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ADAPTIVE ALGORITHM FOR STOCHASTIC GALERKIN METHOD

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Abstract. We introduce a new tool for obtaining efficient a posteriori estimates of errors of approximate solutions of differential equations the data of which depend linearly on random parameters. The solution method is the stochastic Galerkin method. Polynomial chaos expansion of the solution is considered and the approximation spaces are tensor products of univariate polynomials in random variables and of finite element basis functions. We derive a uniform upper bound to the strengthened Cauchy-Bunyakowski-Schwarz constant for a certain hierarchical decomposition of these spaces. Based on this, an adaptive algorithm is proposed. A simple numerical example illustrates the efficiency of the algorithm. Only the uniform distribution of random variables is considered in this paper, but the results obtained can be modified to any other type of random variables.

Keywords: stochastic Galerkin method; a posteriori error estimate; strengthened Cauchy-Bunyakowski-Schwarz constant; adaptive refinement

MSC 2010: 65N22, 65C20

1. Introduction

Theoretical and engineering problems can be affected by uncertainties in input data. Numerical solution methods should then provide quantification of uncertainty of the approximate solution. The most popular such methods are the Monte Carlo method, collocation methods and the stochastic Galerkin method (SGM). The SGM is especially useful for solution of elliptic or parabolic problems dependent on random parameters. Approximation spaces are usually tensor products of finite element (FE) function spaces of physical variables and of spans of orthogonal polynomials of random variables. This leads to a huge dimensionality of the resulting systems of linear equations. Various methods can be applied to reduce the number of unknowns or for preconditioning these systems.

Hierarchical reduction of approximation spaces is a well known approach for preconditioning of the Galerkin method for numerical solution of partial differential

equations. Hierarchical bases of approximation spaces have been used for many decades [1], [2], [6]. Relations between levels of hierarchy can be also used for a posteriori error estimates of approximate solutions. An important role is played by the strengthened Cauchy-Bunyakowski-Schwarz (CBS) constant γ. If the upper bound to γ is sufficiently small, efficient a posteriori error estimates can be evaluated. In this paper we apply these ideas to a new context: to hierarchical approximation spaces of polynomials of random variables in the SGM.

First ideas in this direction were presented in [18], [21], where a hierarchical block preconditioning for approximation spaces using sets of complete polynomials was introduced. In our present paper we focus on tensor products of orthogonal polynomials. As for the bases of these sets, double orthogonal polynomials can be rather used [4], because they result in block diagonal matrices and non-intrusive methods. We show that for a certain types of hierarchy, the strengthened CBS constants are sufficiently small and the corresponding refinements of approximation spaces can be used for a posteriori error estimates.

Some approaches to a posteriori error estimates and adaptive algorithms can be found in the literature. They are based on the idea that the error estimates with respect to spatial and stochastic approximation spaces can be separated in some sense [4], [9], [13], [12]. Eigel et al. in [12], [13] describe and prove residual based a posteriori error estimates derived from the adequate approaches for deterministic problems. A marking strategy for both physical and stochastic degrees of freedom is based on the Dorfler property [12]. For dealing with the stochastic part of the error, the equivalence between the energy norm of the underlying problem and the energy norm of some related deterministic problem is used. Bryant et al. [9] and Butler et al. [10] use the adjoint-based methodology of the a posteriori error estimates and introduce adaptive algorithms based on evaluating linear quantities of interest. Bespalov et al. study hierarchical refinements of physical and of stochastic approximation spaces in [7]. They introduce several types of parameter-free two-sided error estimates. Instead of the energy scalar product connected to the problem including both types of variables, they use the scalar product associated with the deterministic problem and employ the strengthened CBS constant with respect to the spatial discretization spaces. In our present paper we propose the a posteriori error estimates based on a splitting of the stochastic approximation spaces and use the strengthened CBS constants regarding these spaces as well. This is a novel approach in the a posteriori estimation in the SGM. We assume that tensor products of polynomials of random variables are used in the SGM and prove a uniform upper bound to the strengthened CBS constants for certain hierarchical splittings of them. Based on this, we can show that projections of current errors onto refined subspaces can be used as error indicators. Using these estimates we define an adaptive algorithm.

This paper is organized as follows. In the next section we introduce the problem and recall the SGM. Namely, we describe the structures of the underlying systems of linear equations. In Section 3 we remind the idea of the a posteriori error estimates based on a hierarchical splitting of approximation spaces. We apply this approach to the sets of tensor products of orthonormal polynomials in random variables and prove a uniform upper bound to the strengthened CBS constant for this splitting. In Section 4 we introduce a simple adaptive algorithm based on the devised error estimates. A simple numerical example demonstrates that using the introduced adaptivity can reduce the computational cost of the SGM. Some concluding remarks are presented in Section 5.

2. Stochastic Galerkin method

Let $a(x, y)$ be a scalar random field represented by a finite sum

$$a(x,y) = a_0(x) + \sum_{k=1}^{N} a_k(x) y_k, \tag{2.1}$$

where $x \in \mathcal{D}$, $\mathcal{D}$ is a bounded domain with Lipschitz boundary, $\mathcal{D} \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, and $y = (y_1(\omega), \dots, y_N(\omega))\colon \Omega \to \mathbb{R}^N$ is a vector of N random variables which are defined by a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is a sample space with σ-algebra $\mathcal{F}$ and probability measure $\mathcal{P}$. Let the random variables y_k, $k = 1, \dots, N$, be independent and identically distributed and have zero mean and bounded variance and let us denote by ϱ the probability density function of each of y_k, $k = 1, \dots, N$. Thus the mean value of y_k is obtained by

$$\mathrm{E}[y_k] = \int_\Omega y_k(\omega)\,\mathrm{d}\mathcal{P}(\omega) = \int_{\mathbb{R}} z\,\varrho(z)\,\mathrm{d}z.$$

We assume that there exist a_1 and a_2 such that

$$0 < a_1 \leqslant a(x,y) \leqslant a_2 < \infty \quad \text{a.e. in } D \times \Omega. \tag{2.2}$$

Let us solve the elliptic equation in almost sure sense [23]

$$-\nabla \cdot (a(x,y)\nabla u(x,y)) = b(x), \tag{2.3}$$

with homogeneous Dirichlet condition on $\partial\mathcal{D} \times \Omega$, where ∂D is the boundary of $\mathcal{D}$, and where $b \in L^2(D)$. The gradient symbol ∇ denotes the differentiation with respect to the spatial (physical) variables x. Let us denote the Hilbert space $H =$

$H_0^1(\mathcal{D}) \times L^2_{\bar\varrho}(\mathbb{R}^N) = \{u(x,y);\ \int_{\mathbb{R}^N}\int_D |\nabla u(x,y)|^2\,\bar\varrho(y)\,\mathrm{d}x\,\mathrm{d}y < \infty,\ u(x,y) = 0, (x,y) \in \partial D \times \Omega\}$, where $\bar\varrho(y) = \Pi_{k=1}^N \varrho(y_k)$. The weak form of (2.3) then reads [4], [5], [11]: find $u(x,y) \in H$ such that

$$\int_{\mathbb{R}^N}\int_{\mathcal{D}} a(x,y)\nabla u(x,y)\nabla v(x,y)\bar\varrho(y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^N}\int_{\mathcal{D}} b(x)v(x,y)\bar\varrho(y)\,\mathrm{d}x\,\mathrm{d}y \tag{2.4}$$

for all $v(x,y) \in H$.

