Siqin Yao; Jiong Sun; Anton Zettl
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THE STURM-LIOUVILLE FRIEDRICHS EXTENSION

Siqin Yao, Jiong Sun, Hohhot, Anton Zettl, DeKalb

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Abstract. The characterization of the domain of the Friedrichs extension as a restriction of the maximal domain is well known. It depends on principal solutions. Here we establish a characterization as an extension of the minimal domain. Our proof is different and closer in spirit to the Friedrichs construction. It starts with the assumption that the minimal operator is bounded below and does not directly use oscillation theory.

Keywords: Sturm-Liouville operator; Friedrichs extension

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1. Introduction

Given any symmetric bounded below operator S in a Hilbert space H, Friedrichs, in a seminal paper [1] in 1933, constructed a self-adjoint extension S_F of S in H which has the same lower bound as S. This extension has come to be known as the Friedrichs extension.

A Sturm-Liouville (S-L) equation

$$My = w^{-1}[(-py')' + qy] = \lambda y \quad \text{on } J = (a,b),\ -\infty \leqslant a < b \leqslant \infty, \tag{1.1}$$

with coefficients satisfying

$$\frac{1}{p}, q, w \in L_{\rm loc}(J,\mathbb{R}),\ p > 0,\ w > 0,\ \text{a.e. on } J \tag{1.2}$$

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generates a symmetric densely defined minimal operator $S_{\min}$ in the Hilbert space $H = L^2(J, w)$ with inner product $(y, z) = \int_J y\overline{z}w$ whose adjoint is called the maximal operator of (1.1) and is denoted by $S_{\max} = S_{\min}^*$. Here $L_{\mathrm{loc}}(J, \mathbb{R})$ denotes the real-valued functions which are Lebesgue integrable on all compact subintervals of J, $D(S)$ denotes the domain of S.

In general $S_{\min}$ has an uncountable number of self-adjoint extensions S. These satisfy

$$S_{\min} \subset S = S^* \subset S_{\max}. \tag{1.3}$$

Thus each operator S satisfying (1.3) can be considered as an extension of the minimal operator $S_{\min}$ and, equivalently, as a restriction of the maximal operator $S_{\max}$. Clearly these operators S are distinguished from each other only by their domains. These domains can be characterized in terms of two point boundary conditions specified at the two endpoints a, b of the interval J. For details of this characterization as well as other basic results, definitions, notation, etc. used below, see the book [16].

To get the Friedrichs extension of $S_{\min}$ one must use the Friedrichs construction or some equivalent version of it. This construction makes no explicit use of boundary conditions. Thus the natural question arises: Of the, in general, uncountable number of boundary conditions which one determines the Friedrichs extension?

Friedrichs himself considered this question in [2] in 1935 and showed that the Friedrichs extension is determined by the Dirichlet boundary condition

$$y(a) = 0 = y(b), \tag{1.4}$$

when $p = 1 = w$ and q is continuous on a compact interval $[a, b]$.

It is now known that the Dirichlet boundary condition (1.4) determines the Friedrichs extension in the general regular case. This is the case when $L_{\mathrm{loc}}(J, \mathbb{R})$ in (1.2) can be replaced by $L^1(J, \mathbb{R})$. If one endpoint of J is singular, then the Dirichlet boundary condition (1.4) is not well defined because, in general, solutions and maximal domain functions y do not have a finite limit at a singular endpoint. (See Section 2 below for a definition of regular and singular endpoints.)

In 1992 Niessen and Zettl [11], building on the work of Rellich [12], Kalf [5], Rosenberger [13], and others, characterized the Friedrichs extension S_F for the general equation (1.1), (1.2) in terms of singular boundary conditions determined by principal solutions u_a, u_b at the endpoints:

$$[y, u_a](a) = 0 = [y, u_b](b). \tag{1.5}$$

In (1.5) the Lagrange form $[\cdot, \cdot]$ is defined for all $y, z \in D_{\max}$ by $[y, z] = y(p\overline{z}') - \overline{z}(py')$.

This characterization and its proof (see [11] and Theorem 10.5.1 in [16]) is based on the oscillation theory and regularization of singular problems. In the general regular case (1.5) reduces to (1.4).

Note that, although this characterization is customarily described as an extension of $S_{\min}$, it is actually a restriction of $S_{\max}$:

$$D(S_F) = \{y \in D_{\max}\colon\ [y, u_a](a) = 0 = [y, u_b](b)\}. \tag{1.6}$$

In this paper we construct maximal domain functions U_a, U_b which give our characterization of $D(S_F)$ the following form:

$$D(S_F) = D_{\min} \dotplus \operatorname{span}\{U_a, U_b\}, \tag{1.7}$$

and thus it is an actual extension of the minimal operator $S_{\min}$. (Of course the characterizations (1.7) and (1.6) are equivalent, since the Friedrichs extension is unique.) For both the characterizations there is no u_a, U_a when a is in the limit-point case and no u_b, U_b when b is in the limit-point case. Thus both the characterizations reduce to $S_{\min} = S_{\max}$ and $S_{\min}$ is its own Friedrichs extension if it is bounded below and both the endpoints are in the limit-point case.

Our proof is different from the proof in [11]. It has the following features:

- ▷ It is based directly on the assumption that $S_{\min}$ is bounded below.
- ▷ It makes no direct use of the oscillation or non-oscillation theory.
- ▷ It is not based on regularization.
- ▷ Since it does not depend on the oscillation theory we believe there is a better chance of extending it to higher order problems, Hamiltonian systems, difference equations, etc.

