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Josef Talacko

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Méthode B':

$$\sum_{i=1}^{q} L_{x_i} P_{x_i} \, a_{x_i \overline{n}|}^{aa} = a_{y_0 \overline{n}|}^{aa} \sum_{i=1}^{q} L_{x_i} P_{x_i},$$

où

$$y_0 = \frac{y_1 c^n + y_2 G^n}{c^n + G^n}.$$

Méthode C':

$$\sum_{i=1}^{q} L_{x_i} P_{x_i} \, a_{x_i \overline{n}|}^{aa} = \frac{c^n a_{y_1 \overline{n}|}^{aa} + G^n a_{y_2 \overline{n}|}^{aa}}{c^n + G^n} \sum_{i=1}^{q} L_{x_i} P_{x_i}.$$

Les résultats auxquels on arrive sont donnés par le tableau XI.

On voit de ce tableau que les méthodes B' et C' donnent, elles-aussi, de très bons résultats: dans tous les exemples considérés, l'erreur relative est inférieure à 0,4%. Dans l'ensemble de tous ces exemples elle diminue jusqu'à 0,04%, et devient donc absolument negligeable.

MATHEMATICAL THEORY OF GROWTH WITH SPECIAL REGARD TO POPULATION PROBLEMS

By RNDr Josef Talacko

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I. INTRODUCTION

§ 1. Biometrics as an exact science.

The population problem, which is now engaging the attention of experts more than ever before, needs more perfect methods for the exact investigation of demographic phenomena. With the progress of natural sciences and in particular of mathematics, since the 17th century, the science of population which began with mere description and simple empirical numbers has succeeded, within recent decades, in defining exact demographic characteristics. In population studies we are no longer satisfied with elementary measures such as birth-rate, crude death-rate, matrimonial and vital indexes etc. These measures which depend on the age distribution of the population often give us a distorted picture of underlying facts and must, therefore, be replaced by more precise characteristics such as measures derived from a mortality table, rates of fecundity, of reproduction, laws of evolution etc.

Besides the description of the state of population at a given instant of time there is the dynamic aspect of demographic phenomena which, at present, is of the greatest importance. While, for centuries, there was no distinct secular trend in the fluctuations of biometric functions, a persistent striking decline in mortality — especially in infantile mortality — has been observed over a period of time and this decline has led, in the 19th and 20th centuries, to an increase of population in all civilized countries. This increase, however, has been retarded by an equally remarkable decline in the birth-rate during past few decades. These seem to be the main exceptional features in recent population development.

Biometrics is the branch of science which creates methods for the study of population problems by means of consistent application of mathematics and especially of mathematical statistics. Like astronomy and physics, after years of empiricism, biometrics has now become an exact science with considerable application of deductive methods.

It is obvious that the changes in population which we have experienced are not without far-reaching economic, social and political consequences. Declining death rates cause changes in the age distribution and they result — along with falling birth rates — in a higher proportion of aged persons in the population. How deep these changes have been is to be seen from the fact that the life expectation of a live-born infant has been doubled in the course of 150 years.

Statisticians are faced with the task not only of reviewing the present status, but also of estimating the probable future development of population. The long-standing character of population phenomena, richer sources of statistical data and improved methods enable them to perform this task. Such estimates are necessary in economic planning and in the building up of social insurance schemes and play an important part in international relations.

The investigations into population development have been carried out in two directions. Firstly, efforts to discover the mortality law have resulted in establishing the life table and in defining the biometric functions; secondly, the whole population has been investigated in regular censuses and attempts have been made to foretell the future population trend by analytical extrapolation of the given data.

The study of population development as a function of time brings many more interesting problems, some of which we have treated in papers ([55]), ([56]).*) There exists abundant literature on the subject.

*) Small figures in brackets refer to the bibliography at the beginning.

§ 2. *Summary.*

The lay-out of the present paper is as follows:

In section II we shall recapitulate the main laws of growth, especially those with a general application.

In section III we present some methods for the application of the normal and generalized logistic curves and point out how the least squares and successive approximation methods may be used in evaluation of the parameters. Further we shall mention a method of estimating limits for the probable development of the phenomenon in question. Then we shall deal with a method developed by Aitken and present a simple numerical evaluation, taking advantage of Tchebysheff's polynomials.

