ is continuous on $\mathcal{LF}(E, F)$, and hence it is constant on the connected components of $\mathcal{LF}(E, F)$.*

Now we define a Lipschitz-Fredholm vector field on M with respect to a connection on M.

Definition 3. A smooth vector field $\xi : M \to TM$ is called Lipschitz-Fredholm with respect to a connection $\mathcal{K} : T(TM) \to TM$ if for each $p \in M$, $\nabla\xi(p) : T_pM \to T_pM$

is a linear Lipschitz-Fredholm operator. The index of ξ at p is defined to be the index of $\nabla\xi(p)$, that is

$$\operatorname{Ind}\nabla\xi(p) = \dim\ker\nabla\xi(p) - \operatorname{codim}\operatorname{Img}\nabla\xi(p).$$

By Theorem 1, if M is connected, then the index is independent of the choice of p, and the common value is called the index of ξ. If M is not connected, then the index is constant on its components, and we shall require it to be the same on all these components.

Remark 2. Note that the notion of a Lipschitz-Fredholm vector field depends on the choice of the connection $\mathcal{K}$. If p is a *zero* of ξ, $\xi(p) = 0$, then by Definition 1 we have $\nabla\xi(p) = \mathrm{D}\,\xi(p)$, and hence the covariant derivative at p does not depend on $\mathcal{K}$. In this case, the derivative of ξ at p, $\mathrm{D}\,\xi(p)$, can be viewed as a linear endomorphism from T_pM into itself.

4 Finsler structures

A Finsler structure on a bounded Fréchet manifold M is defined in the same way as in the case of Fréchet manifolds (see [1] for the definition of Fréchet-Finsler manifolds). However, we need a countable family of seminorms on its Fréchet model space F which defines the topology of F. As mentioned in the Preliminaries, we always define the topology of a Fréchet space by a metric with absolutely convex balls. One reason for this consideration is that a metric with this property can give us back the original seminorms. More precisely:

Remark 3. ([8], Theorem 3.4) Assume that (E, g) is a Fréchet space and g is a metric with absolutely convex balls. Let $B^g_{\frac{1}{i}}(0) \coloneqq \{y \in E \mid g(y,0) < \frac{1}{i}\}$, and suppose that $(U_i)_{i\in\mathbb{N}}$ is a family of convex subsets of $B^g_{\frac{1}{i}}(0)$. Define the Minkowski functionals

$$\|v\|_i \coloneqq \inf\left\{\epsilon > 0 \;\middle|\; \epsilon \in \mathbb{R},\ \frac{1}{\epsilon}\cdot v \in U_i\right\}.$$

These Minkowski functionals are continuous seminorms on E. A collection $(\|v\|_i)_{i\in\mathbb{N}}$ of these seminorms gives the topology of E.

Definition 4. Let F be as before. Let X be a topological space and $V = X \times F$ the trivial bundle with fibre F over X. A Finsler structure for V is a collection of functions $\|\cdot\|^n : V \to \mathbb{R}^+$, $n \in \mathbb{N}$, such that

1. For any fixed $b \in X$, $\|(b,x)\|^n = \|x\|^n_b$ is a collection of seminorms on F which gives the topology of F.
2. Given $K > 1$ and $x_0 \in X$, there exits a neighborhood $\mathcal{U}$ of x_0 such that
$$\frac{1}{K}\|f\|^n_{x_0} \leqq \|f\|^n_x \leqq K\|f\|^n_{x_0} \tag{4}$$
for all $x \in \mathcal{U}$, $n \in \mathbb{N}$, $f \in F$.