Although the oscillation and non-oscillation theory has a long history and voluminous literature with many known sufficient and necessary conditions it is still an open problem in the sense that there is no known necessary and sufficient condition which could be verified in each case. Higher order oscillation theory is much more complicated and less developed than in the second order case.

In the higher order case the Friedrichs extension has been characterized for very general regular problems in [10], [8] and for a large class of singular problems in [7].

Following this Introduction, in Section 2 we review a version of the Friedrichs construction in abstract Hilbert space and introduce some notation. Applications of our adaptation of this construction are given in Section 3 for the LC/LC case and in Section 4 for LC(R)/LP case. In Section 5 we compare our characterization with the one in [11] and make some comments.

2. The Friedrichs extension

In this section we present the construction of the Friedrichs extension of a symmetric bounded below operator S in a Hilbert space $(H, (.,.))$ as given in [15].

Definition 1. A symmetic operator S is bounded below if there is a real number c such that $(Sf,f) \geqslant c(f,f)$ for all $f \in D(S)$. The number c is called a lower bound for S and the smallest such c is called the lower bound for S.

Let ν denote a lower bound of S. Define a semi-bounded sesquilinear form s and the associated inner product $\langle\cdot,\cdot\rangle_s$ on $D(S)$ by

$$s(f,g) = (Sf,g) \quad \text{and} \quad \langle f,g\rangle_s = (Sf,g) + (1-\nu)(f,g),\ \ f,g \in D(S);$$

then $(D(S), \langle\cdot,\cdot\rangle_s)$ is a pre-Hilbert space.

Let $\|\cdot\|_s$ denote the norm induced by the inner product $\langle\cdot,\cdot\rangle_s$ on H; then

$$\|f\|_s^2 = \langle f,f\rangle_s = (Sf,f) + (1-\nu)\|f\|^2,$$

and $\|f\|_s \geqslant \|f\|$ for all $f \in D(S)$.

Let H_s be the $\|\cdot\|_s$ completion of $D(S)$. Then H_s is a Hilbert space and we define the sesquilinear form $\bar{s}$ on H_s by

$$\bar{s}(f,g) = \langle f,g\rangle_s - (1-\nu)(f,g) \quad \text{for } f,g \in H_s. \tag{2.1}$$

Then $\bar{s}(f,g) = s(f,g)$ for $f,g \in D(S)$, and $\bar{s}$ is the closure of s.

Lemma 1 ([15])**.** *The norm $\|\cdot\|_s$ is compatible with the norm $\|\cdot\|$, i.e. if $\{f_n\}$ is a $\|\cdot\|_s$ Cauchy sequence in $D(S)$ and $\|f_n\| \to 0$, then we also have $\|f_n\|_s \to 0$.*

P r o o f. See page 123 in [15] for a proof. □

R e m a r k 1 ([15]). If H_s is the $\|\cdot\|_s$-completion of $D(S)$, then H_s may be viewed as a subspace of H, if the embedding of H_s into H is defined as follows: Let $\{f_n\}$ be a $\|\cdot\|_s$-Cauchy sequence in $D(S)$. Then $\{f_n\}$ is a Cauchy sequence in H. Let the element $\lim f_n$ from H correspond to the element $[\{f_n\}]$ of H_s. From Lemma 1, this correspondence is injective and the embedding is continuous with norm $\leqslant 1$.

Lemma 2. *For* $\{f_n\} \subseteq H_s$, $f \in H_s$, $\|f_n - f\|_s \to 0$ *if and only if*

$$\|f_n - f\| \to 0 \quad \text{in } H, \quad \text{and} \quad (S(f_n - f_m), f_n - f_m) \to 0, \quad \text{as } n, m \to \infty.$$

Proof. From $\|f\|_s \geqslant \|f\|$ for all $f \in H_s$ and

$$\|f_n - f_m\|_s^2 = (S(f_n - f_m), f_n - f_m) + (1 - \nu)\|f_n - f_m\|^2,$$

the conclusion follows. □

From Lemma 2 and ([15], Theorem 5.38) we get the following theorem:

Theorem 1. *Let* H_s *be the* $\|\cdot\|_s$ *completion of* $D(S)$. *Then the Friedrichs extension* S_F *of* S *is defined as follows:*

$$D(S_F) = H_s \cap D(S^*), \quad S_F y = S^* y, \quad y \in D(S_F). \tag{2.2}$$

The operator S_F *is the only self-adjoint extension of* S *with the property* $D(S_F) \subset H_s$.

Furthermore, $D(S_F)$ *can be characterized as*

$$D(S_F) = \{y \in D(S^*)\colon\ \exists \{y_k\} \subseteq D(S),\ \text{s.t.}\ \lim_{k\to\infty} \bar{s}(y - y_k, y - y_k) = 0\},$$

and

$$D(S_F) = \{y \in D(S^*)\colon\ \exists \{y_k\} \subseteq D(S),\ \text{s.t.}\ \{y_k\} \to y \text{ in } H \text{ and } (S(y_k - y_m), y_k - y_m) \to 0,\ \text{as } k, m \to \infty\}.$$

3. The LC/LC case

In this section we characterize the Friedrichs extension S_F of the Sturm-Liouville equation (1.1) for the case when each endpoint is in the limit-circle (LC) case. This case essentially includes the cases when one or both endpoints are regular (R). Throughout this section we assume that the minimal operator $S_{\min}$ is bounded below with lower bound ν. For convenience we start with a number of well known lemmas which are used below.

Lemma 3 ([11]). *For any* $\lambda < \nu$ *every nontrivial real solution* y *of* $My = \lambda y$ *has at most one zero in* (a, b).

Lemma 4 ([15])**.** *A self-adjoint operator T is bounded below if and only if its spectrum $\sigma(T)$ is bounded from below. The greatest lower bound of T is equal to* $\min \sigma(T)$.