The truncated polynomial chaos approximation [8], [23] to the exact solution $u(x,y)$ of (2.4) is defined as

$$u(x,y) = \sum_{i=1}^{M} u_i(x)\Phi_i(y), \tag{2.5}$$

where $\Phi_1(y), \ldots, \Phi_M(y)$ are N-variate polynomials orthogonal in $L^2_{\bar\varrho}(\mathbb{R}^N)$. The polynomials $\Phi_i(y)$ can be chosen in the form of products of univariate polynomials orthogonal in $L^2_\varrho(\mathbb{R})$,

$$\Phi_i(y) = \prod_{k=1}^{N} \varphi_{i_k}(y_k),$$

where the degree of $\varphi_j(z)$ is equal to $j \in \{0, 1, \ldots\}$, and

$$\int_{\mathbb{R}} \varphi_i(z)\varphi_j(z)\varrho(z)\,\mathrm{d}z = \delta_{ij}.$$

The Hilbert space H is the completion of the sums $\sum\limits_{i=1}^{M} \Psi_i(x)\Phi_i(y)$, where $\Psi_i(x) \in H_0^1(\mathcal{D})$. We refer to [4], [11], [13] for the detailed convergence theory and a priori error estimates of the SGM.

The physical parts $u_i(x)$ of the expansion (2.5) are approximated by some finite element basis functions $\psi_r(x)$, $r = 1, \ldots, F$,

$$u_j(x) = \sum_{r=1}^{F} u_{jr}\psi_r(x). \tag{2.6}$$

A discretization space of the SGM for approximation of the solution $u(x,y)$ of (2.4) is then a tensor product of some finite element space $V_{\mathcal{D}} \subset H_0^1(D)$ of a dimension F and of a set of M orthogonal multivariate polynomials of N random variables. Basis functions are of the type $\psi_r(x)\Phi_i(y)$, $r = 1, \ldots, F$, $i = 1, \ldots, M$.

The coefficient $a(x,y)$ in (2.3) can be considered in a more general form than in (2.1). For example, the terms y_k can be substituted by polynomials in y_k of

higher orders. For example, the log-normal distribution of y in $a(x,y)$ leads to this type of the expansion of $a(x,y)$, see [22]. In this paper we consider only the linear case (2.1). Such an expression can be obtained as a truncated Karhunen-Loeve expansion of a general scalar random field $a(x,y)$ with some given covariance function $C(x,\tilde{x})$. Then $a_k(x)$ would be the normalized eigenfunctions of $C(x,\tilde{x})$ multiplied by square roots of the corresponding eigenvalues [8].

In this paper we assume that y_k are uniformly distributed on $\langle -1,1\rangle$. To satisfy (2.2), we assume that (see [4], [5], [13], [14], [17])

$$\sum_{k=1}^{N} \|a_k(x)\|_\infty < \inf_{x\in\mathcal{D}} a_0(x). \tag{2.7}$$

Note that this condition also guarantees positive definiteness of the associated Galerkin matrix, see Lemma 3.2.

For the polynomial chaos expansion (2.5) of $u(x,y)$ usually one of the following two sets of orthogonal N-variate polynomials is used: a tensor product of orthogonal univariate polynomials $\varphi_j(y_k)$, where the degrees of $\varphi_j(y_k)$ do not exceed p_k and where $p=(p_1,\dots,p_N)$ is a prescribed vector, or complete polynomials, which are products of univariate orthogonal polynomials, the total degree of which does not exceed a given constant q. In this paper we will consider the former type of the approximation polynomials and we denote

$$V_{p_1,\dots,p_N} = \left\{ \prod_{k=1}^{N} \varphi_{i_k}(y_k);\ \deg(\varphi_{i_k}) \leqslant p_k,\ k=1,\dots,N \right\}, \tag{2.8}$$

where $\deg(\varphi_j)$ means the degree of φ_j. The dimension of $V_{p_1,\dots,p_N}$ is

$$M = \dim V_{p_1,\dots,p_N} = \prod_{k=1}^{N} (p_k+1).$$

In practical problems, the tensor products of polynomials should be used rather than the complete sets of polynomials, if the impact of some of the variables y_k is rather greater than the influence of the others. This can happen, for example, if the magnitude of $a_k(x)$ is much larger than the magnitude of the other $a_j(x)$ on D. Then choosing the bounds p_k greater than the others may lead to a more precise approximation of the solution than by complete polynomials and thus to reducing the computational cost. In other words, if the magnitudes of $a_k(x)$ decay fast with growing k then p_k should decay correspondingly, see for example [15].

R e m a r k 2.1. Instead of the orthogonal polynomials $\varphi_j(z)$, a set of double-orthogonal polynomials $\tilde{\varphi}_j(z)$ can be used in the definition (2.8) of $V_{p_1,\dots,p_N}$, see [4]. For $i \neq j$ we then have

$$\int_{\mathbb{R}} \tilde{\varphi}_i(z)\tilde{\varphi}_j(z)\varrho(z)\,\mathrm{d}z = 0 \quad \text{and} \quad \int_{\mathbb{R}} z\tilde{\varphi}_i(z)\tilde{\varphi}_j(z)\varrho(z)\,\mathrm{d}z = 0.$$

Interestingly, this choice of the basis of $V_{p_1,\dots,p_N}$ results in the same set of linear equations as if we use the collocation method with some special choice of nodes [4]. The Galerkin matrix obtained from (2.4), (2.5) and (2.6) is then block diagonal.

The orthogonal polynomials $\varphi_j(z)$ satisfy a three-term recurrence formula [14], [19]

$$\varphi_{k+1}(z) = (q_k z + r_k)\varphi_k(z) + s_k\varphi_{k-1}(z) \tag{2.9}$$

for some appropriate q_k, r_k, s_k. If the weight function ϱ is symmetric, then $r_k = 0$ [14], [19], and thus

$$\int_{\mathbb{R}} \varphi_k^2(z)\varrho(z)\,\mathrm{d}z = 1, \quad \int_{\mathbb{R}} z\varphi_k^2(z)\varrho(z)\,\mathrm{d}z = 0, \tag{2.10}$$

$$\int_{\mathbb{R}} z\varphi_k(z)\varphi_{k+1}\varrho(z)\,\mathrm{d}z = \frac{1}{q_k} = -\frac{s_{k+1}}{q_{k+1}}. \tag{2.11}$$

Since we consider uniform distribution of the random variables y_k on $\langle -1, 1\rangle$ and $\varrho(z) = \frac{1}{2}$, $\varphi_j(z)$ are the orthogonal Legendre polynomials. The recursive formula for the normalized Legendre polynomials reads

$$(k+1)\varphi_{k+1}(z) = \sqrt{(2k+1)(2k+3)}z\varphi_k(z) - \frac{k\sqrt{2k+3}}{\sqrt{2k-1}}\varphi_{k-1}(z),$$

where $\varphi_0(z) = 1$ and $\varphi_1(z) = z\sqrt{3}$. Then instead of (2.11), we have

$$\int_{-1}^{1} z\varphi_k(z)\varphi_{k+1}(z)\frac{1}{2}\,\mathrm{d}z = \frac{k+1}{\sqrt{(2k+1)(2k+3)}}.$$

In this paper we do not distinguish between a function u and its vector representation with respect to some basis of the approximation space $V_{p_1,\dots,p_N} \times V_{\mathcal{D}}$. Galerkin discretization of (2.4) leads to the set of $M \times F$ linear equations with $M \times F$ unknowns,

$$Au = B, \tag{2.12}$$

where the elements of A and B are

$$\begin{aligned}
A_{ir,js} &= \int_{\mathbb{R}^N}\int_{\mathcal{D}} a(x,y)\nabla\psi_r(x)\nabla\psi_s(x)\Phi_i(y)\Phi_j(y)\bar{\varrho}(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^N}\int_{\mathcal{D}} a_0(x)\nabla\psi_r(x)\nabla\psi_s(x)\Phi_i(y)\Phi_j(y)\bar{\varrho}(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \sum_{k=1}^{N}\int_{\mathbb{R}^N}\int_{\mathcal{D}} a_k(x)y_k\nabla\psi_r(x)\nabla\psi_s(x)\Phi_i(y)\Phi_j(y)\bar{\varrho}(y)\,\mathrm{d}x\,\mathrm{d}y \\
&=: (A_0)_{ir,js} + \sum_{k=1}^{N}(A_k)_{ir,js}, \\
B_{ir} &= \int_{\mathbb{R}^N}\int_{\mathcal{D}} b(x)\psi_r(x)\Phi_i(y)\bar{\varrho}(y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Let us define matrices K_m, G_m, $m = 0, 1, \ldots, N$, by

$$(K_0)_{rs} = \int_{\mathcal{D}} a_0(x)\nabla\psi_s(x)\nabla\psi_r(x)\,\mathrm{d}x, \tag{2.13}$$

$$(K_m)_{rs} = \int_{\mathcal{D}} a_m(x)\nabla\psi_s(x)\nabla\psi_r(x)\,\mathrm{d}x,$$

$$(G_0)_{ij} = \int_{\mathbb{R}^N} \Phi_i(y)\Phi_j(y)\bar{\varrho}(y)\,\mathrm{d}y = \delta_{ij}, \tag{2.14}$$

$$(G_m)_{ij} = \int_{\mathbb{R}^N} y_m\Phi_i(y)\Phi_j(y)\bar{\varrho}(y)\,\mathrm{d}y.$$