Let $\pi_M : TM \to M$ be the tangent bundle of M and let $\|\cdot\|^n : TM \to \mathbb{R}^+$, $n \in \mathbb{N}$, be a family of functions. We say that $(\|\cdot\|^n)_{n\in\mathbb{N}}$ is a *Finsler structure* for TM if for a given $m_0 \in M$ and any open neighborhood U of m_0 which trivializes the tangent bundle TM, i.e., there exists a diffeomorphism

$$\psi : \pi_M^{-1}(U) \approx U \times \big(F_{m_0} := \pi_M^{-1}(m_0)\big),$$

the family $(\|\cdot\|^n \circ \psi^{-1})_{n\in\mathbb{N}}$ is a Finsler structure for $U \times F_{m_0}$.

Definition 5. A bounded Fréchet-Finsler manifold is a bounded Fréchet manifold together with a Finsler structure on its tangent bundle.

Proposition 1. *Let N be a paracompact bounded Fréchet manifold modelled on a Fréchet space (E, g). If all seminorms $\|\cdot\|_i$, $i \in \mathbb{N}$, (which are defined as in Remark 3) are smooth maps on $E \setminus \{0\}$, then N admits a partition of unity. Moreover, N admits a Finsler structure.*

Proof. See [1], Propositions 3 and 4. □

If $(\|\cdot\|^n)_{n\in\mathbb{N}}$ is a Finsler structure for M then eventually we can obtain a graded Finsler structure, denoted again by $(\|\cdot\|^n)_{n\in\mathbb{N}}$, for M (see [1]). Let $(\|\cdot\|^n)_{n\in\mathbb{N}}$ be a graded Finsler structure for M. We define the length of piecewise MC^1-curve $\gamma : [a, b] \to M$ by

$$L^n(\gamma) = \int_a^b \|\gamma'(t)\|^n_{\gamma(t)}\, \mathrm{d}t.$$

On each connected component of M, the distance is defined by

$$\rho^n(x, y) = \inf_{\gamma} L^n(\gamma),$$

where the infimum is taken over all continuous piecewise MC^1-curve connecting x to y. Thus, we obtain an increasing sequence of pseudometrics $\rho^n(x, y)$ and define the distance ρ by

$$\rho(x, y) = \sum_{n=1}^{n=\infty} \frac{1}{2^n} \cdot \frac{\rho^n(x, y)}{1 + \rho^n(x, y)}. \tag{5}$$

Lemma 1. *([1], Lemma 2) A family $(\sigma^i)_{i\in\mathbb{N}}$ of pseudometrics on F defines a unique topology $\mathcal{T}$ such that for every sequence $(x_n)_{n\in\mathbb{N}} \subset F$, we have $x_n \to x$ in the topology $\mathcal{T}$ if and only if $\sigma^i(x_n, x) \to 0$, for all $i \in \mathbb{N}$. The topology is Hausdorff if and only if $x = y$ when all $\sigma^i(x, y) = 0$. In addition,*

$$\sigma(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\sigma^n(x, y)}{1 + \sigma^n(x, y)}$$

is a pseudometric on F, which defines the same topology.

With the aid of this lemma, the proof of the following theorem is quite similar to the proof given for Banach manifolds (cf. [10]).

Theorem 2. *Suppose M is a connected manifold endowed with a Finsler structure $(\|\cdot\|^n)_{n\in\mathbb{N}}$. Then the distance ρ defined by (5) is a metric for M. Furthermore, the topology induced by this metric coincides with the original topology of M.*

Proof. The distance ρ is pseudometric by Lemma 1. We prove that $\rho(x_0, y_0) > 0$ if $x_0 \neq y_0$. Let $(\|\cdot\|_n)_{n\in\mathbb{N}}$ be the family of all seminorms on F (which are defined as in Remark 3). Given $x_0 \in M$, let $\varphi : U \to F$ be a chart for M with $x_0 \in U$ and $\varphi(x_0) = u_0$. Let $y_0 \in M$, and let $\gamma : [a, b] \to M$ be an MC^1-curve connecting x_0 to y_0. Let $B_r(u_0)$ be a ball with center u_0 and radius $r > 0$. Choose r small enough so that $\mathcal{U} := \varphi^{-1}(B_r(u_0)) \subset U$ and for a given $K > 1$,