Lemma 5. *Assume that the deficiency index of the minimal operator $S_{\min}$ of* (1.1) *is d. For λ real denote by $r(\lambda)$ the number of linearly independent solutions of equation* (1.1) *which lie in $L^2((a,b),w)$. Then $r(\lambda) \leqslant d$ and if $r(\lambda) < d$, then λ is in the essential spectrum $\sigma_e(S)$ for every self-adjoint extension S of the the minimal operator $S_{\min}$.* (*See* [15] *for the definition of $\sigma_e(S)$.*)

Proof. See [4]. □

Next we state the well known GKN characterization of the self-adjoint domains. Although we only use this theorem in the second order, i.e. Sturm-Liouville case we state the general theorem, since this does not involve any extra complications or length.

Lemma 6 ([9], GKN Theorem)**.** *Assume M is a symmetric differential expression with real coefficients and $S_{\min}$, $S_{\max}$ are its minimal and maximal operators with domains $D_{\min}$ and $D_{\max}$, respectively. Then the deficiency indices of $S_{\min}$ are equal with a common value d, say. A linear submanifold $D(S)$ of $D_{\max}$ is the domain of a self-adjoint extension S of $S_{\min}$ if and only if there exist functions $w_1, w_2, \ldots, w_d$ in $D_{\max}$ satisfying the following conditions:*

(1) *$w_1, w_2, \ldots, w_d$ are linearly independent modulo $D_{\min}$;*
(2) $[w_i, w_j](b) - [w_i, w_j](a) = 0$, $i, j = 1, \ldots, d$;
(3) $D(S) = \{y \in D_{\max}\colon\ [y, w_j](b) - [y, w_j](a) = 0,\ j = 1, \ldots, d\}$.

Here $[\cdot,\cdot]$ denotes the Lagrange bracket associated with M.

The next lemma characterizes the maximal domain in terms of solutions for certain real values of the parameter λ.

Lemma 7. *Assume equation* (1.1) *is in the Limit-Circle case at both a and b. Assume $\lambda < \nu$ in equation* (1.1)*. Let $a < c < b$. The following statements hold.*

(1) *The initial conditions*

$$u_1(c) = 0, \quad u_1^{[1]}(c) = 1; \quad u_2(c) = 1, \quad u_2^{[1]}(c) = 0$$

determine two linearly independent solutions u_1, u_2 of equation (1.1) *on (c,b) which lie in $L^2((c,b),w)$, where $u_i^{[1]} = (pu_i')$, $i = 1, 2$.*

(2) *The initial conditions*

$$u_3(c) = 0, \quad u_3^{[1]}(c) = 1; \quad u_4(c) = 1, \quad u_4^{[1]}(c) = 0,$$

determine two linearly independent solutions u_3, u_4 *of equation* (1.1) *on* (a, c) *which lie in* $L^2((a,c), w)$, *where* $u_i^{[1]} = (pu_i')$, $i = 3, 4$.

(3) *There exists* $c_b \in (c, b)$ *such that* $u_i(x) \neq 0$ *for* $x \in I_b = [c_b, b)$, $i = 1, 2$. *Furthermore,* u_1/u_2 *is an increasing function on* I_b.

(4) *There exists* $c_a \in (a, c)$ *such that* $u_i(x) \neq 0$ *for* $x \in I_a = (a, c_a]$, $i = 3, 4$. *Furthermore,* u_3/u_4 *is an increasing function on* I_a.

(5) *The solutions* u_1, u_2, u_3, u_4 *can be extended to* (a, b) *so that the extended functions, also denoted by* u_1, u_2, u_3, u_4, *satisfy* $u_j \in D_{\max}(a, b)$, $j = 1, \ldots, 4$, *and* (i) u_1, u_2 *are identically zero in a right neighborhood of* a, (ii) u_3, u_4 *are identically zero in a left neighborhood of* b. *Moreover,*

$$D_{\max} = D_{\min} \dotplus \operatorname{span}\{u_1, u_2, u_3, u_4\}.$$

P r o o f. From Lemma 3 there exists a subinterval $I_b = [c_b, b)$ of (c, b) such that $u_i(x) \neq 0$ for $x \in I_b$, $i = 1, 2$. Furthermore, we have

$$\left(\frac{u_1}{u_2}\right)'(x) = \frac{u_1^{[1]} u_2 - u_1 u_2^{[1]}}{pu_2^2}(x) = \frac{1}{pu_2^2} \quad \text{for } x \in I_b \text{ a.e.}$$

Hence, u_1/u_2 is an increasing function on I_b, i.e., item (3) is proved. Similarly, item (4) can be proved. From the Patching Lemma and Theorem 4.6 in [3] item (5) follows. □

Definition 2. Let the notation and hypotheses of Lemma 7 hold. Then

$$\lim_{x\to b} -\frac{u_1(x)}{u_2(x)} = l_b, \quad \lim_{x\to a} -\frac{u_3(x)}{u_4(x)} = l_a, \quad -\infty \leqslant l_b, l_a \leqslant \infty.$$

If l_b, l_a are finite numbers, we define

$$\tilde{u}_b := u_1 + l_b u_2, \quad \tilde{u}_a := u_3 + l_a u_4. \tag{3.1}$$

If $l_b = -\infty$, then $\lim\limits_{x\to b} -u_2(x)/u_1(x) = \tilde{l}$, $\tilde{l} = 0$, we define

$$\tilde{u}_b := u_2 + \tilde{l} u_1 = u_2.$$

Similarly, if $l_a = \infty$, define

$$\tilde{u}_a := u_4.$$

R e m a r k 2. In fact $\tilde{u}_b$ is the principal solution at the endpoint b and $\tilde{u}_a$ is the principal solution at the endpoint a, as in [11].