Then the Galerkin matrix of the problem (2.4) is

$$A = G_0 \otimes K_0 + \sum_{m=1}^{N} G_m \otimes K_m. \tag{2.15}$$

Since the polynomials $\Phi_i(y)$ are tensor products of the univariate normalized orthogonal polynomials $\varphi_k(y_m)$, the structure of A can be even more specified. Let us order the basis functions $\psi_r(x)\Phi_j(y)$ lexicographically, where the indices at the physical basis functions $\psi_r(x)$ are changing fastest. Let the polynomials $\Phi_i(y) = \prod_{k=1}^{N} \varphi_{i_k}(y_k)$ be lexicographically ordered in such manner that the indices at the polynomials of the random variables y_k are changing faster than the polynomials of the random variables y_j whenever $k < j$. Let $G_{k,0}$ and $G_{k,1}$ be $(k+1)\times(k+1)$ matrices with elements

$$(G_{k,0})_{ij} = \int_{\mathbb{R}} \varphi_i(z)\varphi_j(z)\varrho(z)\,\mathrm{d}z = \delta_{ij},$$

$$(G_{k,1})_{ij} = \int_{\mathbb{R}} z\varphi_i(z)\varphi_j(z)\varrho(z)\,\mathrm{d}z = \delta_{|i-j|,1}\frac{1}{q_n}, \quad n = \min\{i,j\}. \tag{2.16}$$

Then $G_{k,0}$ is the $(k+1)\times(k+1)$ identity matrix and $G_{k,1}$ is a $(k+1)\times(k+1)$ non-negative tridiagonal matrix. In the sequel, we will denote the $m\times m$ identity matrix by I_m. Sometimes we will use only I if its dimension clearly follows from the context. For the introduced ordering of the basis functions $\psi_r(x)\Phi_i(y)$, the matrix A can be written in the form of a sum of tensor products, cf. [14],

$$\begin{aligned}A = {} & G_{p_N,0}\otimes G_{p_{N-1},0}\otimes\ldots\otimes G_{p_2,0}\otimes G_{p_1,0}\otimes K_0\\ & + G_{p_N,0}\otimes G_{p_{N-1},0}\otimes\ldots\otimes G_{p_2,0}\otimes G_{p_1,1}\otimes K_1\\ & + G_{p_N,0}\otimes G_{p_{N-1},0}\otimes\ldots\otimes G_{p_2,1}\otimes G_{p_1,0}\otimes K_2\\ & + \ldots + G_{p_N,1}\otimes G_{p_{N-1},0}\otimes\ldots\otimes G_{p_2,0}\otimes G_{p_1,0}\otimes K_N,\end{aligned}$$

or, more precisely,

$$\begin{aligned}A = {} & I_{p_N+1}\otimes I_{p_{N-1}+1}\otimes\ldots\otimes I_{p_2+1}\otimes I_{p_1+1}\otimes K_0\\ & + I_{p_N+1}\otimes I_{p_{N-1}+1}\otimes\ldots\otimes I_{p_2+1}\otimes G_{p_1,1}\otimes K_1\\ & + I_{p_N+1}\otimes I_{p_{N-1}+1}\otimes\ldots\otimes G_{p_2,1}\otimes I_{p_1+1}\otimes K_2\\ & + \ldots + G_{p_N,1}\otimes I_{p_{N-1}+1}\otimes\ldots\otimes I_{p_2+1}\otimes I_{p_1+1}\otimes K_N.\end{aligned}\tag{2.17}$$

Let us introduce some examples of the nonzero structures of the matrix A. For the uniformly distributed random variables y_m on $\langle -1,1\rangle$ and for $N=1$, $p_1=3$, the block structure of A is

$$A = \begin{pmatrix} K_0 & \frac{1}{\sqrt{3}}K_1 & 0 & 0\\ \frac{1}{\sqrt{3}}K_1 & K_0 & \frac{2}{\sqrt{15}}K_1 & 0\\ 0 & \frac{2}{\sqrt{15}}K_1 & K_0 & \frac{3}{\sqrt{35}}K_1\\ 0 & 0 & \frac{3}{\sqrt{35}}K_1 & K_0\end{pmatrix},$$

and for $N=2$, $p_1=2$, $p_2=1$, the block structure of A is

$$A = \begin{pmatrix} K_0 & \frac{1}{\sqrt{3}}K_1 & 0 & \frac{1}{\sqrt{3}}K_2 & 0 & 0\\ \frac{1}{\sqrt{3}}K_1 & K_0 & \frac{2}{\sqrt{15}}K_1 & 0 & \frac{1}{\sqrt{3}}K_2 & 0\\ 0 & \frac{2}{\sqrt{15}}K_1 & K_0 & 0 & 0 & \frac{1}{\sqrt{3}}K_2\\ \frac{1}{\sqrt{3}}K_2 & 0 & 0 & K_0 & \frac{1}{\sqrt{3}}K_1 & 0\\ 0 & \frac{1}{\sqrt{3}}K_2 & 0 & \frac{1}{\sqrt{3}}K_1 & K_0 & \frac{2}{\sqrt{15}}K_1\\ 0 & 0 & \frac{1}{\sqrt{3}}K_2 & 0 & \frac{2}{\sqrt{15}}K_1 & K_0\end{pmatrix},$$

Due to the orthogonality of the polynomials φ_j, the matrix A is block sparse and the nonzero structure of A depends on the numbering of the basis functions $\psi_r(x)\Phi_i(y)$. According to our ordering, A is a block matrix where each block is of the size $F\times F$. Such a block of A with coordinates i,j corresponds to a pair of polynomials Φ_i and Φ_j.

An example of the nonzero block-structure of A can be seen in Figure 1 for $p = (2, 2, 4)$. This scheme is obtained for the uniformly distributed random variables y_k on $\langle -1, 1\rangle$ and for the Legendre polynomials $\varphi_i(y_k)$. The same nonzero scheme can be obtained for normally distributed random variables y_k and for the Hermite polynomials. Two dashed lines split the matrix A into four blocks, the left upper diagonal block of A corresponds to $p = (2, 2, 3)$. Every small rectangle stands for a block matrix of the size $F \times F$. There are 237 such $F \times F$ nonzero blocks in this A. The small rectangles represent some multiples of matrices K_i, thus the nonzero structure of these small blocks is the same as the nonzero structure of the stiffness matrix of the corresponding deterministic problem. The nonzero block structure of A depends on the properties of the approximation polynomials $\Phi_j(y)$ and on the expansion of $a(x, y)$. Since for many types of basic random variables a three term recursive formula is available, the nonzero block structure of A remains the same but, of course, the spectral properties of A may change. See, for example, [14].