In section IV we shall analyse the functional relations between various biometric functions and apply integral and integro-differential equations to a special problem. We pay special attention to the deduction of the natality law in a closed logistic population with a constant or declining mortality.

In section V we apply integral and integro-differential equations to the estimation of reproduction and evolution coefficients and discuss a method for solving the fundamental integral equation.

In section VI we pay attention to some other branches of applied mathematics in which the methods mentioned above may be used analogically, and in particular we deduce the function of industrial renovation.

We shall always bear in mind the possibility of generalization and wider application of the methods under discussion.

II. LAWS OF GROWTH

§ 3. *Determination of growth laws from differential relations.*

In the study of demographic phenomena, an accurate estimate of the total population is often necessary. When we know the results of several censuses, the interpolation for the years between each two census-years is not a difficult problem. But when we are confronted with the task to foretell the future course of population, we cannot simply extrapolate the data from the latest census-year without considering the growth law of the population. The extrapolation, however, is indispensable not only in forecasting the future growth, but also in estimating the present status of the population, as the censuses are carried out only at ten year intervals and the necessary computations require much time.

Many extrapolation formulae have been set up in attempts to find an analytical expression of growth laws. Most of these formulae may be easily deduced from elementary differential equations as we shall demonstrate in several examples.

Let us assume that $y(t)$ denoting a total of population at a time t is a continuous function of t and that there exists the derived function $\frac{dy}{dt}$.

α) Let us suppose

$$\frac{dy}{dt} = a = \text{const.} \tag{2,01}$$

On integrating we get y as a linear function of t

$$y = at + k, \tag{2,02}$$

where k is a constant. For example, by substituting $t = 0$, we obtain $k = y_0$

$$y = at + y_0. \tag{2,03}$$

This is the simplest formula for short-term linear interpolation and extrapolation and is frequently used.

β) Let us suppose

$$\frac{dy}{dt} = a - c\,\frac{1}{t}. \tag{2,04}$$

On integration we get the equation for logarithmic growth

$$y = k + at - c \lg t \tag{2,05}$$

clearly illustrating growth with decreasing intensity.

γ) Similarly by integration of

$$\frac{dy}{dt} = a + bt - c\,\frac{1}{t}, \tag{2,06}$$

we obtain

$$y = k + at + bt^2 - c \lg t. \tag{2,07}$$

δ) If growth is considered as a linear function of population, we can write

$$\frac{dy}{dt} = ay + b. \tag{2,08}$$

After separating variables and appropriate substitutions we get

$$y = A + Be^{at}, \tag{2,09}$$

which is the well known and commonly used geometrical law of growth. This exponential function has an extensive application in natural science ([11]).

ε) Finally, like Verhulst ([52]), ([53]), we can assume that growth is given by the relation

$$\frac{dy}{dt} = a + by - cy^2, \tag{2,10}$$

where $-cy^2$ is a term expressing conditions and influences checking growth.

Assuming that $a > 0$, $b > 0$, $c > 0$ this equation gives the same solution arrived at by Čupr ([11]). He obtained logistic growth as did Lotka ([37]), ([38]) who expressed the result in terms of the hyperbolic tangent.

We then obtain the equation

$$y = A + B \tanh c\,(t - \delta), \tag{2,11}$$

where A, B, C and δ are constants.

Let us write

$$f(t) = \frac{1}{1 + e^{\delta t}} = \tfrac{1}{2} + \tfrac{1}{2} \tanh \tfrac{1}{2}\delta t. \tag{2,12}$$

We then have

$$y = \varkappa + \frac{K}{1 + e^{a_0 + a_1 t}} \tag{2,13}$$

§ 4. Derivation of rules of growth from the differential equations for the intensity of growth.

Let y be the number of inhabitants in a certain place at time t. The intensity of change in y is measured by $a(t)$, defined by the equation

$$a(t) = \frac{1}{y} \frac{\mathrm{d}y}{\mathrm{d}t}. \tag{2,14}$$

By integration we obtain a general formula for the size of the population at time t

$$y = y_0 e^{\int_0^t a(t)\mathrm{d}t} \tag{2,15}$$

where y_0 signifies the number of inhabitants at time $t = 0$.