Here we consider the case when l_b, l_a are finite numbers. The other cases can be investigated similarly.

Lemma 8. *Let $b_n \to b$, $a_n \to a$ as $n \to \infty$. Define $\tilde{u}_{b_n}$, $\tilde{u}_{a_n}$ as*

$$\tilde{u}_{b_n} = u_1 + l_{b_n} u_2, \quad l_{b_n} = -\frac{u_1(b_n)}{u_2(b_n)},$$
$$\tilde{u}_{a_n} = u_3 + l_{a_n} u_4, \quad l_{a_n} = -\frac{u_3(a_n)}{u_4(a_n)}.$$

Then it is obvious that $\lim_{n\to\infty} \tilde{u}_{b_n} = \tilde{u}_b$, $\lim_{n\to\infty} \tilde{u}_{a_n} = \tilde{u}_a$.

Let $\varrho > 0$ be large enough to satisfy $3/(2\varrho) < c_b - c_a$. Extend $\tilde{u}_b$ to (a,b) denoting it as u_b, so that $u_b \in D_{\max}$ as follows:

$$u_b := \begin{cases} 0, & a \leqslant x < c_b - 1/\varrho - \varphi, \\ g, & c_b - 1/\varrho - \varphi \leqslant x \leqslant c_b, \\ \tilde{u}_b, & c_b < x \leqslant b, \end{cases} \tag{3.2}$$

where

$$g = A[\cos \varrho\pi(x - c_b + \varphi) + 1], \quad -\frac{1}{2\varrho} \leqslant \varphi \leqslant \frac{1}{2\varrho} \tag{3.3}$$

and

$$g(c_b) = \tilde{u}_b(c_b), \quad g'(c_b) = \tilde{u}'_b(c_b), \tag{3.4}$$

where A, φ are constants. Similarly, extend $\tilde{u}_a$ to (a,b) denoting it as u_a so that $u_a \in D_{\max}$ as follows:

$$u_a := \begin{cases} \tilde{u}_a, & a \leqslant x < c_a, \\ f, & c_a \leqslant x \leqslant c_a + \dfrac{1}{\varrho} - \varphi, \\ 0, & c_a + \dfrac{1}{\varrho} - \varphi < x \leqslant b, \end{cases} \tag{3.5}$$

where

$$f = B[\cos \varrho\pi(x - c_a + \varphi) + 1], \quad -\frac{1}{2\varrho} \leqslant \varphi \leqslant \frac{1}{2\varrho}, \tag{3.6}$$

and

$$f(c_a) = \tilde{u}_a(c_a), \;\; f'_a(c_a) = \tilde{u}'_a(c_a), \tag{3.7}$$

where B, φ are constants. Define $\{u_{b_n}\}$ as follows:

$$(3.8) \qquad u_{b_n} := \begin{cases} 0, & a \leqslant x < c_b - 1/\varrho - \varphi_n, \\ g_n, & c_b - 1/\varrho - \varphi_n \leqslant x \leqslant c_b, \\ \tilde{u}_{b_n}, & c_b < x \leqslant b_n, \\ 0, & b_n < x \leqslant b, \end{cases}$$

where

$$(3.9) \qquad g_n = A_n[\cos \varrho\pi(x - c_b + \varphi_n) + 1], \quad -\frac{1}{2\varrho} \leqslant \varphi_n \leqslant \frac{1}{2\varrho},$$

and

$$(3.10) \qquad g_n(c_b) = \tilde{u}_{b_n}(c_b), \quad g_n'(c_b) = \tilde{u}_{b_n}'(c_b).$$

Define $\{u_{a_n}\}$ as follows:

$$(3.11) \qquad u_{a_n} := \begin{cases} 0, & a \leqslant x < a_n, \\ \tilde{u}_{a_n}, & a_n \leqslant x < c_a, \\ f_n, & c_a \leqslant x \leqslant c_a + 1/\varrho - \varphi_n, \\ 0, & c_a + 1/\varrho - \varphi_n < x \leqslant b, \end{cases}$$

where

$$(3.12) \qquad f_n = B_n[\cos \varrho\pi(x - c_a + \varphi_n) + 1], \quad -\frac{1}{2\varrho} \leqslant \varphi_n \leqslant \frac{1}{2\varrho},$$

and

$$(3.13) \qquad f_n(c_a) = \tilde{u}_{a_n}(c_a), \quad f_n'(c_a) = \tilde{u}_{a_n}'(c_a).$$

Then the definitions of u_b, u_{b_n} and u_a, u_{a_n} are meaningful.

P r o o f. Consider u_b defined as in (3.2)–(3.4).

For convenience let $\sin \varrho\pi\varphi = x$, $\cos \varrho\pi\varphi = \sqrt{1 - x^2}$ and $\tilde{u}_b(c_b) = \alpha$, $\tilde{u}_b'(c_b) = \beta$. Then from (3.4) obtain

$$A(\sqrt{1 - x^2} + 1) = \alpha, \quad -\varrho\pi A x = \beta,$$

where $A \neq 0$, since $\alpha \neq 0$.

If $\beta = 0$, then $x = 0$, i.e. $\varphi = 0$, and $A = \alpha/2$.