3. A posteriori error estimates

Let us assume a decomposition of some general finite dimensional approximation space V of (2.4) into a direct sum $V = U \oplus W$ and the resulting Galerkin system of linear equations with a positive definite matrix A in the form

$$Au = \begin{pmatrix} A_U & A_{UW} \\ A_{UW}^{\mathrm{T}} & A_W \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} B_U \\ B_W \end{pmatrix} = B. \tag{3.1}$$

The strengthened Cauchy-Bunyakowski-Schwarz (CBS) constant $\gamma \in \langle 0, 1)$, see [2], for subspaces U and W with respect to the energy scalar product $(u, v)_A = u^{\mathrm{T}} A v$ is the smallest $\gamma \geqslant 0$ satisfying

$$(u_1, u_2)_A^2 \leqslant \gamma^2 (u_1, u_1)_A (u_2, u_2)_A, \quad u_1 \in U,\ u_2 \in W,$$

or, equivalently,

$$(v_1^{\mathrm{T}} A_{UW} v_2)^2 \leqslant \gamma^2 v_2^{\mathrm{T}} A_U v_2 v_1^{\mathrm{T}} A_W v_1,$$

where v_2 and v_1 are any real vectors of appropriate dimensions.

Let us consider the original problem (3.1) and the coarse problem $A_U u = B_U$ and their solutions u_V and u_U, respectively,

$$Au_V = B \quad \text{and} \quad A_U u_U = B_U.$$

Let $\hat{u}$ be the exact solution of (2.4) and let $e_V = u_V - \hat{u}$ and $e_U = u_U - \hat{u}$ be the discretization errors of the solutions u_V and u_U, respectively. Let $\|\cdot\|_A = \sqrt{(\cdot,\cdot)_A}$ denote the energy norm. Let $\tilde{e}_W$ be the solution of

$$A_W \tilde{e}_W = B_W - A_{UW}^{\mathrm{T}} u_U.$$

Then from the Galerkin orthogonality we obtain [2], [7]

$$\|e_V\|_A^2 = \|e_U\|_A^2 - \|u_V - u_U\|_A^2 \tag{3.2}$$

and

$$\|\tilde{e}_W\|_A^2 \leqslant \|u_V - u_U\|_A^2 \leqslant \frac{1}{1-\gamma^2}\|\tilde{e}_W\|_A^2. \tag{3.3}$$

This means that for a sufficiently small γ, the error decay in the energy norm obtained after some refinement $V = U \oplus W$ of the approximation space U can be estimated by the energy norm of the solution $\tilde{e}_W$ of a small problem with the matrix A_W. Note that $\tilde{e}_W$ is the projection of e_U onto W with respect to the energy scalar product. The space W can be of a much smaller dimension than U, thus $\|\tilde{e}_W\|_A$ can be relatively easy to obtain. Moreover, if the saturation assumption [1] holds with some constant $\beta \in \langle 0, 1)$,

$$\|e_V\|_A \leqslant \beta \|e_U\|_A,$$

the energy norm of e_U can be estimated by [1]

$$\|\tilde{e}_W\|_A^2 \leqslant \|e_U\|_A^2 \leqslant \frac{1}{1-\gamma^2}\frac{1}{1-\beta^2}\|\tilde{e}_W\|_A^2,$$

which means that if γ and β are sufficiently small, the energy error of u_U is well approximated by $\|\tilde{e}_W\|_A$. To the best of our knowledge, no estimates of β are available for the approximation spaces V_p in the literature, cf. [7]. Some asymptotic convergence estimates [4] could provide ideas of what type the estimates could be.

Preconditioning and a posteriori error estimates using upper bounds to the strengthened CBS constant γ (algebraic multilevel methods, hierarchical Schur complement reduction) are well applicable to the finite element methods for deterministic problems, see for example [1], [2], [3]. To reduce the approximation error of the solution of (2.4), the physical or stochastic approximation spaces or both of them can be refined [7], [9], [10], [13]. It appears that the estimates of the physical and stochastic parts of the error can be separated in some sense. This follows from theoretical results [4], [7], [9], [13], and from computational experiments [7] as well. Moreover, the energy norm of the error generated by the scalar product on the

left-hand side of the weak form (2.4) is usually approximated by a norm derived form the related deterministic problem. The equivalence of these two norms follows from the strengthened assumption (2.7), namely, it is assumed that there exists $c_1 \in (0,1)$ such that [13], [17]

$$\sum_{k=1}^{N} \|a_k(x)\|_\infty \leqslant c_1 \inf_{x\in\mathcal{D}} a_0(x).$$

Our approach is different and has not appeared in the literature yet. We suppose that some algorithm for the refinement of the physical unknowns is available and we focus only on the stochastic part of the solution. Our aim is to find an efficient algorithm for refining the current stochastic approximation space $V_{p_1,\dots,p_N}$ to reduce the energy norm of the error as much as possible. Since we consider the tensor product of univariate polynomials φ_i, any refinement of a current approximation space means increasing the degree of some of the polynomials φ_i. Let us denote

$$W_{m;p_1,\dots,p_N} = \left\{ \prod_{k=1}^{N} \varphi_{i_k}(y_k);\ \deg(\varphi_{i_m}) = p_m + 1,\ \deg(\varphi_{i_k}) \leqslant p_k,\ k \neq m \right\}. \tag{3.4}$$

An example of the nonzero block structure of A can be seen in Figure 1 for $N = 3$. The two dashed lines split the matrix A according to the approximation spaces $V_{2,2,3}$ and $W_{3;2,2,3}$. The adaptive algorithm which we propose in this paper is based on an estimate of the error reduction using (3.2) and (3.3) and on proving a sufficiently small upper bound to the CBS constant γ for spaces $V_{p_1,\dots,p_N}$ and refining spaces $W_{k;p_1,\dots,p_N}$. We suggest to update the current solution space $V_{p_1,\dots,p_N}$ according to the largest estimate of the decay of the error.

In the following two lemmas we first show that under the assumption (2.7), the matrix A is positive definite.

Lemma 3.1. *Let the matrices K_m, $m = 0,\dots,N$, be defined by* (2.13) *and let the assumption* (2.7) *hold. Then for $m = 1,\dots,N$*

$$\|K_0^{-1/2} K_m K_0^{-1/2}\| \leqslant \frac{\|a_m(x)\|_\infty}{\inf\limits_{x\in D} a_0(x)}$$

and thus

$$\sum_{m=1}^{N} \|K_0^{-1/2} K_m K_0^{-1/2}\| < 1.$$

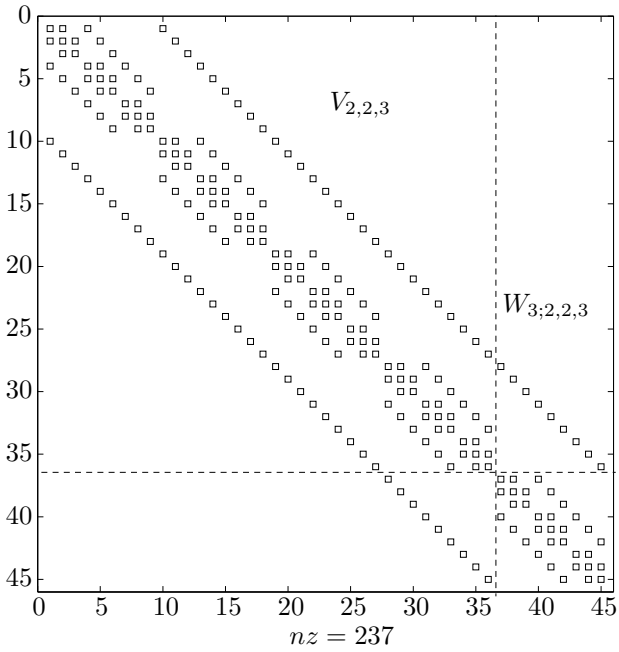

Figure 1. Example of a nonzero block scheme of $A = A_{2,2,4}$. The two dashed lines split the matrix according to the discretization of (2.4) in random variables of $V_{2,2,3}$ and $W_{3;2,2,3}$, where $V_{2,2,4} = V_{2,2,3} \oplus W_{3;2,2,3}$. The first diagonal block is $A_{2,2,3}$.