$$\frac{1}{K}\|f\|^n_{x_0} \leqq \|f\|^n_x \leqq K\|f\|^n_{x_0},$$

for all $x \in \mathcal{U}$, $n \in \mathbb{N}$, $f \in F$. Let $I = [a, b]$ and $\mu(t) := \varphi \circ \gamma(t)$. If $\gamma(I) \subset \mathcal{U}$, then let $\beta = b$. Otherwise, let β be the first $t > 0$ such that $\|\mu(t) - u_0\|_n = r$ for all $n \in \mathbb{N}$. Then, since for every $x \in U$ the map $\phi(x) : T_xM \to F$ given by $j^1_x \mapsto \varphi(x)$ is a homeomorphism, it follows that for all $n \in \mathbb{N}$ we have

$$\begin{aligned}\int_a^b \|\gamma'(t)\|^n_{\gamma(t)}\,\mathrm{d}t &\geq \frac{1}{K}\int_a^\beta \|\phi^{-1}(x)\circ\mu'(t)\|^n_{x_0}\mathrm{d}t \geq k_1\int_a^\beta \|\mu'(t))\|_n\mathrm{d}t\\ &\geq k_1\Big\|\int_a^\beta \mu'(t)\mathrm{d}t\Big\|_n = k_1\|\mu(\beta) - \mu(a)\|_n \quad \text{for some } k_1 > 0.\end{aligned}$$

(The last inequality follows from [7, Theorem 2.1.1].) Thereby, if $x_0 \neq y_0$ then $\rho^n(x_0, y_0) > 0$ and hence $\rho(x_0, y_0) > 0$. Now we prove that the topology induced by ρ coincides with the topology of M. By virtue of Lemma 1, we only need to show that $(\rho^n)_{n\in\mathbb{N}}$ induces the topology which is consistent with the topology of M. If $x_i \to x_0$ in M then eventually $x_i \in \mathcal{U}$. Define $\lambda_i : [0, 1] \to \mathcal{U}$, an MC^1-curve connecting x_0 to x_i, by $\lambda_i(t) := t\varphi(x_i)$. Then, for all $n \in \mathbb{N}$

$$\begin{aligned}\rho^n(x_i, x_0) \leq L^n(\lambda_i) &= \int_0^1 \|\lambda_i'\|^n_{\lambda_i(t)}\mathrm{d}t = \int_0^1 \|\varphi(x_i)\|^n_{t\varphi(x_i)}\mathrm{d}t\\ &\leq K\int_0^1 \|\varphi(x_i)\|^n_{x_0}\mathrm{d}t = K\|\varphi(x_i)\|_n.\end{aligned}$$

But $\varphi(x_i) \to 0$ as $x_i \to x_0$, thereby $\rho^n(x_i, x_0) \to 0$ for all $n \in \mathbb{N}$. Conversely, if for all $n \in \mathbb{N}$, $\rho^n(x_i, x_0) \to 0$ then eventually we can choose r small enough so that $x_i \in \mathcal{U}$. Then, for all $n \in \mathbb{N}$ we have $\|\varphi(x_i)\|^n_{x_0} \leq K\rho^n(x_i, x_0)$ so $\|\varphi(x_i)\|^n_{x_0} \to 0$ in $T_{x_0}M$, whence $\varphi(x_i) \to 0$. Therefore, $x_i \to x_0$ in $\mathcal{U}$ and hence in M. □

The metric ρ is called the *Finsler metric* for M.

5 Morse-Sard-Brown Theorem

In this section we prove the Morse-Sard-Brown theorem for functionals on bounded Fréchet-Finsler manifolds. The proof relies on the following inverse function theorem.