Assume $\beta \neq 0$, then

$$\frac{\sqrt{1-x^2}+1}{-\varrho\pi x} = \frac{\alpha}{\beta}, \tag{3.14}$$

and by solving equation (3.14) we get

$$x = \frac{-2\alpha\beta\varrho\pi}{\varrho^2\pi^2\alpha^2+\beta^2}; \tag{3.15}$$

obviously, $\left|-2\alpha\beta\varrho\pi/(\varrho^2\pi^2\alpha^2+\beta^2)\right| \leqslant 1$, so the solution is well defined,

$$A = \frac{\varrho^2\pi^2\alpha^2+\beta^2}{2\alpha\varrho^2\pi^2}. \tag{3.16}$$

From $\sin\varrho\pi\varphi = x$, $-1/(2\varrho) \leqslant \varphi \leqslant 1/(2\varrho)$,

$$\varphi = \frac{1}{\varrho\pi}\arcsin\Big(\frac{-2\alpha\beta\varrho\pi}{\varrho^2\pi^2\alpha^2+\beta^2}\Big). \tag{3.17}$$

The above results show that for every $\tilde{u}_b(d) \neq 0$, $\tilde{u}_b'(d) \neq 0$, $d \in [c_b, b)$, there exist exactly one A and φ such that

$$A = \frac{\varrho^2\pi^2(\tilde{u}_b(d))^2+(\tilde{u}_b'(d))^2}{2\varrho^2\pi^2\tilde{u}_b(d)}, \tag{3.18}$$

$$\varphi = \frac{1}{\varrho\pi}\arcsin\Big(\frac{-2\varrho\pi\tilde{u}_b(d)\tilde{u}_b'(d)}{\varrho^2\pi^2(\tilde{u}_b(d))^2+(\tilde{u}_b'(d))^2}\Big). \tag{3.19}$$

So u_b is well defined. Similarly, we can prove u_{b_n}, u_a, u_{a_n} are well defined. □

Lemma 9. *Let H_s be the $\|\cdot\|_s$ completion of $D_{\min}$ and let u_b, u_{b_n} and u_a be defined as in Definition* 8. *Then*

(1) $u_b, u_a \in D_{\max}$ *are linearly independent modulo* $D_{\min}$;

(2) $[u_a, u_b](b) = [u_a, u_b](a) = 0$;

(3) $\{u_{b_n}\}, \{u_{a_n}\} \subseteq H_s$.

P r o o f. Parts (1) and (2) are clear, and for a proof of part (3) see the proof of Lemma 8 in [7]. □

Lemma 10. (1) *Let* g, g_n *be defined as in* (3.3)–(3.4) *and* (3.9)–(3.10). *Then*

$$A_n \to A, \quad \varphi_n \to \varphi.$$

Consequently,

$$g_n \to g, \quad g_n' \to g',$$

and $\{g_n\}, \{g_n'\}$ *are uniformly bounded on* $[c_b - 1/\varrho - \varphi_n, c_b]$ *and* $-1/(2\varrho) \leqslant \varphi_n \leqslant 1/(2\varrho)$.

(2) *Similarly, for* f, f_n *defined in* (3.6)–(3.7) *and* (3.12)–(3.13), *we have*

$$B_n \to B, \quad \varphi_n \to \varphi.$$

Consequently,

$$f_n \to f, \quad f_n' \to f',$$

and $\{f_n\}, \{f_n'\}$ *are uniformly bounded on* $[c_a, \leqslant c_a + 1/\varrho - \varphi_n]$ *and* $-1/(2\varrho) \leqslant \varphi_n \leqslant 1/(2\varrho)$.

P r o o f. We prove item (1), item (2) can be proved similarly. Equations (3.18)–(3.19) define functions F and G:

$$A = F(\tilde{u}_b(c_b), \tilde{u}_b'(c_b)), \quad \varphi = G(\tilde{u}_b(c_b), \tilde{u}_b'(c_b)).$$

Similarly,

$$A_n = F(\tilde{u}_{b_n}(c_b), \tilde{u}_{b_n}'(c_b)), \quad \varphi_n = G(\tilde{u}_{b_n}(c_b), \tilde{u}_{b_n}'(c_b)).$$

From (3.18) and (3.19), F and G are continuous functions. Hence

$$\begin{aligned} \lim_{n\to\infty} A_n &= \lim_{n\to\infty} F(\tilde{u}_{b_n}(c_b), \tilde{u}_{b_n}'(c_b)) \\ &= F\Big(\lim_{n\to\infty} \tilde{u}_{b_n}(c_b), \lim_{n\to\infty} \tilde{u}_{b_n}'(c_b)\Big) = F(\tilde{u}_b(c_b), \tilde{u}_b'(c_b)) = A. \end{aligned}$$

Similarly,

$$\lim_{n\to\infty} \varphi_n = G(\tilde{u}_b(c_b), \tilde{u}_b'(c_b)) = \varphi.$$

Then from (3.3) and (3.9), we have

$$g_n \to g, \quad g_n' \to g'.$$

Since the trigonometric functions $\{g_n\}$ and $\{g_n'\}$ are all bounded, and since A is a fixed number determined by $\tilde{u}_b(c_b)$ and $\tilde{u}_b'(c_b)$, we conclude that $\{g_n\}$ and $\{g_n'\}$ are uniformly bounded functions. □

Theorem 2. *Let the notation and hypotheses of Lemma 7 hold. Then the domain of the Friedrichs extension of the minimal operator $S_{\min}$ generated by the differential expression M on $J=(a,b)$ is given by:*

$$D(S_F)=D_{\min}\dotplus \operatorname{span}\{u_a,u_b\}. \tag{3.20}$$

P r o o f. First we prove $u_{b_n}\to u_b$ in H, i.e.