P r o o f. The matrix K_0 is the stiffness matrix of the corresponding deterministic problem, hence it is positive definite. We have for $m = 1, \dots, N$,

$$\begin{aligned}
\|K_0^{-1/2} K_m K_0^{-1/2}\| &= \sup_{u \in \mathbb{R}^F, u \neq 0} \left| \frac{u^{\mathrm{T}} K_0^{-1/2} K_m K_0^{-1/2} u}{u^{\mathrm{T}} u} \right| = \sup_{u \in \mathbb{R}^F, u \neq 0} \left| \frac{u^{\mathrm{T}} K_m u}{u^{\mathrm{T}} K_0 u} \right| \\
&\leqslant \sup_{u \in H_0^1(\mathcal{D}), u \neq 0} \left| \frac{\int_{\mathcal{D}} a_m(x) (\nabla u)^2 \, \mathrm{d}x}{\int_{\mathcal{D}} a_0(x) (\nabla u)^2 \, \mathrm{d}x} \right| \\
&\leqslant \sup_{u \in H_0^1(\mathcal{D}), u \neq 0} \left| \frac{\|a_m\|_\infty \int_{\mathcal{D}} (\nabla u)^2 \, \mathrm{d}x}{\inf\limits_{x \in \mathcal{D}} a_0(x) \int_{\mathcal{D}} (\nabla u)^2 \, \mathrm{d}x} \right| = \frac{\|a_m\|_\infty}{\inf\limits_{x \in \mathcal{D}} a_0(x)},
\end{aligned}$$

where u stands for vectors and for functions of $H_0^1(\mathcal{D})$ as well. □

In the following we will deal with the Galerkin matrices A defined by (2.17). Let us denote them according to the spaces used for the Galerkin projections: let $A_{p_1,\dots,p_N}$ be the matrix arising from the Galerkin projection of (2.4) onto $V_{p_1,\dots,p_N} \times V_{\mathcal{D}}$, where $V_{\mathcal{D}}$ is a finite element space, $V_{\mathcal{D}} \subset H_0^1(\mathcal{D})$. Then from (2.17), $A_{p_1,\dots,p_N}$ can be

generated recursively as the $(p_N+1)\times(p_N+1)$ block tridiagonal matrix

$$A_{p_1,\dots,p_N} = \begin{pmatrix} A_{p_1,\dots,p_{N-1}} & B_{N;1} & 0 & \dots & 0 \\ B_{N;1} & A_{p_1,\dots,p_{N-1}} & B_{N;2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & B_{N;p_N-1} & A_{p_1,\dots,p_{N-1}} & B_{N;p_N} \\ 0 & \dots & 0 & B_{N;p_N} & A_{p_1,\dots,p_{N-1}} \end{pmatrix}, \tag{3.5}$$

where $B_{N;k}$ is a block diagonal matrix

$$B_{N;k} = \frac{k}{\sqrt{(2k-1)(2k+1)}} I \otimes K_N$$

and

$$A_{p_1} = I \otimes K_0 + G_1 \otimes K_1.$$

Note that the Galerkin matrix $A_{p_1,\dots,p_N-1}$ associated with the projection onto $V_{p_1,\dots,p_N-1}\times V_D$ is equal to the left upper part of $A_{p_1,\dots,p_N}$, namely to the matrix composed from the left upper $p_N\times p_N$ blocks. We would like to point out the difference between $A_{p_1,\dots,p_N-1}$ and $A_{p_1,\dots,p_{N-1}}$. The matrix $A_{p_1,\dots,p_{N-1}}$ is a diagonal block of $A_{p_1,\dots,p_N-1}$.

Matrix K_0 is positive definite. Let $D_0 = I_M \otimes K_0^{-1/2}$ and denote

$$\widetilde{A}_{p_1,\dots,p_N} = D_0 A_{p_1,\dots,p_N} D_0.$$

Lemma 3.2. *Let y_m, $m=1,\dots,N$, be random variables uniformly distributed on $\langle -1,1\rangle$. Then for any N and for any vector $p=(p_1,\dots,p_N)$ the matrix $A_{p_1,\dots,p_N}$ is symmetric and positive definite and for all vectors $v\in\mathbb{R}^{(M\times F)\times 1}$, $v\neq 0$,*

$$0 < \frac{v^{\mathrm{T}} A_{p_1,\dots,p_N} v}{v^{\mathrm{T}} v} \leqslant 2\|K_0\|.$$

Especially, every matrix

$$A_{p_1,\dots,p_{N-1},1} = \begin{pmatrix} A_{p_1,\dots,p_{N-1}} & B_{N;1} \\ B_{N;1} & A_{p_1,\dots,p_{N-1}} \end{pmatrix}$$

is positive definite and thus $r(A^{-1}_{p_1,\dots,p_{N-1}} B_{N;1}) < 1$, where r denotes the spectral radius.

P r o o f. The symmetry of $A_{p_1,\dots,p_N}$ follows from its definition (2.15) and from relations (2.13) and (2.14). Let $c=(c_1,\dots,c_N)$, $\|c\|\neq 0$, and let

$$u = \sum_{k=0}^{m} c_k \varphi_k(z),$$

where $\varphi_k(z)$ are the normalized orthogonal Legendre polynomials. Then for $G_{m,1}$ defined by (2.16)

$$|c^{\mathrm{T}}G_{m,1}c| = \left|\int_{-1}^{1} zu^2(z)\frac{1}{2}\,\mathrm{d}z\right| \leqslant \int_{-1}^{1} |z|u^2(z)\frac{1}{2}\,\mathrm{d}z \leqslant \int_{-1}^{1} u^2(z)\frac{1}{2}\,\mathrm{d}z = c^{\mathrm{T}}c$$

and the equality cannot be achieved except for the case where $u(z) = 0$ on $\langle -1, 1\rangle$, which means $c = 0$. Thus we have $\|G_{m,1}\| < 1$ for $m = 0, 1, 2, \ldots$ Then we have

$$\begin{aligned}
\frac{v^{\mathrm{T}}A_{p_1,\ldots,p_N}v}{v^{\mathrm{T}}v} &= \frac{v^{\mathrm{T}}(I\otimes K_0)v + \sum_{k=1}^{N} v^{\mathrm{T}}(I\otimes G_{p_k,1}\otimes I\otimes K_k)v}{v^{\mathrm{T}}v}\\
&= \frac{v^{\mathrm{T}}v + \sum_{k=1}^{N} v^{\mathrm{T}}(I\otimes G_{p_k,1}\otimes I\otimes K_0^{-1/2}K_kK_0^{-1/2})v}{v^{\mathrm{T}}(I\otimes K_0^{-1})v}\\
&= \frac{v^{\mathrm{T}}v + \sum_{k=1}^{N} v^{\mathrm{T}}(I\otimes G_{p_k,1}\otimes I\otimes K_0^{-1/2}K_kK_0^{-1/2})v}{v^{\mathrm{T}}v}\,\frac{v^{\mathrm{T}}v}{v^{\mathrm{T}}(I\otimes K_0^{-1})v},
\end{aligned}$$

where I stands for the identity matrices of the appropriate sizes. Thus from Lemma 3.1

$$\begin{aligned}
\sup_{v\neq 0}\frac{v^{\mathrm{T}}A_{p_1,\ldots,p_N}v}{v^{\mathrm{T}}v} &\leqslant \left(1 + \sum_{k=1}^{N}\|G_{p_k,1}\|\|K_0^{-1/2}K_kK_0^{-1/2}\|\right)\|K_0\|\\
&\leqslant \left(1 + \max_{i=1,\ldots,N}\|G_{p_k,1}\|\right)\|K_0\| \leqslant 2\|K_0\|,\\
\inf_{v\neq 0}\frac{v^{\mathrm{T}}A_{p_1,\ldots,p_N}v}{v^{\mathrm{T}}v} &\geqslant \left(1 - \sum_{k=1}^{N}\|G_{p_k,1}\|\|K_0^{-1/2}K_kK_0^{-1/2}\|\right)\|K_0^{-1}\|^{-1}\\
&\geqslant \left(1 - \max_{k=1,\ldots,N}\|G_{p_k,1}\|\right)\|K_0^{-1}\|^{-1} > 0.
\end{aligned}$$

□

In the next lemma we prove an auxiliary result for the main theorem of this paper.

