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Jiakuan Lu; Yanyan Qiu

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ON SOLVABILITY OF FINITE GROUPS WITH SOME ss-SUPPLEMENTED SUBGROUPS

Jiakuan Lu, Yanyan Qiu, Guilin

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Abstract. A subgroup H of a finite group G is said to be ss-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is s-permutable in K. In this paper, we first give an example to show that the conjecture in A. A. Heliel's paper (2014) has negative solutions. Next, we prove that a finite group G is solvable if every subgroup of odd prime order of G is ss-supplemented in G, and that G is solvable if and only if every Sylow subgroup of odd order of G is ss-supplemented in G. These results improve and extend recent and classical results in the literature.

Keywords: ss-supplemented subgroup; solvable group; supersolvable group

MSC 2010: 20D10, 20D20

1. Introduction

All groups considered in this paper are finite. Recall that a subgroup H of a group G is said to be s-permutable in G if H permutes with every Sylow subgroup P of G, that is, $HP = PH$ (see [13]); H is said to be c-supplemented in G if G has a subgroup K such that $G = HK$ and $H \cap K \leqslant H_G$, where H_G is the normal core of H in G (see [3]); H is said to be ss-quasinormal in G if there is a subgroup K of G such that $G = HK$ and H permutes with every Sylow subgroup of K (see [14]). Recently, Guo and Lu in [7] introduced the following concept, which covers both the ss-quasinormality and c-supplementation concepts.

Definition 1.1. A subgroup H of G is said to be ss-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is s-permutable in K.

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It is clear that each of the c-supplementation and ss-quasinormality concepts implies ss-supplementation. The following example shows that the ss-supplementation is a true generalization of the ss-quasinormality and c-supplementation concepts.

Example 1.2 ([7], Example 2.3)**.** Let $G = S_4 \times P$, where S_4 is the symmetric group of degree 4 and $P = \langle x, y\colon\ x^{16} = y^4 = 1,\ x^y = x^3\rangle$, and let $H = C_2 \times P_1$, $K = A_4 \times P$, where $C_2 = \langle(34)\rangle$, $P_1 = \langle y^2\rangle$ and A_4 is the alternating group on four symbols. Then $G = HK$ and $H \cap K$ is s-permutable in K since $H \cap K \cong P_1$. Hence H is ss-supplemented in G. However, H is neither c-supplemented nor ss-quasinormal in G.

In the literature, many authors have investigated the structure of the group G under the assumption that some subgroups of G are well-situated in G. For example, Hall in [9] proved that a group G is solvable if and only if each Sylow subgroup of G is complemented in G. Arad and Ward in [1] obtained a nice generalization of Hall's theorem. In fact, they proved that a group G is solvable if the Sylow 2-subgroups and Sylow 3-subgroups of G are complemented in G. Moreover, Hall in [10] proved that a group G is supersolvable with elementary abelian Sylow subgroups if and only if every subgroup of G is complemented in G. Ballester-Bolinches and Guo in [4] analysed the class of groups for which every subgroup of prime order is complemented. In fact, they proved that G is supersolvable if every subgroup of prime order of G is complemented in G.

In [3], Ballester-Bolinches, Wang and Guo proved that a group G is solvable if and only if every Sylow subgroup of G is c-supplemented in G. Some related results can also be found by Wang in [18]. Asaad and Ramadan in [2] proved that G is solvable if every subgroup of prime order of G is c-supplemented in G.

Recently, Guo and Lu in [7] proved that a group G is solvable if and only if every Sylow subgroup of G is ss-supplemented in G. Lu, Guo and Li in [16] proved that G is solvable if every subgroup of prime order of G is ss-supplemented in G. In [11], Heliel improved and extended some of the classical and recent results mentioned above, and he proposed the following conjecture.

Question 1.3 ([11])**.** *Let G be a group such that every noncyclic Sylow subgroup P of odd order of G has a subgroup D such that $1 < |D| \leqslant |P|$ and all subgroups H of P with $|H| = |D|$ are c-supplemented in G. Is G solvable?*

The following example shows that in general the answer to Question 1.3 is negative.

Example 1.4. Let B be an elementary abelian group of order 5^n for some non-negative integer n, and let $G = A_5 \times B$, where A_5 is the alternating group on five symbols. Now, let P be the Sylow 5-subgroup of G. Then for any subgroup D of P

with $1 < |D| \leqslant |P|$, all subgroups H of P with $|H| = |D|$ are complemented in G. However, G is not solvable.

In this paper, we take the studies mentioned above a bit further. More precisely, we improve and generalize the results of Hall [9], Arad and Ward [1], Ballester-Bolinches et al. [3], Asaad and Ramadan [2], Guo et al. [7], [16], and Heliel [11] as follows.