$$\|u_b-u_{b_n}\|\to 0; \tag{3.21}$$

here we may assume that $c_b-1/\varrho-\varphi_n<c_b-1/\varrho-\varphi$. We have

$$\begin{aligned}\|u_b-u_{b_n}\|^2&=\int_a^b|u_b-u_{b_n}|^2w\,\mathrm{d}x\\&=\int_{c_b-1/\varrho-\varphi_n}^{c_b-1/\varrho-\varphi}|g_n|^2w\,\mathrm{d}x+\int_{c_b-1/\varrho-\varphi}^{c_b}|g-g_n|^2\omega\,\mathrm{d}x\\&\quad+(l_b-l_{b_n})^2\int_{c_b}^{b_n}|u_2|^2w\,\mathrm{d}x+\int_{b_n}^{b}|u_b|^2w\,\mathrm{d}x.\end{aligned}$$

By virtue of Lemma 10, $g_n\to g$, $\varphi_n\to\varphi$, and $\{g_n\}$ are uniformly bounded functions. And note $l_{b_n}\to l_b$ and $u_2,u_b\in D_{\max}$. Then

$$\lim_{n\to\infty}\|u_b-u_{b_n}\|=0.$$

Secondly, let us prove

$$(S_{\max}(u_{b_n}-u_{b_m}),u_{b_n}-u_{b_m})\to 0. \tag{3.22}$$

Assume $c_b-1/\varrho-\varphi_m<c_b-1/\varrho-\varphi_n$ and $b_m<b_n$. Then

$$\begin{aligned}&(S_{\max}(u_{b_n}-u_{b_m}),u_{b_n}-u_{b_m})\\&\quad=\int_{c_b-1/\varrho-\varphi_m}^{c_b-1/\varrho-\varphi_n}S_{\max}(g_m)\bar{g_m}w\,\mathrm{d}x\\&\qquad+\int_{c_b-1/\varrho-\varphi_n}^{c_b}S_{\max}(g_n-g_m)\overline{(g_n-g_m)}w\,\mathrm{d}x\\&\qquad+\lambda(l_{b_n}-l_{b_m})^2\int_{c_b}^{b_m}|u_2|^2w\,\mathrm{d}x+\lambda\int_{b_m}^{b_n}|u_{b_n}|^2w\,\mathrm{d}x.\end{aligned}$$

Since $l_{b_n} \to l_b$ and $u_2, u_{b_n} \in D_{\max}$, the last two integrals converge to zero. The first two integrals are

$$\tag{3.23} \begin{aligned} &\int_{c_b-1/\varrho-\varphi_m}^{c_b-1/\varrho-\varphi_n} S_{\max}(g_m)\bar{g_m} w \,\mathrm{d}x \\ &\qquad = \int_{c_b-1/\varrho-\varphi_m}^{c_b-1/\varrho-\varphi_n} (p|g_m'|^2 + q|g_m|^2)\,\mathrm{d}x - [pg_m' g_m]_{c_b-1/\varrho-\varphi_m}^{c_b-1/\varrho-\varphi_n}, \end{aligned}$$

$$\tag{3.24} \begin{aligned} &\int_{c_b-1/\varrho-\varphi_n}^{c_b} S_{\max}(g_n - g_m)\overline{(g_n - g_m)} w \,\mathrm{d}x \\ &\qquad = \int_{c_b-1/\varrho-\varphi_n}^{c_b} (p|g_n' - g_m'|^2 + q|g_n - g_m|^2)\,\mathrm{d}x \\ &\qquad\quad - [p(g_n' - g_m')(g_n - g_m)]_{c_b-1/\varrho-\varphi_n}^{c_b}. \end{aligned}$$

Also from Lemma 10, $\varphi_n \to \varphi$, $g_n \to g$, $g_n' \to g'$, and $\{g_n\}, \{g_n'\}$ are uniformly bounded functions. Combining it with $p^{-1}, q \in L_{\mathrm{loc}}(J, \mathbb{R})$ and the Hölder inequality, we conclude that integrals (3.23), (3.24) both converge to zero. So

$$(S_{\max}(u_{b_n} - u_{b_m}), u_{b_n} - u_{b_m}) \to 0, \quad n, m \to \infty.$$

Combining (3.21) and (3.22), we obtain

$$\tag{3.25} \|u_b - u_{b_n}\|_s \to 0.$$

Then from Theorem 1, $u_b \in D(S_F)$. Similarly, we can prove $u_a \in D(S_F)$. Hence,

$$D_{\min} \dotplus \operatorname{span}\{u_a, u_b\} \subseteq D(S_F).$$

Lemma 6 (GKN Theorem) and Lemma 9 yield that

$$D(T) = \{y \in D_{\max} \mid [y, u_a]_a^b = 0, [y, u_b]_a^b = 0\}$$

is a domain of a self-adjoint extension T of the minimal operator $M_{\min}$.

It is clear that $D_{\min} \dotplus \operatorname{span}\{u_a, u_b\} \subseteq D(T)$. Now we prove

$$D_{\min} \dotplus \operatorname{span}\{u_a, u_b\} = D(T).$$

From item (5) in Lemma 7 and the construction of u_a, u_b, we have

$$D_{\max} = D_{\min} \dotplus \operatorname{span}\{u_1, u_2, u_3, u_4\} = D_{\min} \dotplus \operatorname{span}\{u_b, u_2, u_a, u_4\}.$$

For all $f = f_0 + c_1 u_b + c_2 u_2 + c_3 u_a + c_4 u_4 \in D(T)$, $f_0 \in D_{\min}$,

$$0 = [f, u_b]_a^b = c_2[u_2, u_b](b) = c_2[u_2, u_1](b), \quad \text{and} \quad [u_2, u_1](b) \neq 0,$$

so we get $c_2 = 0$; similarly, we get $c_4 = 0$, hence,

$$f = f_0 + c_1 u_b + c_3 u_a \in D_{\min} \dotplus \operatorname{span}\{u_a, u_b\},$$

i.e. $D(T) \subseteq D_{\min} \dotplus \operatorname{span}\{u_a, u_b\}$. So $D_{\min} \dotplus \operatorname{span}\{u_a, u_b\} = D(T)$.