Lemma 3.3. *Let matrix M_m be of the $m\times m$ block tridiagonal form*

$$M_m = \begin{pmatrix}
I & \frac{1}{\sqrt{3}}\widetilde{M} & 0 & \ldots & 0\\
\frac{1}{\sqrt{3}}\widetilde{M} & I & \frac{2}{\sqrt{15}}\widetilde{M} & \ldots & 0\\
\ldots & \ldots & \ldots & \ldots & \ldots\\
0 & \ldots & \frac{m-2}{\sqrt{(2m-5)(2m-3)}}\widetilde{M} & I & \frac{m-1}{\sqrt{(2m-3)(2m-1)}}\widetilde{M}\\
0 & \ldots & 0 & \frac{m-1}{\sqrt{(2m-3)(2m-1)}}\widetilde{M} & I
\end{pmatrix},$$

where $\widetilde{M}$ *is symmetric and* $\|\widetilde{M}\| \leqslant 1$. *Then the strengthened CBS constant* γ *for the splitting of* M_m *into the* 2×2 *blocks*

$$M_m = \begin{pmatrix} M_{m-1} & \begin{matrix} 0 \\ \vdots \\ 0 \\ \frac{m-1}{\sqrt{(2m-3)(2m-1)}}\widetilde{M} \end{matrix} \\ \begin{matrix} 0, \dots, 0, \frac{m-1}{\sqrt{(2m-3)(2m-1)}}\widetilde{M} \end{matrix} & I \end{pmatrix} \tag{3.6}$$

is bounded by

$$\gamma^2 \leqslant \frac{m-1}{2m-1}.$$

P r o o f. Let M be a block matrix and let $(M)_{\mathrm{ldb}}$ denote its last (lower right) diagonal block. For the strengthened CBS constant γ corresponding to the 2×2 decomposition (3.6) of M_m we have

$$\gamma^2 \leqslant \Big\| \frac{(m-1)^2}{(2m-3)(2m-1)} r\big(\widetilde{M}(M_{m-1}^{-1})_{\mathrm{ldb}}\widetilde{M}\big) \Big\| \leqslant \frac{(m-1)^2}{(2m-3)(2m-1)} \|(M_{m-1}^{-1})_{\mathrm{ldb}}\|,$$

where $r(M)$ denotes the spectral radius of M. The recursive evaluation of the norm $\|(M_{m-1}^{-1})_{\mathrm{ldb}}\|$ can start with

$$\|(M_2^{-1})_{\mathrm{ldb}}\| = \Big\| \Big(I - \frac{1}{3}\widetilde{M}^2\Big)^{-1} \Big\| \leqslant \sum_{k=0}^{\infty} \frac{1}{3^k} \|\widetilde{M}^2\|^k \leqslant \frac{3}{2}.$$

By induction we can prove that

$$\|(M_m^{-1})_{\mathrm{ldb}}\| \leqslant \frac{2m-1}{m}.$$

Indeed, supposing

$$\|(M_{m-1}^{-1})_{\mathrm{ldb}}\| \leqslant \frac{2m-3}{m-1},$$

we get

$$\begin{aligned} \|(M_m^{-1})_{\mathrm{ldb}}\| &= \Big\| \Big(I - \frac{(m-1)^2}{(2m-3)(2m-1)} \widetilde{M}(M_{m-1}^{-1})_{\mathrm{ldb}}\widetilde{M}\Big)^{-1} \Big\| \\ &\leqslant \sum_{k=0}^{\infty} \Big(\frac{(m-1)^2}{(2m-3)(2m-1)} \big\|\widetilde{M}(M_{m-1}^{-1})_{\mathrm{ldb}}\widetilde{M}\big\| \Big)^k \\ &\leqslant \sum_{k=0}^{\infty} \Big(\frac{(m-1)^2}{(2m-3)(2m-1)} \frac{2m-3}{m-1} \Big)^k = \frac{1}{1-(m-1)/(2m-1)} = \frac{2m-1}{m}. \end{aligned}$$

Finally, we obtain

$$\gamma^2 \leqslant \frac{(m-1)^2}{(2m-3)(2m-1)} \frac{2m-3}{m-1} = \frac{m-1}{2m-1}.$$

□

The next theorem contains the main result of this paper. It proves the upper bound to the strengthened CBS constant γ for the spaces $V_{p_1,\ldots,p_N} \times V_{\mathcal{D}}$ and $W_{m;p_1,\ldots,p_N} \times V_{\mathcal{D}}$ for any $m = 1, \ldots, N$, and for the energy scalar product defined by the left-hand side of (2.4).

Theorem 3.1. *Let the random variables y_m, $m = 1, \ldots, N$, be uniformly distributed on $\langle -1, 1\rangle$. Then the strengthened CBS constant $\gamma_{k;p_1,\ldots,p_k,\ldots,p_N}$ for spaces*

$$V_{p_1,\ldots,p_k,\ldots,p_N} = V_{p_1,\ldots,p_k-1,\ldots,p_N} \oplus W_{k;p_1,\ldots,p_k-1,\ldots,p_N}$$

is bounded by

$$\gamma^2_{k;p_1,\ldots,p_N} \leqslant \frac{p_k}{2p_k+1}$$

P r o o f. Without any loss of generality we can assume $k = N$. Thus we consider the splitting $V_{p_1,\ldots,p_N} = V_{p_1,\ldots,p_N-1} \oplus W_{N;p_1,\ldots,p_N-1}$. Let us consider the scheme of the corresponding matrix $A_{p_1,\ldots,p_N}$ as in (3.5). Then to obtain the upper bound to $\gamma_{N;p_1,\ldots,p_N}$, we need to find the maximum singular value of the matrix

$$Q = A^{-1/2}_{p_1,\ldots,p_{N-1},p_N-1}(0,\ldots,0,B_{N;p_N})^{\mathrm{T}} A^{-1/2}_{p_1,\ldots,p_{N-1}},$$

or, equivalently,

$$\gamma^2_{N;p_1,\ldots,p_N} \leqslant r(Q^{\mathrm{T}}Q) = r(A^{-1/2}_{p_1,\ldots,p_{N-1}} B_{N;p_N} (A^{-1}_{p_1,\ldots,p_{N-1},p_N-1})_{\mathrm{ldb}} B_{N;p_N} A^{-1/2}_{p_1,\ldots,p_{N-1}}),$$

where $r(M)$ is the spectral radius of M and $(M)_{\mathrm{ldb}}$ stands for the last diagonal block of M. Notice the difference between $p_1, \ldots, p_N - 1$ and $p_1, \ldots, p_{N-1}$. Let us also point out that

$$B_{N;k} = \frac{k}{\sqrt{(2k-1)(2k+1)}} I \otimes K_N$$

and that all of the diagonal blocks of $A_{p_1,\ldots,p_N}$ are the same matrices $A_{p_1,\ldots,p_{N-1}}$. Let us denote

$$D_A = A^{-1/2}_{p_1,\ldots,p_{N-1}}$$

and

$$\widetilde{B}_{N;k} = D_A B_{N;k} D_A, \quad \widetilde{A}_{p_1,\ldots,p_{N-1}} = (I_{p_N+1} \otimes D_A) A_{p_1,\ldots,p_{N-1}} (I_{p_N+1} \otimes D_A).$$