This exponential function has a wide validity especially in natural science. It is a general rule in biology, chemistry as well as in physics in all cases, where a given number of units is continually changing in such a way that the rate of change is proportional to the number.

In order to solve the given equation, the intensity of change $a(t)$ must be an integrable function of t. Assuming that $a(t) = c$, where $c > 0$ is a constant we obtain the well known geometrical rule of growth:

$$y = y_0 e^{c\int_0^t \mathrm{d}t} = K \cdot r^t. \tag{2,16}$$

This rule, first formulated by Euler, clearly shows the quantitative growth of population, for example in the USA and in Australia in 18th and 19th centuries. It cannot, however, be applied to populations which do not have unlimited means of existence available [52], [53]. Even Euler perceived that the geometrical law of growth in such cases leads to absurdity. Scepticism of Euler's calculations led Malthus to his well known criticism, and to the idea, that geometrical growth is checked by a certain retarding factor. This idea was formulated philosophically by Quételet and then mathematically about a hundred years ago by the Dutch mathematician Verhulst [52], [53], who thus discovered the logistic formula of growth. His works were forgotten and only after the work of Pearl and Reed [53] were they again studied. Verhulst's discovery is important not only because of its priority, but also because of the fact that he reached it by deduction thus demonstrating the importance of mathematical methods in biometrics.

Quételet's formula, in terms of the initial differential equation, is given by equation (2,10), and the result by equation (2,11). Verhulst, who presented the solution in various forms, assumed that constant intensity of growth is decreased by a retarding, linear function of population.

Mathematically written, this gives

$$\frac{1}{y}\frac{dy}{dt} = b - cy \tag{2,17}$$

and after integration [see ref. [53]] we get

$$y = \frac{\frac{b}{c}e^{b(t+k)}}{1 + e^{b(t+k)}}. \tag{2,18}$$

By rearrangement and substitution we obtain the normal logistic law in the form

$$y = \frac{K}{1 + e^{a_0 + a_1 t}}. \tag{2,19}$$

or in Verhulst's notation

$$y = \frac{L}{1 + e^{\frac{\beta - t}{\alpha}}}. \tag{2,20}$$

It is clear that the differential equation (2,17) is only a special case of equation (2,10), where $a = 0$, $b > 0$, $c > 0$.

We have gone into the details of the properties of logistic curves and the significance of the parameters [53], [54] and can assume the results to be correct.

§ 5. *Derivation of rules of quantitative growth from the differential equations for acceleration.*

If we assume again, according to Quételet, that the resulting acceleration of growth in a population is made up of two opposite components then we can express this by the equation

$$\frac{d^2y}{dt^2} = \varepsilon \frac{dy}{dt} - \varphi(t). \tag{2,21}$$

In order to solve this equation, we require to make certain assumptions about $\varphi(t)$.

Let us suppose [see Delevsky ([54])] that

$$\varphi(t) = -a + by + \alpha y' + \beta y'^2.$$

Then, by substitution, the law of growth is given in its most general form by a differential equation of the second order and second degree,

$$y'' = a - by + (\varepsilon - \alpha) y' - \beta y'^2, \tag{2,22}$$

where the quantities a, b, α, β, ε are considered to be positive and constant.

If we wish to find the conditions under which the important property

$$\lim_{t\to\infty} y(t) = y > 0 \tag{2,23}$$

holds, we substitute in (2,22)

$$y' = v,$$

$$y'' = \frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = v \frac{dv}{dy}$$

and thus obtain Ricatti's differential equation. We can solve the equation

$$v \frac{dv}{dy} = a - by + (\varepsilon - \alpha) v - \beta v^2 \tag{2,24}$$

by making certain assumptions about parameters and under the condition (2,23), and thus obtain a number of solutions of which the following four are the most important.

1. If $\beta=0$, $(\varepsilon-\alpha)<0$, the solution represents development approaching a state of equilibrium which, after rearrangement, can be written

$$y = \varkappa - c_1e^{At} - c_2e^{Bt}. \tag{2,25}$$

2. If $-(\varepsilon-\alpha) > 0$, $\beta = 0$, $b \neq 0$, the solution is given by the equation

$$y = \varkappa + c_1e^{-\frac{(\varepsilon-\alpha)t}{2}} \cos(At + c_2) \tag{2,26}$$

which indicates damped periodic motion.