From the uniqueness of the Friedrichs extension shown in Theorem 1, the domain of the Friedrichs extension is

$$D(S_F) = D_{\min} \dotplus \operatorname{span}\{u_a, u_b\}.$$

□

To make Theorem 2 more general and convenient to use, we give the following theorem:

Theorem 3. *Let the notation and hypotheses of Lemma* 7*, Theorem* 2*, and Definition* 2 *hold. Then the domain of the Friedrichs extension of the minimal operator* $S_{\min}$ *is*

$$D(S_F) = D_{\min} \dotplus \operatorname{span}\{U_a, U_b\}, \tag{3.26}$$

where

$$U_b := \begin{cases} 0, & a \leqslant x < c_a \\ g_b, & c_a \leqslant x \leqslant c_b, \\ \tilde{u}_b, & c_b < x \leqslant b, \end{cases} \qquad U_a := \begin{cases} \tilde{u}_a, & a \leqslant x < c_a \\ g_a, & c_a \leqslant x \leqslant c_b, \\ 0, & c_b < x \leqslant b, \end{cases} \tag{3.27}$$

where g_b, g_a *are smooth functions which connect* $\tilde{u}_b$ *and* 0*,* $\tilde{u}_a$ *and* 0 *so that* $U_b, U_a \in D_{\max}$*.*

Proof. Since U_b, U_a defined above and u_b, u_a defined in (3.2), (3.5) satisfy

$$U_b - u_b = h_0 \in D_{\min}, \quad U_a - u_a = e_0 \in D_{\min}, \tag{3.28}$$

we have

$$D_{\min} \dotplus \operatorname{span}\{U_a, U_b\} = D_{\min} \dotplus \operatorname{span}\{u_a, u_b\} = D(S_F).$$

In fact, if $f \in D_{\min} \dotplus \operatorname{span}\{U_a, U_b\}$, we have

$$f = f_0 + \alpha U_a + \beta U_b,$$

where $f_0 \in D_{\min}$, and by (3.28),

$$f = (f_0 + \alpha e_0 + \beta h_0) + \alpha u_a + \beta u_b \in D_{\min} \dotplus \operatorname{span}\{u_a, u_b\}.$$

We have $D_{\min} \dotplus \operatorname{span}\{U_a, U_b\} \subseteq D_{\min} \dotplus \operatorname{span}\{u_a, u_b\}$.

Similarly we have $D_{\min} \dotplus \operatorname{span}\{U_a, U_b\} \supseteq D_{\min} \dotplus \operatorname{span}\{u_a, u_b\}$. □

To compare our method of construction with those in [6] and [11] we consider the following example:

Example 1. Let us consider the Friedrichs extension of the Legendre equation

$$-(py')' = \lambda y \quad \text{on } J = (-1,1), \quad p(t) = 1 - t^2, \quad -1 < t < 1. \tag{3.29}$$

This equation is singular at both -1 and 1 and the deficiency index of its minimal operator is 2.

The equation

$$-(py')' = 0, \quad -1 < t < 1,$$

has two linearly independent solutions z_1, z_2 on $(-1,0]$ and v_1, v_2 on $[0,1)$:

$$\begin{aligned} z_1(t) &= -\frac{1}{2}\ln\left|\frac{1-t}{1+t}\right|, \quad z_2(t) = 1, \quad -1 < t \leqslant 0,\\ v_1(t) &= -\frac{1}{2}\ln\left|\frac{1-t}{1+t}\right|, \quad v_2(t) = 1, \quad 0 \leqslant t < 1. \end{aligned}$$

Then we have

$$\tilde{u}_a = z_2 \equiv 1, \quad \tilde{u}_b = v_2 \equiv 1.$$

To define U_1, U_{-1}, let $-1 < c_{-1} < 0 < c_{+1} < 1$, and let

$$U_1 := \begin{cases} 0, & -1 < x < c_{-1},\\ g_b, & c_{-1} \leqslant x \leqslant 0,\\ 1, & 0 < x \leqslant 1, \end{cases} \qquad U_{-1} := \begin{cases} 1, & -1 < x \leqslant 0,\\ g_a, & 0 \leqslant x \leqslant c_{+1},\\ 0, & c_{+1} < x < 1, \end{cases}$$

where g_b, g_a are smooth functions such that U_{-1}, U_1 lie in the maximal domain.

From Theorem 3, the domain of the Friedrichs extension of the Legendre equation on $J = (-1,1)$ is

$$D(S_F) = D_{\min} \dotplus \operatorname{span}\{U_{-1}, U_1\}. \tag{3.30}$$

This is equivalent to the well-known boundary condition determining the Legendre Friedrichs extension [6]:

$$(py')(-1) = 0 = (py')(1).$$

In fact, let

$$\mathcal{D}(S_{\mathcal{F}}) = \{y \in D_{\max}\colon\ (py')(-1) = 0 = (py')(1)\}, \tag{3.31}$$

then on the one hand $D(S_{\mathcal{F}}) \subseteq \mathcal{D}(S_{\mathcal{F}})$, and on the other hand $\mathcal{D}(S_{\mathcal{F}}) \subseteq D(S_{\mathcal{F}})$, i.e. we have $D(S_{\mathcal{F}}) = \mathcal{D}(S_{\mathcal{F}})$.