Then

$$r(Q^{\mathrm{T}}Q) = r(\widetilde{B}_{N;p_N}(\widetilde{A}^{-1}_{p_1,\ldots,p_{N-1},p_N-1})_{\mathrm{ldb}}\widetilde{B}_{N;p_N}),$$

where

$$\widetilde{A}_{p_1,\ldots,p_N} = \begin{pmatrix} I & \widetilde{B}_{1;N} & 0 & \ldots & 0 \\ \widetilde{B}_{1;N} & I & \widetilde{B}_{2;N} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \widetilde{B}_{p_N-1;N} & I & \widetilde{B}_{p_N;N} \\ 0 & \ldots & 0 & \widetilde{B}_{p_N;N} & I \end{pmatrix}$$

$$= \begin{pmatrix} I & \frac{1}{\sqrt{3}}\widetilde{B} & 0 & \ldots & 0 \\ \frac{1}{\sqrt{3}}\widetilde{B} & I & \frac{2}{\sqrt{15}}\widetilde{B} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \frac{p_N-1}{\sqrt{(2p_N-3)(2p_N-1)}}\widetilde{B} & I & \frac{p_N}{\sqrt{(2p_N-1)(2p_N+1)}}\widetilde{B} \\ 0 & \ldots & 0 & \frac{p_N}{\sqrt{(2p_N-1)(2p_N+1)}}\widetilde{B} & I \end{pmatrix},$$

where $\widetilde{B} = D_A(I \otimes K_N)D_A$. From Lemma 3.1 and Lemma 3.2 we get $\|\widetilde{B}\| < 1$. Thus $r(Q^{\mathrm{T}}Q)$ can be estimated recursively and from Lemma 3.3 we obtain $\gamma^2_{N;p_1,\ldots,p_N} \leqslant p_N/(2p_N+1)$. □

R e m a r k 3.1. We would like to emphasize that the refining spaces $W_{m;p_1,\ldots,p_N} \times V_D$, $m = 1,\ldots,N$, are pairwise orthogonal with respect to the energy scalar product. Indeed, for example, for $N = 3$ and $\Phi_i(y) \in W_{k,p_1+1,p_2,p_3}$, $\Phi_j(y) \in W_{m,p_1,p_2+1,p_3}$, $k \neq m$, we have

$$\int_{\mathbb{R}^3} \Phi_i(y)\Phi_j(y)\bar{\varrho}(y)\,\mathrm{d}y = 0$$

and

$$\int_{\mathbb{R}^3} y_m\Phi_i(y)\Phi_j(y)\bar{\varrho}(y)\,\mathrm{d}y = 0$$

for any $m = 1,2,3$. This means that the projection of the error of u obtained in $V_{p_1,\ldots,p_N} \times V_D$ onto the span of $\bigcup_{k=1}^{N} W_{k;p_1,\ldots,p_N} \times V_D$ can be decomposed into N orthogonal components, which are projections onto the spaces $W_{m;p_1,\ldots,p_N} \times V_D$, $m = 1,\ldots,N$.

4. Adaptive algorithm and numerical example

The derived uniform upper bounds to the strengthened CBS constants γ allow us to use the projections of current errors onto the spaces $W_{m;p_1,\ldots,p_N} \times V_D$ as reliable estimates of the discretization error associated with each particular random variable y_m, $m = 1,\ldots,N$, and to guess what refinement would decrease the energy norm of the error as much as possible. Based on such estimates we propose an adaptive algorithm. The error of a current solution is projected onto the spaces $W_{m;p_1,\ldots,p_N} \times$

V_D for every $m = 1, \ldots, N$. Since $\gamma^2 < 1/2$, we obtain from (3.2) and (3.3) the estimates of the error decay for each m and thus we can decide the degree of which polynomial $\varphi_i(y_m)$ should be increased. Moreover, assuming β be sufficiently small, we obtain a quite accurate estimate of the energy norm of the current error.

Adaptive algorithm.

1. Choose an initial vector $p = (p_1, \ldots, p_N)$.
2. Compute the Galerkin solution of (2.4) in $V_{p_1,\ldots,p_N} \times V_{\mathcal{D}}$.
3. Find a projection of the current error onto the spaces $W_{k;p_1,\ldots,p_N} \times V_D$, $k = 1, \ldots, N$, with the largest energy norm. Denote the corresponding index by m.
4. Update $(p_1, \ldots, p_m, \ldots, p_N) := (p_1, \ldots, p_m + 1, \ldots, p_N)$ and go to Step 2.

To examine the proposed algorithm let us consider a simple 1D problem. Let us emphasize that the efficiency of the algorithm does not depend on the physical dimensionality of the problem. Let us solve the equation $-(au')' = 1$ on $(0, 1)$, $u(0) = 0$, $u(1) = 0$, where $a(x, y) = a_0 + a_1(x)y_1 + a_2(x)y_3 + a_3(x)y_3$, thus $N = 3$. Let $a_0 = 1$ and let y_k be independent and uniformly distributed random variables on $\langle -1, 1\rangle$. Let $a_k(x)$ be piecewise constant,

$$\begin{aligned} a_1(x) &= 0.95, \quad && x \in (0, 1/3), \\ a_2(x) &= 0.1, && x \in (1/3, 2/3), \\ a_3(x) &= 0.5, && x \in (2/3, 1), \\ a_k(x) &= 0, && \text{otherwise}, \quad k = 1, 2, 3. \end{aligned}$$

We use the uniform mesh on $(0, 1)$ and the piecewise linear FE basis functions, $F = 20$ or $F = 41$; thus some nodes coincide with the discontinuity points of $a_k(x)$. Then we can suppose that the physical discretization error is relatively small compared to the stochastic discretization error. We compute the energy norm of the error e_U of the current approximate solution both for the adaptive refinement of the tensor product (TP) of polynomials and for the sets of complete polynomials (CP) with growing total degree. These norms are plotted by the solid lines in Figure 2. The dashed lines indicate the largest energy norm of the projections of e_U onto the subspace $W_{k;p_1,p_2,p_3} \times V_D$, $k = 1, 2, 3$, in the case of the TP. For the CP scheme, the dashed line depicts the energy norm of the projection of e_U onto the space of polynomials of the total degree equal to $q + 1$, where q is the largest total degree of the current approximation polynomials. The sequences of vectors p^n such that the spaces $V_{p^n} \times V_D$ are used in the Adaptive algorithm are the same for 20 and for 41 spatial nodal points:

$$p^n\colon\ (111), (211), (311), (411), (412), (512), (612), (712), (812), (813).$$

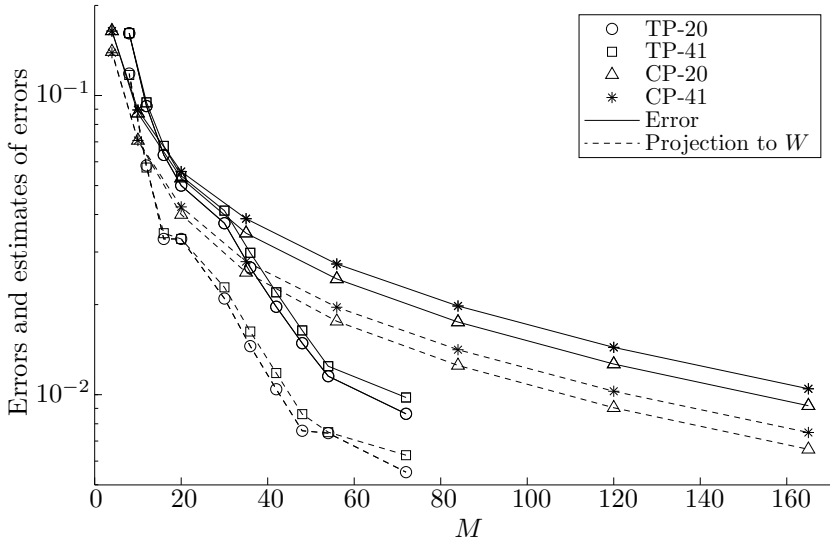

Figure 2. Solid lines: Energy norms of current errors $\|e_U\|_A$ of the solutions in $V_{p_1,p_2,p_3} \times V_D$ for the adaptively refined TP scheme and of the solutions using the CP schemes dependent on the number of stochastic basis function M. Dashed lines: Energy norms of the largest projections of the current error e_U onto $W_{k;p_1,p_2,p_3} \times V_D$ for the TP scheme and energy norm of the projection of e_U onto the space of polynomials of the total degree equal to $q+1$, where q is the largest total degree of the current approximation polynomials for the CP scheme. Numbers of physical nodes are $F = 20$ or $F = 41$.