Theorem 3 (Inverse Function Theorem for MC^k-maps). *([6], Proposition 7.1) Let (E, g) be a Fréchet space with standard metric g. Let $U \subset E$ be open, $x_0 \in U$ and $f : U \subset E \to E$ an MC^k-map, $k \geq 1$. If $f'(x_0) \in \operatorname{Aut}(E)$, then there exists an open neighbourhood $V \subseteq U$ of x_0 such that $f(V)$ is open in E and $f|_V : V \to f(V)$ is an MC^k-diffeomorphism.*

The following consequence of this theorem is an important technical tool.

Proposition 2 (Local representation). *Let F_1, F_2 be Fréchet spaces and U an open subset of $F_1 \times F_2$ with $(0,0) \in U$. Let E_2 be another Fréchet space and $\phi : U \to F_1 \times E_2$ an MC^∞-map with $\phi(0,0) = (0,0)$. Assume that the partial derivative $\mathrm{D}_1 \phi(0,0) : F_1 \to F_1$ is linear isomorphism. Then there exists a local MC^∞-diffeomorphism ψ from an open neighbourhood $V_1 \times V_2 \subseteq F_1 \times F_2$ of $(0,0)$ onto an open neighbourhood of $(0,0)$ contained in U such that*

$$\phi \circ \psi(u, v) = (u, \mu(u, v)),$$

where $\mu : V_1 \times V_2 \to E_2$ is an MC^∞-mapping.

Proof. Let $\phi = \phi_1 \times \phi_2$, where $\phi_1 : U \to F_1$ and $\phi_2 : U \to E_2$. By assumption we have $\mathrm{D}_1 \phi_1(0,0) = \mathrm{D}_1 \phi(0,0)|_{F_1} \in \operatorname{Iso}(F_1, F_1)$. Define the map

$$g : U \subset F_1 \times F_2 \to F_1 \times E_2, \quad g(u_1, u_2) \coloneqq \big(\phi_1(u_1, u_2), u_2\big)$$

locally at $(0,0)$. Then, for all $u = (u_1, u_2) \in U$, $f_1 \in F_1$, $f_2 \in F_2$ we have

$$\mathrm{D}\, g(u) \cdot (f_1, f_2) = \begin{pmatrix} \mathrm{D}_1 \phi_1(u) & \mathrm{D}_2 \phi_1(u) \\ 0 & \mathrm{Id}_{E_2} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

and hence $\mathrm{D}\, g(u)$ is a linear isomorphism at $(0,0)$. By the inverse function theorem, there are open sets U' and $V = V_1 \times V_2$ and an MC^∞-diffeomorphism $\Psi : V \to U'$ such that $(0,0) \in U' \subset U$, $g(0,0) \in V \subset F_1 \times E_2$, and $\Psi^{-1} = g|_{U'}$. Hence if $(u, v) \in V$, then

$$(u, v) = (g \circ \Psi)(u, v) = g\big(\Psi_1(u, v), \Psi_2(u, v)\big) = \big(\phi_1 \circ \Psi_1(u, v), \Psi_2(u, v)\big),$$

where $\Psi = \Psi_1 \times \Psi_2$. This shows that $\Psi_2(v, v) = v$ and $(\phi_1 \circ \Psi)(u, v) = u$. If $\eta = \phi_2 \circ \Psi$, then

$$(\phi \circ \Psi)(u, v) = \big(\phi_1 \circ \Psi(u, v), \phi_2 \circ \Psi(u, v)\big) = (u, \eta(u, v)).$$

This completes the proof. $\square$

In the sequel, we assume that M is connected and it is endowed with a Finsler structure $(\|\cdot\|^n)_{n \in \mathbb{N}}$ and the induced Finsler metric ρ.

Definition 6. Let $l : M \to \mathbb{R}$ be an MC^∞-functional and $\xi M \to TM$ a smooth vector field. By saying that l and ξ are associated we mean $\mathrm{D}\, l(p) = 0$ if and only if $\xi(p) = 0$. A point $p \in M$ is called a critical point for l if $\mathrm{D}\, l(p) = 0$. The corresponding value $l(p)$ is called a critical value. Values other than critical are called regular values. The set of all critical points of l is denoted by Crit_l.