Theorem 1.5. *Let G be a group. Then G is solvable if and only if every Sylow subgroup of odd order of G is ss-supplemented in G.*

Theorem 1.6. *Let G be a group. Then G is solvable if and only if all Sylow 2-subgroups and Sylow 3-subgroups of G are ss-supplemented in G.*

Theorem 1.7. *Let G be a group. If each subgroup of odd prime order of G is ss-supplemented in G, then G is solvable and possesses a normal 2-subgroup S such that G/S is supersolvable.*

2. Preliminaries

Lemma 2.1 ([7], Lemma 2.4)**.** *Let H be an ss-supplemented subgroup of G. Then the following statements hold:*

(1) *If K is a subgroup of G and $H \leqslant K$, then H is ss-supplemented in K.*
(2) *If N is a normal subgroup of G and $N \leqslant H$, then H/N is ss-supplemented in G/N.*
(3) *Let π be a set of primes. If H is a π-subgroup of G and N is a normal π'-subgroup of G, then HN/N is ss-supplemented in G/N.*

Lemma 2.2 ([13])**.** *Let G be a group and $H \leqslant G$. If H is s-permutable in G, then H is subnormal in G.*

Lemma 2.3 ([17], Lemma A)**.** *If H is a p-subgroup of G for some prime p, then H is s-permutable in G if and only if $O^p(G) \leqslant N_G(H)$.*

Let $\mathcal{U}$ denote the class of supersolvable groups. Then the $\mathcal{U}$-hypercenter of a group G, denoted by $Z_{\mathcal{U}}(G)$, is the product of all normal subgroups N of G such that each chief factor of G below N has prime order.

Lemma 2.4 ([15], Theorem 3.3)**.** *Suppose that P is a normal p-subgroup of G, where p is an odd prime number. If every subgroup of P of order p is s-permutable in G, then $P \leqslant Z_{\mathcal{U}}(G)$.*

Lemma 2.5. *Suppose that P is a normal p-subgroup of G, where p is an odd prime number. If every subgroup of P of order p is ss-supplemented in G, then $P \leqslant Z_{\mathcal{U}}(G)$.*

Proof. In view of Lemma 2.4, we may assume that P has a minimal subgroup H such that H is not s-permutable in G. By assumption, there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is s-permutable in K. Since H is not s-permutable in G, we see that $H \cap K = 1$. It is easy to see that $P = H(P \cap K)$ and $P \cap K$ is normal in G. Since every subgroup of $P \cap K$ of order p is ss-supplemented in G, it follows that $P \cap K \leqslant Z_{\mathcal{U}}(G)$ by induction. As $P/(P \cap K)$ is a normal subgroup of $G/(P \cap K)$ of order p, we have that $P/(P \cap K) \leqslant Z_{\mathcal{U}}(G/(P \cap K))$. Since $P \cap K \leqslant Z_{\mathcal{U}}(G)$, it follows that $Z_{\mathcal{U}}(G/(P \cap K)) = Z_{\mathcal{U}}(G)/(P \cap K)$ and so $P \leqslant Z_{\mathcal{U}}(G)$ as desired. □

3. The proofs

Proof of Theorem 1.5. If the group G is solvable, then, by Hall's theorem in [9], every Sylow subgroup of G is complemented and hence is ss-supplemented in G. In particular, every Sylow subgroup of odd order of G is ss-supplemented in G.

Conversely, we assume that every Sylow subgroup of odd order of G is ss-supplemented in G. We claim that every Sylow subgroup of odd order of G is, in fact, complemented in G. Let P be any Sylow subgroup of odd order of G. Then, by definition, there exists $K \leqslant G$ such that $PK = G$ and $P \cap K$ is S-quasinormal in K. Clearly, $P \cap K$ is a Sylow subgroup of K. By Lemma 2.2, $P \cap K$ is subnormal in K, and therefore $P \cap K$ is normal in K. By applying the Schur-Zassenhaus theorem in [6], Theorem 6.2.1, we have $K = (P \cap K)K_{p'}$, where $K_{p'}$ is a Hall p-subgroup of K. Now $G = PK = PK_{p'}$ and $P \cap K_{p'} = 1$. Hence P is complemented in G, as claimed.

Now we show G is not simple. Assume false. By Burnside's theorem, we may assume that $|\pi(G)| \geqslant 3$. Since every Sylow subgroup of odd order of G is complemented in G, we conclude that G possesses two subgroups H and K such that $|G : H| = p^s$ and $|G : K| = q^t$, where p and q are different odd primes with $p < q$. By checking the simple groups with subgroups of prime power index (see [8], Theorem 1), we have that $G \cong \mathrm{PSL}(2, 7)$. Therefore, $|G : H| = 3$, and consequently, G has nontrivial normal subgroups, a contradiction. Thus G is not simple.

