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BANACH SPACES OF HOMOGENEOUS POLYNOMIALS WITHOUT THE APPROXIMATION PROPERTY

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Dedicated to the memory of Pierre Lelong (1912–2011)

Abstract. We present simple proofs that spaces of homogeneous polynomials on $L_p[0,1]$ and ℓ_p provide plenty of natural examples of Banach spaces without the approximation property. By giving necessary and sufficient conditions, our results bring to completion, at least for an important collection of Banach spaces, a circle of results begun in 1976 by R. Aron and M. Schottenloher (1976).

Keywords: Banach space; approximation property; linear operator; homogeneous polynomial; holomorphic function

MSC 2010: 46G25, 46G20, 46B28

1. Introduction

The approximation property was introduced by Grothendieck [18]. Enflo [14] gave the first example of a Banach space without the approximation property. Enflo's counterexample is an artificially constructed Banach space. The first naturally defined Banach space without the approximation property was given by Szankowski [27], who proved that the space $\mathcal{L}(\ell_2;\ell_2)$ of continuous linear operators on ℓ_2 does not have the approximation property. More recently Godefroy and Saphar [17] proved that, if $\mathcal{L}_K(\ell_2;\ell_2)$ denotes the subspace of all compact members of $\mathcal{L}(\ell_2;\ell_2)$, then the quotient $\mathcal{L}(\ell_2;\ell_2)/\mathcal{L}_K(\ell_2;\ell_2)$ does not have the approximation property.

There was a widespread belief that almost all Banach spaces without the approximation property are artificially constructed. See for instance the comments of Defant and Floret in [6], page 59, and Pietsch in [25], page 283.

Nevertheless, in this paper we present simple proofs that spaces of homogeneous polynomials on $L_p[0,1]$ and ℓ_p provide plenty of natural examples of Banach spaces without the approximation property.

This paper is organized as follows. Section 2, of preparatory character, is devoted to the study of spaces of linear operators. There we observe that the space $\mathcal{L}(L_p[0,1]; L_q[0,1])$ does not have the approximation property whenever $1 < p$, $q < \infty$, whereas the space $\mathcal{L}(\ell_p; \ell_q)$ does not have the approximation property whenever $1 < p \leqslant q < \infty$.

Section 3, devoted to the study of spaces of homogeneous polynomials, contains our main results. There we show that the space $\mathcal{P}(^n L_p[0,1])$ does not have the approximation property whenever $1 < p < \infty$ and $n \geqslant 2$, whereas the space $\mathcal{P}(^n \ell_p)$ does not have the approximation property whenever $1 < p < \infty$ and $n \geqslant p$.

Finally, Section 4 is devoted to the study of spaces of holomorphic functions. There we show that neither the space $(\mathcal{H}(U), \tau_\omega)$ nor the space $(\mathcal{H}(U), \tau_\delta)$ have the approximation property when U is an open subset of $L_p[0,1]$ or of ℓ_p, with $1 < p < \infty$.

Our proofs are based on important results of several authors. Among them we mention Szankowski's famous counterexample [27], the complementation properties of L_p spaces established by Pełczyński [23], the isomorphisms between spaces of operators on L_p spaces and spaces of operators on ℓ_p spaces discovered by Arias and Farmer [2], the isomorphisms between spaces of multilinear forms and spaces of symmetric multilinear forms on stable Banach spaces discovered by Diaz and Dineen [7], the complementation properties of spaces of homogeneous polynomials obtained by Aron and Schottenloher [3], and the complementation properties of tensor products of ℓ_p spaces, obtained also by Arias and Farmer [2]. These results play a key role in our proofs, and some of them are applied several times.

2. Spaces of linear operators

Let E, F, G, E_j, F_j denote Banach spaces over $\mathbb{K}$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. Let E' denote the dual of E, and let $\mathcal{L}(E; F)$ denote the Banach space of all continuous linear operators from E into F. If $T \in \mathcal{L}(E; F)$, then $T' \in \mathcal{L}(F'; E')$ denotes the dual operator.