3. If $(\varepsilon - \alpha) = -n = 0$, $a = \sqrt{b}$, then the solution is an even periodic function of time, given by the equation

$$y = \varkappa + c_1 \cos(\sqrt{b}t + c_2). \tag{2,27}$$

4. For the special case of equation (2,21), where $a = b = \beta = 0$, $\alpha = cy$ we obtain the solution

$$y = \frac{L}{1 + \lambda e^{-ct}} \tag{2,28}$$

which is the equation for the logistic curve similar to those given in equations (2,13) and (2,20).

It is clear that, by similar considerations, the rules of growth stated in section II can be deduced from the differential equation (2,22). We have dealt here with only a few of the many rules which can be derived and have practical applications. As has already been pointed out ([54]), also Rhodes's equations, which are considered as a generalization of the population growth curve and are given by:

$$\frac{1}{y} = a + be^{-rt} + ce^{rt} \tag{2,29}$$

or

$$y = k \sec h^2 \frac{\beta - t}{2} \tag{2,30}$$

are only special cases of the solution to the general differential equation (2,22).

III. THE GENERALIZED LOGISTIC CURVE AND ITS APPLICATION

§ 6. *Derivation of generalized logistic curve.*

Let us start out from the intensity of the change in population and let us assume it to be given generally by the equation

$$\frac{1}{y}\frac{dy}{dt} = \varphi(y, t). \tag{3,01}$$

In order to solve this equation, we must again make a certain assumption about the nature of $\varphi(y, t)$. Let us suppose, according to Reed ([53]), that

$$\varphi(y, t) = (a - by)\,\varphi(t) \tag{3,02}$$

and substitute in the equation (3,01). After integration we get

$$y = \frac{\dfrac{a}{b} e^{a\int_0^t \varphi(t)dt + ac}}{1 + e^{a\int_0^t \varphi(t)dt + ac}} \tag{3,03}$$

which, after rearrangement, and the substitutions

$$\frac{a}{b} = L,$$

$$-a\int_0^t \varphi(t)\,dt = \Phi(t),$$

$$e^{-ac} = \lambda,$$

gives the general formula for quantitative growth as

$$y = \frac{L}{1 + \lambda e^{\Phi(t)}}. \tag{3,04}$$

We may assume that the function $\Phi(t)$ can be expressed in terms of Taylor's series and the equation (3,04) then becomes

$$y = \frac{L}{1 + \lambda e^{a_1 t + a_2 t^2 + \ldots + a_n t^n}} \tag{3,05}$$

where $L, \lambda, a_1, a_2, \ldots, a_n$ are parameters.

It is clear that we can also replace the function $\Phi(t)$ by Tchebysheff's series of orthogonal polynomials. We have tried this successfully in several practical cases with a view mainly to ease in calculation of the constants — in particular, by expressing the orthogonal polynomials in the form developed by Lorenz.

If we can write

$$\Phi(t) = A_0 + A_1 X_1 + A_2 X_2 + \ldots + A_n X_n$$

where A_k are constants and X_k are the known Lorenz's polynomials, then equation (3,04) becomes

$$y = \frac{L}{1 + e^{A_0 + A_1 X_1 + A_2 X_2 + \ldots + A_n X_n}}. \tag{3,06}$$

In this paper we do not concern ourselves with the analysis and characteristics of the normal and generalized logistic curve. This was thoroughly dealt with in refs. ([53]), ([54]) and we therefore consider the properties derived there to hold true or refer to what has already been discussed. In practice, the calculation of the constants from empirical data is not an easy problem. In ref. ([53]), we have shown the current methods of determining the constants as well as the conditions of application of the normal logistic curve.