Of course, the first elements p_1^n of $p^n = (p_1^n, p_2^n, p_3^n)$ are the largest ones due to the largest magnitude of $a_1(x)$. From Figure 2 we can see that the adaptive refinement of the approximation stochastic spaces leads to a significant memory saving. We can also notice that the largest error projections onto $W_{k;p_1,p_2,p_3}$ well indicate what refinement should be made in each step of the Adaptive algorithm. It can be also seen that different spatial meshes almost do not influence the error estimates and particular refining steps of the Adaptive algorithm.

4.1. Discussion. Function spaces used to approximate the stochastic part of the solution obtained by the SGM have some properties different from spaces usually used in the FE methods. This is caused by the following reasons: the domains of random variables are very regular, usually hypercubes, and thus classes of orthogonal polynomials can be used as the basis functions; many random variables can be employed, even tens or hundreds; no derivatives with respect to the random variables are considered. This is why completely new forms of preconditioning, a posteriori estimates and adaptive strategies can be devised. In this paper, we introduce an efficient algorithm of refinement of the approximation spaces of the SGM in which

tensor products of polynomials in random variables are used. Following the first results [18], we here introduce another form of exploiting the uniform upper bound to the strengthened CBS constant for a certain hierarchical decomposition of the stochastic approximation spaces. We prove that for the uniformly distributed random variables this upper bounds are sufficiently small for any degrees of polynomials and for any numbers of random variables. Then a kind of hierarchical a posteriori error estimates can be applied to define an adaptive algorithm.

Let us emphasize that instead of the orthogonal polynomials $\varphi_k(y_m)$, $k = 1, \ldots, p_m$, $m = 1, \ldots, N$, one can use the sets of double orthogonal polynomials [4] which lead to a non-intrusive computational scheme, because the matrix $A_{p_1,\ldots,p_N}$ becomes block diagonal with diagonal blocks of the same size as the underlying deterministic problem. In this case, the Galerkin projection of current errors onto the spaces $W_{m;p_1,\ldots,p_N}$ can be computed in the same way as in the case of orthogonal polynomials. But, of course, after any refinement of an approximation space of double orthogonal polynomials, all of the current polynomials in some variable y_m must be substituted to obtain a new set of double orthogonal polynomials of a higher degree.

In this paper, only the uniform distribution of the random parameters is considered. Of course, any other distribution can be used and new upper bounds to the strengthened CBS constants can be obtained. The main limitation of the presented approach is the linearity of $a(x, y)$ with respect to random parameters. If the coefficient $a(x, y)$ were in a more general form than in (2.1), the matrix A of the resulting system of linear equations would have a different structure and can be even full [20]. Instead of the orthogonal polynomials, we can use wavelets, piecewise polynomials or other functions to approximate the solution [16]. For all such problems, new techniques of hierarchical a posteriori error estimates can be studied.

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References

[1] *M. Ainsworth, J. T. Oden*: A Posteriori Error Estimation in Finite Element Analysis. Pure and Applied Mathematics. A Wiley-Interscience Series of Texts, Monographs, and Tracts, Wiley, Chichester, 2000.

[2] *O. Axelsson*: Iterative Solution Methods. Cambridge Univ. Press, Cambridge, 1994.

[3] *O. Axelsson, R. Blaheta, M. Neytcheva*: Preconditioning of boundary value problems using elementwise Schur complements. SIAM J. Matrix Anal. Appl. *31* (2009), 767–789.

[4] *I. Babuška, F. Nobile, R. Tempone*: A stochastic collocation method for elliptic partial differential equations with random input data. SIAM Rev. *52* (2010), 317–355.

[5] *I. Babuška, R. Tempone, G. E. Zouraris*: Galerkin finite element approximations of stochastic elliptic partial differential equations. SIAM J. Numer. Anal. *42* (2004), 800–825.

[6] *R. E. Bank, T. F. Dupont, H. Yserentant*: The hierarchical basis multigrid method. Numer. Math. *52* (1988), 427–458.
[7] *A. Bespalov, C. E. Powell, D. Silvester*: A priori error analysis of stochastic Galerkin mixed approximations of elliptic PDEs with random data. SIAM J. Numer. Anal. *50* (2012), 2039–2063.
[8] *D. Bosq*: Linear Processes in Function Spaces. Theory and Applications. Lecture Notes in Statistics 149, Springer, New York, 2000.
[9] *C. M. Bryant, S. Prudhomme, T. Wildey*: A posteriori error control for partial differential equations with random data. ICES Report 13-08, 2013, `https://www.ices.utexas.edu/media/reports/2013/1308.pdf`.
[10] *T. Butler, P. Constantine, T. Wildey*: A posteriori error analysis of parameterized linear systems using spectral methods. SIAM J. Matrix Anal. Appl. *33* (2012), 195–209.
[11] *M. K. Deb, I. M. Babuška, J. T. Oden*: Solution of stochastic partial differential equations using Galerkin finite element techniques. Comput. Methods Appl. Mech. Eng. *190* (2001), 6359–6372.
[12] *M. Eigel, C. J. Gittelson, C. Schwab, E. Zander*: A convergent adaptive stochastic Galerkin finite element method with quasi-optimal spatial meshes. Preprint No. 1911. Weierstrass Institute für Angewadte Analysis und Stochastic, Berlin, 2013.
[13] *M. Eigel, C. J. Gittelson, C. Schwab, E. Zander*: Adaptive stochastic Galerkin FEM. Comput. Methods Appl. Mech. Eng. *270* (2014), 247–269.
[14] *O. G. Ernst, E. Ullmann*: Stochastic Galerkin matrices. SIAM J. Matrix Anal. Appl. *31* (2010), 1848–1872.
[15] *P. Frauenfelder, C. Schwab, R. A. Todor*: Finite elements for elliptic problems with stochastic coefficients. Comput. Methods Appl. Mech. Eng. *194* (2005), 205–228.
[16] *O. P. Le Maître, O. M. Knio, H. N. Najm, R. G. Ghanem*: Uncertainty propagation using Wiener-Haar expansions. J. Comput. Phys. *197* (2004), 28–57.
[17] *C. E. Powell, H. C. Elman*: Block-diagonal preconditioning for spectral stochastic finite-element systems. IMA J. Numer. Anal. *29* (2009), 350–375.
[18] *I. Pultarová*: Hierarchical preconditioning for the stochastic Galerkin method: upper bounds to the strengthened CBS constants. Submitted. Available in ERC-CZ project LL1202 database, `http://more.karlin.mff.cuni.cz`.
[19] *A. Ralston*: A First Course in Numerical Analysis. McGraw-Hill Book, New York, 1965.
[20] *B. Sousedík, R. G. Ghanem*: Truncated hierarchical preconditioning for the stochastic Galerkin FEM. Int. J. Uncertain. Quantif. *4* (2014), 333–348.
[21] *B. Sousedík, R. G. Ghanem, E. T. Phipps*: Hierarchical Schur complement preconditioner for the stochastic Galerkin finite element methods. Numer. Linear Algebra Appl. *21* (2014), 136–151.
[22] *E. Ullmann, H. C. Elman, O. G. Ernst*: Efficient iterative solvers for stochastic Galerkin discretizations of log-transformed random diffusion problems. SIAM J. Sci. Comput. *34* (2012), A659–A682.
[23] *D. Xiu*: Numerical Methods for Stochastic Computations. A Spectral Method Approach. Princeton University Press, Princeton, 2010.

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