Let $\mathcal{L}(E_1, \ldots, E_n; F)$ denote the Banach space of all continuous n-linear mappings from $E_1 \times \ldots \times E_n$ into F. We omit F when $F = \mathbb{K}$. When $E_1 = \ldots = E_n = E$, we write $\mathcal{L}(^n E; F)$ instead of $\mathcal{L}(E, \overset{(n)}{\ldots}, E; F)$. Let $E_1 \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi E_n$ denote the complete projective tensor product of $E_1, \ldots, E_n$. We have the canonical isomorphisms

$$\mathcal{L}(E_1 \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi E_n; F) = \mathcal{L}(E_1, \ldots, E_n; F) = \mathcal{L}(E_1; \mathcal{L}(E_2, \ldots, E_n; F)).$$

In particular

$$(E_1 \widehat{\otimes}_\pi E_2)' = \mathcal{L}(E_1, E_2) = \mathcal{L}(E_1; E_2').$$

Let us recall that E is said to have the *approximation property* if given $K \subset E$ compact and $\varepsilon > 0$, there is a finite rank operator $T \in \mathcal{L}(E; E)$ such that $\|Tx - x\| < \varepsilon$ for every $x \in K$. If E has the approximation property, then every complemented subspace of E also has the approximation property.

E is isomorphic to a complemented subspace of F if and only if there are $A \in \mathcal{L}(E; F)$ and $B \in \mathcal{L}(F; E)$ such that $B \circ A = I$. If E is isomorphic to a complemented subspace of F, and F is isomorphic to a complemented subspace of G, then E is isomorphic to a complemented subspace of G. If E is isomorphic to a complemented subspace of F, then E' is isomorphic to a complemented subspace of F'. If E_j is isomorphic to a complemented subspace of F_j for $j = 1, 2$, then $E_1 \widehat{\otimes}_\pi E_2$ is isomorphic to a complemented subspace of $F_1 \widehat{\otimes}_\pi F_2$. These simple remarks will be repeatedly used throughout this paper.

Proposition 2.1. *If $1 < p$, $q < \infty$, then $\mathcal{L}(L_p[0,1]; L_q[0,1])$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2; \ell_2)$. In particular $\mathcal{L}(L_p[0,1]; L_q[0,1])$ does not have the approximation property.*

Proof. Let $1 < p < \infty$. By a result of Pełczyński (see [23], Proposition 5, or [8], page 13) $L_p[0,1]$ contains a complemented subspace isomorphic to ℓ_2. Hence there are $A_p \in \mathcal{L}(\ell_2; L_p[0,1])$ and $B_p \in \mathcal{L}(L_p[0,1]; \ell_2)$ such that $B_p \circ A_p = I$. Given $1 < p$, $q < \infty$, we define

$$C_{pq}\colon\ S \in \mathcal{L}(\ell_2; \ell_2) \to A_q \circ S \circ B_p \in \mathcal{L}(L_p[0,1]; L_q[0,1])$$

and

$$D_{pq}\colon\ T \in \mathcal{L}(L_p[0,1]; L_q[0,1]) \to B_q \circ T \circ A_p \in \mathcal{L}(\ell_2; \ell_2),$$

then $D_{pq} \circ C_{pq} = I$ and the desired conclusion follows. □

Proposition 2.2. *If $1 < p \leqslant q < \infty$, then $\mathcal{L}(\ell_p; \ell_q)$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2; \ell_2)$. In particular $\mathcal{L}(\ell_p; \ell_q)$ does not have the approximation property.*

Proof. By a result of Arias and Farmer [2], Theorem 2.1, $\mathcal{L}(\ell_p; \ell_q)$ is isomorphic to $\mathcal{L}(L_p[0,1]; L_q[0,1])$. Then the desired conclusion follows from Theorem 2.1. □

Remark 2.3. If $1 < q < p < \infty$, then $\mathcal{L}(\ell_p; \ell_q)$ is a reflexive Banach space with a Schauder basis. In particular the restriction $p \leqslant q$ in Theorem 2.2 cannot be deleted. Indeed, if $1/q + 1/q' = 1$, then

$$\mathcal{L}(\ell_p; \ell_q) = \mathcal{L}(\ell_p, \ell_{q'}) = (\ell_p \widehat{\oplus}_\pi \ell_{q'})'.$$

The space $\ell_p \widehat{\otimes}_\pi \ell_{q'}$ has a Schauder basis, by a result of Gelbaum and Gil de Lamadrid [16], and $\mathcal{L}(\ell_p; \ell_{q'})$ is reflexive, by [9], page 248, Corollary 5. It follows that $\mathcal{L}(\ell_p; \ell_q)$ has a Schauder basis as well.