§ 7. Derivation of the criterion of application, and further methods in the determination of constants.

a) If we are given a series of n equidistant empirical quantities

$$y_0, y_1, \ldots, y_{n-1}$$

which we wish to express analytically by a logistic function, it is necessary first to be quite sure that it is logistic growth which is implicated, and then to determine parameters and theoretical values. The empirical series of values under consideration must comply with certain conditions if we are to express it by means of a normal logistic curve. The preliminary conditions stated in refs. ([53]), ([54]) are not sufficient since it is not always possible to apply a logistic curve even when they are fulfilled.

The best method of procedure is to use the so called criterion of application, derived on the basis of the characteristics of the reciprocal values of the logistic curve by the elimination of constants.

The derivation of the criterion of application and the necessary conditions for applicability can be found in the references.

The criterion is

$$\gamma = e^{\frac{h}{\varkappa}} + 1 + e^{-\frac{h}{\varkappa}}$$

derived from 4 equidistant values, where $\varkappa$ is a constant characteristic of the curve, and h is the size of the interval between the observed values. In other words, the criterion is

$$\Gamma = e^{\frac{rh}{\varkappa}} + 1 + e^{-\frac{rh}{\varkappa}}$$

where r denotes the number of observations made at these intervals and calculated by the method of sums and has necessarily a value between the limits $(3, \infty)$. The ease of calculation and also the fact that further constants can immediately be determined from the value of the criterion γ or Γ has undoubted advantages.

In a similar manner Čupr ([11]) derives a criterion and seeks the necessary and sufficient conditions that the logistic curve should be determined by five points.

b) Besides the above methods we can also compute the constants of the normal logistic curve by the method of least squares. Let us write the equation of a logistic curve (2,19) in the form

$$y = \frac{1}{\varkappa + \lambda e^{at}} \tag{3,07}$$

and then substitute

$$\frac{1}{y} = z.$$

We then have

$$z = \varkappa + \lambda e^{at}. \tag{3,08}$$

By a suitable transformation, we can change this into a linear function of time to which we then apply the method of least squares. First we take the derivative

$$\frac{dz}{dt} = a\lambda e^{at}$$

and then the logarithm

$$\log \frac{dz}{dt} = \log (a\lambda) + a \log e \,.\, t. \tag{3,09}$$

By the substitutions

$$\log \left(\frac{dz}{dt}\right) = Y \quad \text{or} \quad \log \left(\frac{\Delta z}{\Delta t}\right) = Y,$$

$$\log (a\lambda) = A, \; a \log e = B$$

we get

$$Y = A + Bt. \tag{3,10}$$

After calculating the constants a, λ by the method of least squares from equation (3,10), we easily calculate $\varkappa$ from the equation (3,08).

c) If it is not possible to use this method, especially where the relationships

$$y_0 < y_1 < y_2 < \dots < y_{n-1}$$

or

$$y_0 > y_1 > y_2 > \dots > y_{n-1}$$

do not hold, we use the method of successive approximations.

By means of the criterion of application we calculate as the first approximation the constant a_0 for the equation

$$y = \frac{K}{\varkappa + e^{at}} \tag{3,11}$$

i. e. for the equation

$$y = \frac{K}{\varkappa + e^{(a_0 + \varepsilon)t}}. \tag{3,12}$$

Now we have only to determine the constants K, $\varkappa$, ε or a since $a = a_0 + \varepsilon$. We call ε the correction term.

In equation (3,12) we expand $e^{\varepsilon t}$ using Taylor's series. For small ε, we can neglect terms from the second degree upwards. We then get

$$y = \frac{K}{\varkappa + e^{a_0 t}(1 + \varepsilon t)} \tag{3,13}$$

and after rearrangement

$$y\varkappa + ye^{a_0 t} + ye^{a_0 t}\varepsilon t = K. \tag{3,14}$$

That is

$$K - y\varkappa - yt\varepsilon e^{a_0 t} = ye^{a_0 t}. \tag{3,15}$$

We now apply the method of least squares to equation (3,15). That is, if we have the function

$$y = a_0 A_0(t) + a_1 A_1(t) + a_2 A_2(t) + \ldots + a_\nu A_\nu(t) \tag{3,16}$$

where a_k are constants, and $A_k(t)$ are known functions of t, then, in order to calculate the constants by the method of least squares, we can easily derive a set of normal equations:

$$\begin{array}{l} a_0 [A_0 A_0] + a_1 [A_0 A_1] + \ldots + a_\nu [A_0 A_\nu] = [A_0 Y_k] \\ a_0 [A_1 A_0] + a_1 [A_1 A_1] + \ldots + a_\nu [A_1 A_\nu] = [A_1 Y_k] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_0 [A_\nu A_0] + a_1 [A_\nu A_1] + \ldots + a_\nu [A_\nu A_\nu] = [A_\nu Y_k] \end{array} \tag{3,17}$$

where the brackets [], according to Gauss, indicate the sums of all values from 0 to $(n - 1)$.