The following is our version of the compactness condition due to Tromba [11].

Condition 1 (CV). *Let $(m_i)_{i\in\mathbb{N}}$ be a bounded sequence in M. We say that a vector field $\xi : M \to TM$ satisfies condition* (CV) *if $\|\xi(m_i)\|^n \to 0$ for all $n \in \mathbb{N}$ implies that $(m_i)_{i\in\mathbb{N}}$ has a convergent subsequence.*

If ξ satisfies condition (CV) then the set of its zeros in any closed bounded set is compact (see [11, Proposition 1, p. 55]). This property turns out to be important. We then say ξ satisfies condition (WCV) if the set of its zeros in any closed bounded set is compact.

A subset G of a Fréchet space E is called topologically complemented or split in E if there is another subspace H of E such that E is homeomorphic to the topological direct sum $G \oplus H$. In this case we call H a topological complement of G in F.

We need the following facts:

Theorem 4. *([8], Theorem 3.14) Let E be a Fréchet space. Then*

1. *Every finite-dimensional subspace of E is closed.*
2. *Every closed subspace $G \subset E$ with $\operatorname{codim}(G) = \dim(E/G) < \infty$ is topologically complemented in E.*
3. *Every finite-dimensional subspace of E is topologically complemented.*
4. *Every linear isomorphism $G \oplus H \to E$ between the direct sum of two closed subspaces and E, is a homeomorphism.*

The proof of the Morse-Sard-Brown theorem requires Proposition 2 and Theorem 4. Except the arguments which involve these results and the Finslerian nature of manifolds, the rest of arguments are similar to that of Banach manifolds case, see [12, Theorem 1].

Theorem 5 (Morse-Sard-Brown Theorem). *Assume that (M, ρ) is endowed with a connection $\mathcal{K}$. Let ξ be a smooth Lipschitz-Fredholm vector field on M with respect to $\mathcal{K}$ which satisfies condition* (WCV). *Then, for any MC^∞-functional l on M which is associated to ξ, the set of its critical values $l(\mathrm{Crit}_l)$ is of first category in $\mathbb{R}$. Therefore, the set of the regular values of l is a residual Baire subset of $\mathbb{R}$.*

Proof. We can assume $M = \bigcup_{i\in\mathbb{N}} M_i$, where all the M_i's are closed bounded balls of radius i about some fixed point $m_0 \in M$. The boundedness and the radii of balls are relative to the Finsler metric ρ. Thus to conclude the proof it suffices to show that the image $l(C_B)$ of the set C_B of the zeros of ξ in some bounded set B is compact without interior. If, in addition, B is closed, then C_B is compact because ξ satisfies condition (WCV).

Let B be a closed bounded set and let C_B as before. If $p \in C_B$ then eventually $\xi(p) = 0$. Since C_B is compact we only need to show that for a bounded neighbourhood U of p, $l(C_B \cap \overline{U})$ is compact without interior. In other words, we can work locally. Therefore, we may assume without loss of generality that $p = 0 \in F$

and ξ, l are defined locally on an open neighbourhood of p. An endomorphism $\mathrm{D}\,\xi(p) : F \to F$ is a Lipschitz-Fredholm operator because ξ is a Lipschitz-Fredholm vector field (see Remark 2). Thereby, in the light of Theorem 4 it has a split image F_1 with a topological complement F_2 and a split kernel E_2 with a topological complement E_1. Moreover, $\mathrm{D}\,\xi(p)$ maps E_1 isomorphically onto F_1 so we can identify F_1 with E_1. Then, by Proposition 2, there is an open neighborhood $U \subset E_1 \times E_2$ of p such that $\xi(u,v) = (u, \eta(u,v))$ for all $(u,v) \in U$, where $\eta : U \to F_2$ is an MC^∞-map. Thus, if $\xi(u,v) = 0 = (u, \eta(u,v))$ then $u = 0$. Therefore, in this local representation, the zeros of ξ (and hence the critical points of l) in $\overline{U}$ are in $\overline{U}_1 := \overline{U} \cap (\{0\} \times E_2)$. The restriction of l, $l_{\overline{U}_1} : \overline{U}_1 \to \mathbb{R}$, is again MC^∞ and $C_B \cap \overline{U} = C_B \cap \overline{U}_1$ so $l(C_B \cap \overline{U}) = l(C_B \cap \overline{U}_1)$.