Propositions 2.1 and 2.2 extend results of Pisier [26], page 316, and Arias and Farmer [2], page 17, respectively, and play a key role in the proofs of our main results, namely Theorems 3.1 and 3.2.

The proof of Proposition 2.1 can be easily adapted to prove the following more general result. We leave the details to the reader.

Proposition 2.4.

(a) *If E and F contain complemented subspaces isomorphic to ℓ_2, then $\mathcal{L}(E; F)$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2; \ell_2)$. In particular, $\mathcal{L}(E; F)$ does not have the approximation property.*

(b) *If E contains a complemented subspace isomorphic to ℓ_2, then $\mathcal{L}(E; E')$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2; \ell_2)$. In particular, $\mathcal{L}(E; E')$ does not have the approximation property.*

3. Spaces of homogeneous polynomials

Let $\mathcal{L}^s({}^nE; F)$ denote the subspace of all symmetric members of $\mathcal{L}({}^nE; F)$, and let $\mathcal{P}({}^nE; F)$ denote the Banach space of all continuous n-homogeneous polynomials from E into F. We omit F when $F = \mathbb{K}$. We have the canonical isomorphism $\mathcal{P}({}^nE; F) = \mathcal{L}^s({}^nE; F)$. We refer to [10] or [19] for background information on the theory of polynomials on Banach spaces.

It follows from a result of Diaz and Dineen [7], Theorem 3, that if E is *stable*, that is if E is isomorphic to its square, then $\mathcal{L}^s({}^nE)$ is isomorphic to $\mathcal{L}({}^nE)$ for every $n \in \mathbb{N}$. That ℓ_p and $L_p[0, 1]$ are stable was already known to Banach [4], page 182. Hence it follows that

$$\mathcal{P}({}^2\ell_2) = \mathcal{L}^s({}^2\ell_2) = \mathcal{L}({}^2\ell_2) = \mathcal{L}(\ell_2; \ell_2') = \mathcal{L}(\ell_2; \ell_2)$$

does not have the approximation property. A result of Aron and Schottenloher [3], Proposition 5.3, asserts that $\mathcal{P}({}^mE)$ is isomorphic to a complemented subspace of $\mathcal{P}({}^nE)$ whenever $m \leqslant n$. Hence it follows that $\mathcal{P}({}^n\ell_2)$ does not have the approximation property for every $n \geqslant 2$. This result is due to Floret (see [15], page 173, or [10], page 467), and was the initial motivation of this paper.

Theorem 3.1. *If $1 < p < \infty$ and $n \geqslant 2$, then $\mathcal{P}(^n L_p[0,1])$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2;\ell_2)$. In particular, $\mathcal{P}(^n L_p[0,1])$ does not have the approximation property.*

P r o o f. If $1/p + 1/p' = 1$, then

$$\mathcal{P}(^2 L_p[0,1]) = \mathcal{L}^s(^2 L_p[0,1]) = \mathcal{L}(^2 L_p[0,1]) = \mathcal{L}(L_p[0,1]; L_{p'}[0,1]).$$

It follows from Proposition 2.1 that $\mathcal{P}(^2 L_p[0,1])$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2;\ell_2)$. Hence $\mathcal{P}(^n L_p[0,1])$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2;\ell_2)$ for every $n \geqslant 2$. □

Theorem 3.2. *If $1 < p < \infty$ and $n \geqslant p$, then $\mathcal{P}(^n \ell_p)$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2;\ell_2)$. In particular, $\mathcal{P}(^n \ell_p)$ does not have the approximation property.*

P r o o f. If $1 < p < \infty$, then $k < p \leqslant k+1$ for a unique $k \in \mathbb{N}$. Let

$$E = \ell_p \widehat{\otimes}_\pi \overset{(k)}{\cdots} \widehat{\otimes}_\pi \ell_p,$$

and let $r = p/k$. Then $1 < r \leqslant p/(p-1)$ and another result of Arias and Farmer [2], Theorem 1.3, implies that E contains a complemented subspace isomorphic to ℓ_r. Hence it follows that

$$E \widehat{\otimes}_\pi \ell_p = \ell_p \widehat{\otimes}_\pi \overset{(k+1)}{\cdots} \widehat{\otimes}_\pi \ell_p$$

contains a complemented subspace isomorphic to $\ell_r \widehat{\otimes}_\pi \ell_p$. Hence it follows that

$$(\ell_p \widehat{\otimes}_\pi \overset{(k+1)}{\cdots} \widehat{\otimes}_\pi \ell_p)' = \mathcal{L}(^{k+1}\ell_p) = \mathcal{L}^s(^{k+1}\ell_p) = \mathcal{P}(^{k+1}\ell_p)$$

contains a complemented subspace isomorphic to

$$(\ell_r \widehat{\otimes}_\pi \ell_p)' = \mathcal{L}(\ell_r, \ell_p) = \mathcal{L}(\ell_r; \ell_{p'}).$$