If we write in equation (3,15)

$$\begin{aligned} A_0 &= 1 \\ A_1 &= - y_k \\ A_2 &= - y_k t_k e^{a_0 t_k} \\ Y_k &= y_k e^{a_0 t_k} \end{aligned}$$

then we have the following three equations for calculating the constants of the logistic curve:

$$\begin{aligned} Kn - \varkappa \Sigma y_k - \varepsilon \Sigma y_k t_k e^{a_0 t_k} &= \Sigma y_k e^{a_0 t_k}, \\ - K \Sigma y_k + \varkappa \Sigma y_k^2 + \varepsilon \Sigma y_k^2 t_k e^{a_0 t_k} &= \Sigma y_k^2 e^{a_0 t_k}, \\ - K \Sigma y_k e^{a_0 t_k} + \varkappa \Sigma y_k^2 t_k e^{a_0 t_k} + \varepsilon \Sigma y_k^2 t_k^2 e^{2a_0 t_k} &= \Sigma y_k^2 e^{2a_0 t_k}. \end{aligned} \tag{3,18}$$

We have then found the necessary constants for the analytical function

$$y = \frac{K}{\varkappa + e^{at}}.$$

In a similar manner constants were calculated by Pacák ([41]), who applied the logistic curve to the estimation of the future development of a certain enterprise.

When a logistic curve is used in practice it may be simplified in different ways, some of which are illustrated in the next paragraph.

Finally it is necessary to mention that the method of least squares is one of several possible methods by which empirical data can be expressed by an analytical law if the number of empirical values exceeds the number of parameters in the analytical relationship. A well known method for example is that of Pearson's χ^2 criterion where the condition is that χ^2 must be a minimum. Another method, elaborate and equally valuable from the theoretical standpoint, is that of Cauchy (Cauchy, Collected Works) which is not used probably only because no-one has taken the trouble to put it into a form suitable for computation (cf. the normal equations derived by Gauss, so useful for computation in the method of least squares).

(To be continued.)

REFORM OF STUDIES IN MATHEMATICAL STATISTICS, ACTUARIAL MATHEMATICS AND ECONOMETRICS IN CZECHOSLOVAKIA

Courses of study in Mathematical Statistics, Actuarial Mathematics and Econometrics in Czechoslovakia are provided in the Faculty of Science of King Charles' University in Prague and at the University of Technical Sciences in Prague; both these courses have been reorganised by the Act of May 16, 1946, No 122 of the Collection of Laws and Ordinances. The two years' course of lectures at the University of Technical Sciences — originally mainly devoted to the study of Insurance Technique and in existence since 1905 — has been fundamentally reformed and transformed into a four years course in the Department of Statistical and Actuarial Engineering. The course of lectures at the University, in existence since 1921 as a four year course, has also been reformed in view of the recent developments in actuarial science. The above mentioned act provides also for some facilities and lectures to be held jointly for students of both courses.

The subject matter covered by the course on Mathematical Statistics, Actuarial Mathematics and Econometrics in the *Faculty of Science of King Charles' University* may be seen from the Regulations concerning Examinations issued by the Ministry of Schools and Education on Feb. 12, 1947, No A-273.711-46-V.

According to these Regulations, the purpose of the course is to equip students with proficiency in carrying out mathematical-statistical, actuarial and econometric work; students must display this proficiency in two State Examinations taken before a Board of Examiners appointed by the Minister of Schools and Education from the ranks of professors and lecturers in the Faculty of Science as well as from outstanding experts active in practical profession.

The First State Examination must be taken not sooner than in the fourth semester (i. e. half-year term) of studies. To be allowed to enter for this exa-