Let N be a minimal normal subgroup of G. For any Sylow subgroup P of odd order of G, by the above argument, we have that $P \cap N$ is complemented in N. As $P \cap N$ is also a Sylow subgroup of N, it follows that every Sylow subgroup of odd order of N is complemented in N. By induction, N is solvable, and so N is an elementary abelian p-group for some prime p. Now, by Lemma 2.1, G/N satisfies the hypothesis of the theorem. By induction, G/N is solvable, and hence G is solvable. This completes the proof. □

P r o o f of Theorem 1.6. If the group G is solvable, then, by Hall's theorem in [9], every Sylow subgroup of G is complemented and hence is ss-supplemented in G. In particular, all Sylow 2-subgroups and Sylow 3-subgroups of G are ss-supplemented in G.

Conversely, assume that the Sylow 2-subgroups and Sylow 3-subgroups of G are ss-supplemented in G. With the same argument as in the proof of Theorem 1.5, we know that the Sylow 2-subgroups and Sylow 3-subgroups of G are complemented in G. By Arad and Ward in [1], G is solvable as desired. □

P r o o f of Theorem 1.7. We first show that G is solvable. Assume false and choose G to be a counterexample of minimal order.

(1) Every proper subgroup of G is solvable.

Let H be any proper subgroup of G. By Lemma 2.1 (1), each subgroup of odd prime order of H is ss-supplemented in H. Thus H is solvable by the choice of G.

(2) For each odd prime p dividing the order of G, there exists a subgroup N of order p such that N is not s-permutable in G.

Assume that there exists an odd prime, say p, such that each subgroup N of G of order p is s-permutable in G. Then, by Lemma 2.3, $O^p(G) \leqslant N_G(L)$. If $O^p(G)$ is a proper subgroup of G, then $O^p(G)$ is solvable by (1) and so is G, a contradiction. Hence we may assume $O^p(G) = G$ and so N is normal in G. Applying the NC-theorem, we have that $G' \leqslant C_G(N)$, where G' is the commutator subgroup of G. Then $\Omega_1(P \cap G') \leqslant Z(G')$, where P is a Sylow p-subgroup of G. It follows from Itô's lemma in [12], Satz 5.5, page 435, that G' is p-nilpotent. This together with (1) implies that G is solvable, a contradiction.

(3) There exist two subgroups H and K of G such that $|G : H| = p$ and $|G : K| = q$, where p and q are distinct odd primes with $p < q$.

Since G is not solvable, by Burnside's theorem, we may assume that $|\pi(G)| \geqslant 3$. Let $p, q \in \pi(G)$ be two distinct odd primes with $p < q$. By (2), there exist two subgroups L_1 and L_2 such that $|L_1| = p$, $|L_2| = q$ and L_1, L_2 are not s-permutable in G. By the hypothesis, L_1 and L_2 are ss-supplemented in G. Since L_1 and L_2 are not s-permutable in G, we claim that L_1 and L_2 are complemented in G. Hence there exist two subgroups H and K of G such that $|G : H| = p$ and $|G : K| = q$.

(4) Final contradiction.

Considering the permutable representation of G on H, we have that G/H_G is isomorphic to a subgroup of S_p, where S_p is the symmetric group on p symbols. Then $|G/H_G|$ divides $|S_p| = p!$. Since $p < q$, we know that H_G contains some Sylow q-subgroup of G. In particular, $H_G \neq 1$. Thus $G = H_G K$ and so $G/H_G = K/(H_G \cap K)$. By (1), we know that H and K are solvable. This implies that G is solvable, a contradiction.

Now, we show that G possesses a normal 2-subgroup S such that G/S is supersolvable. Set $S = O_2(G)$. Assume that $S = 1$. Since G is solvable, we know that $F(G) \neq 1$ and $F(G)$ is of odd order. By Lemma 2.5, each Sylow subgroup of $F(G)$ is contained in $Z_{\mathcal{U}}(G)$ and so $F(G) \leqslant Z_{\mathcal{U}}(G)$. By [5], page 390, Theorem 6.10, $G/C_G(F(G))$ is supersolvable. Since G is solvable, we get $C_G(F(G)) \leqslant F(G)$. This implies that $G/F(G)$ is supersolvable. Hence G is supersolvable and we are done. Assume that $S \neq 1$. By Lemma 2.1(3), we know that G/S satisfies the hypothesis of the theorem. Thus, by induction, G/S possesses a normal 2-subgroup P/S such that $(G/S)/(P/S) = G/P$ is supersolvable. Since $S = O_2(G)$, we have $S = P$ and G/S is supersolvable as desired. □

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Authors' address: Jiakuan Lu, Yanyan Qiu, School of Mathematics and Statistics, No. 15 Yucai Road, Guangxi Normal University, Guilin 541004, Guangxi, P. R. China, e-mail: jklu@mailbox.gxnu.edu.cn, qiuyanyan1988@qq.com.