Since $r \leqslant p/(p-1) = p'$, it follows from Proposition 2.2 that $\mathcal{L}(\ell_r;\ell_{p'})$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2;\ell_2)$. Hence $\mathcal{P}(^{k+1}\ell_p)$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2;\ell_2)$. Hence $\mathcal{P}(^n \ell_p)$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2;\ell_2)$ for every $n \geqslant k+1$, and therefore for every $n \geqslant p$. □

Remark 3.3. As pointed out by the second author in [20], page 25, if $1 < p < \infty$ and $n < p$, then $\mathcal{P}(^n \ell_p)$ is a reflexive Banach space with a Schauder basis. In particular, the restriction $n \geqslant p$ in Theorem 3.2 cannot be deleted. Indeed, every $P \in \mathcal{P}(^n \ell_p)$ is weakly sequentially continuous, by [24], page 178. Hence $\mathcal{P}(^n \ell_p)$ is reflexive, by [10], Proposition 2.30, and has a Schauder basis, by [1], Theorem 8.

Theorem 3.2 proves a conjecture of the second author in [20], page 25.

By using Proposition 2.3 instead of Proposition 2.1 the proof of Theorem 3.1 can be easily adapted to prove the following more general result. We leave the details to the reader.

Theorem 3.4. *If E is stable and contains a complemented subspace isomorphic to ℓ_2, then $\mathcal{P}({}^nE)$ contains a complemented subspace isomorphic to $\mathcal{L}(\ell_2;\ell_2)$ for every $n \geqslant 2$. In particular, $\mathcal{P}({}^nE)$ does not have the approximation property for every $n \geqslant 2$.*

4. Spaces of holomorphic functions

Let $\mathcal{H}(U)$ denote the vector space of all complex-valued holomorphic functions on an open subset U of a complex Banach space E. Let τ_0 denote the compact-open topology, let τ_ω denote the compact-ported topology introduced by Nachbin [22], and let τ_δ denote the bornological topology introduced independently by Coeuré [5] and Nachbin [21]. We refer to [10] for background information on these topologies.

The study of the approximation property on $(\mathcal{H}(U),\tau_0)$, $(\mathcal{H}(U),\tau_\omega)$ and $(\mathcal{H}(U),\tau_\delta)$ was initiated in 1976 by Aron and Schottenloher [3]. They restricted mainly to the case where $U = E$ or where U is a balanced open set. In the articles [11], [12] and [13] Dineen and Mujica extended some of the results of Aron and Schottenloher to the case where U is an arbitrary open set, or where U is at least pseudoconvex. In particular, Dineen and Mujica have given sufficient conditions on E and U for the spaces $(\mathcal{H}(U),\tau_0)$, $(\mathcal{H}(U),\tau_\omega)$ and $(\mathcal{H}(U),\tau_\delta)$ to have the approximation property. In this section we give some counterexamples to the approximation property for the spaces $(\mathcal{H}(U),\tau_\omega)$ and $(\mathcal{H}(U),\tau_\delta)$. Indeed, since $\mathcal{P}({}^nE)$ is a complemented subspace of $(\mathcal{H}(U),\tau_\omega)$ and $(\mathcal{H}(U),\tau_\delta)$, Theorems 3.1 and 3.2 immediately imply the following theorems.

Theorem 4.1. *If $U \subset L_p[0,1]$, where $1 < p < \infty$, then neither $(\mathcal{H}(U),\tau_\omega)$ nor $(\mathcal{H}(U),\tau_\delta)$ have the approximation property.*

Theorem 4.2. *If $U \subset \ell_p$, where $1 < p < \infty$, then neither $(\mathcal{H}(U),\tau_\omega)$ nor $(\mathcal{H}(U),\tau_\delta)$ have the approximation property.*

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