We have for some constant $k \in \mathbb{N}$, $\dim \overline{U}_1 = \dim E_2 = k$ because $\xi(p)$ is a Lipschitz-Fredholm operator and E_2 is its kernel. Thus, by the classical Sard theorem, $l(C_B \cap \overline{U}_1)$ has measure zero (note that MC^k-differentiability implies the usual C^k-differentiability for maps of finite dimensional manifolds). Therefore, since $C_B \cap \overline{U}_1$ is compact it follows that $l(C_B \cap \overline{U}_1)$ is compact without interior and hence $l(C_B \cap \overline{U})$ is compact without interior. □

Remark 4. From the preceding proof we see that $\dim F_2 = m$, where $m \in \mathbb{N}$ is constant. Thus, the index of ξ is the $\operatorname{Ind} \xi = \dim E_2 - \dim F_2 = k - m$.

References

[1] C.L. Bejan: Finsler structures on Fréchet bundles. In: *Proc. 3-rd Seminar on Finsler spaces, Univ. Braşov 1984.* Societatea de ştiinţe Matematice Romania, Bucharest (1984) 49–54.

[2] C.T.J. Dodson: Some recent work in Fréchet geometry. *Balkan J. Geometry and Its Applications* 17 (2) (2012) 6–21.

[3] K. Eftekharinasab: Sard's theorem for mappings between Fréchet manifolds. *Ukrainian Math. J.* 62 (12) (2011) 1896–1905.

[4] K. Eftekharinasab: Geometry of Bounded Fréchet Manifolds. *Rocky Mountain J. Math., to appear*().

[5] H. Eliasson: Geometry of manifolds of maps. *J. Differential Geometry* 1 (1967) 169–194.

[6] H. Glöckner: Implicit functions from topological vector spaces in the presence of metric estimates. *preprint, Arxiv:math/6612673*(2006).

[7] R.S. Hamilton: The inverse function theorem of Nash and Moser. *Bulletin of the AMS* 7 (1982) 65–222.

[8] O. Müller: A metric approach to Fréchet geometry. *Journal of Geometry and Physics* 58 (11) (2008) 1477–1500.

[9] R.S. Palais: Lusternik-Schnirelman theory on Banach manifolds. *Topology* 5 (2) (1966) 115–132.

[10] R.S. Palais: Critical point theory and the minimax principle. *Proc. Symp. Pur. Math.* 15 (1970) 185–212.

[11] A.J. Tromba: A general approach to Morse theory. *J. Differential Geometry* 12 (1) (1977) 47–85.

[12] A.J. Tromba: The Morse-Sard-Brown theorem for functionals and the problem of Plateau. *Amer. J. Math.* 99 (1977) 1251–1256.

[13] A.J. Tromba: The Euler characteristic of vector fields on Banach manifolds and a globalization of Leray-Schauder degree. *Advances in Mathematics* 28 (2) (1978) 148–173.

Author's address:
Kaveh Eftekharinasab, Topology dept., Institute of Mathematics of NAS of Ukraine, Tereshchenkivs'ka str. 3, 01601 Kyiv, Ukraine

E-mail: kaveh@imath.kiev.ua

Received: 25 August, 2014
Accepted for publication: 24 August, 2015
Communicated by: Olga Rossi