Remark 3. We comment on the well-known characterization of the Friedrichs extension for the Legendre equation (3.31) and the characterization given by Theorem 3. As mentioned in Introduction, although the Friedrichs operator $S_{\mathcal{F}}$ is customarily described as an extension of $S_{\min}$ it is actually a restriction of $S_{\max}$, see (3.31). In contrast, the characterization (3.30) of Theorem 3 is an actual extension of the minimal operator. It is interesting to observe that—in the Legendre case—the singular condition (3.31) 'looks like' a regular Neumann boundary condition but is actually the singular analogue of the regular Dirichlet condition.

Although Theorems 2 and 3 are stated for the LC/LC case they also include the case when one or both endpoints are regular. If both endpoints are regular the assumption that $S_{\min}$ is bounded below is not needed, since this is always true in the regular case (with p and w positive) considered here. Next we state the regular case as a corollary. The cases R/LC and LC/R are obtained similarly to the corollaries of Theorems 2 and 3 (but in these cases the assumption that $S_{\min}$ is bounded below is required).

Corollary 1. *Assume equation* (1.1) *is regular at both a and b. Then:*

(1) *The minimal operator $S_{\min}$ is symmetric and bounded below, therefore $S_{\min}$ has a Friedrichs extension. Let ν denote its lower bound.*

(2) *The domain of the Friedrichs extension S_F of $S_{\min}$ is*

$$D(S_F) = D_{\min} \dotplus \operatorname{span}\{u_a, u_b\},$$

where $u_a \in D_{\max}, u_a(a) = 0, u_a'(a) \neq 0$, and u_a is identically zero in a left neighborhood of b; $u_b \in D_{\max}$, $u_b(b) = 0$, $u_b'(b) \neq 0$, and u_b is identically zero in a right neighborhood of a.

Example 2. Let us consider the Friedrichs extension of the Legendre equation

$$-(py')' = \lambda y \quad \text{on } J = [0,1), \quad p(t) = 1 - t^2, \quad 0 \leqslant t < 1.$$

It is regular at 0 and singular at 1, the deficiency index is 2.

The equation

$$-(py')' = 0, \quad 0 \leqslant t < 1,$$

has two linearly independent solutions

$$u_1(t) = -\frac{1}{2}\ln\left|\frac{1-t}{1+t}\right|, \quad u_2(t) = 1,$$

satisfying

$$u_1(0) = 0,\ u_1(t) \neq 0 \quad \text{for every } t \in (0,1), \quad u_2 \equiv 1 \neq 0,$$
$$u_1^{[1]} \equiv 1; \quad u_2^{[1]} \equiv 0.$$

Since

$$\left(\frac{u_1}{u_2}\right)' = \frac{[u_2, u_1]}{pu_2^2} = \frac{1}{1-t^2} > 0,$$

u_1/u_2 is an increasing function on $[0,1)$. Furthermore we have

$$\lim_{t\to 1} -\frac{u_1(t)}{u_2(t)} = \infty, \quad \lim_{t\to 0} -\frac{u_1(t)}{u_2(t)} = 0.$$

By Definition 1 we have

$$\tilde{u}_b = u_2 \equiv 1, \quad \tilde{u}_a = u_1.$$

Define U_0, U_1 as in Theorem 3. Let $0 < c_0 < c_1 < 1$,

$$U_1 := \begin{cases} 0, & 0 \leqslant x < c_0, \\ g_b, & c_0 \leqslant x \leqslant c_1, \\ u_2, & c_1 < x < 1, \end{cases} \quad U_0 := \begin{cases} u_1, & 0 \leqslant x < c_0, \\ g_a, & c_0 \leqslant x \leqslant c_1, \\ 0, & c_1 < x < 1, \end{cases}$$

where g_b, g_a are smooth functions.

The Friedrichs extension of the Legendre equation described by Theorem 3 is

$$D(S_F) = D_{\min} \dotplus \operatorname{span}\{U_0, U_1\}. \tag{3.32}$$

It is easy to check, as in Example 1, that this is equivalent to the well-known boundary condition [6]

$$y(0) = 0 = (py')(1). \tag{3.33}$$

Remark 4. As in Remark 3 we note that the known characterization (3.33) of the Friedrichs 'extension' is actually a restriction of the maximal domain whereas the characterization (3.32) given by Theorem 3 is a genuine extension of the minimal domain.

4. The LC (R)/LP case

At an LP endpoint no boundary condition is required or allowed to determine a self-adjoint realization of (1.1) in $L^2(J, w)$. From the well-known GKN Theorem the following observations follow: If both endpoints are LP, then the deficiency index d is $d = 0$; if exactly one endpoint is LP then $d = 1$; if neither endpoint is LP then $d = 2$. When $d = 0$ there is no boundary condition and $D(S_F) = D_{\min}$. When $d = 1$, then $D(S_F)$ is a one dimensional extension of $D_{\min}$ and a one dimensional restriction of $D_{\max}$. When $d = 2$, then $D(S_F)$ is a two dimensional extension of $D_{\min}$ and a two dimensional restriction of $D_{\max}$.

Theorem 4. *Assume M is LC (or regular) at a and LP at b, the minimal operator $M_{\min}$ being bounded below with a lower bound μ. Then the domain of the Friedrichs extension of $M_{\min}$ is*

$$D(S_F) = D_{\min} \dotplus \operatorname{span}\{z_a\} \quad \text{or} \quad D(S_F) = D_{\min} \dotplus \operatorname{span}\{Z_a\},$$

where $z_a(Z_a)$ is a Limit-Circle solution at a, and z_a is constructed as in Definition 8; and Z_a is constructed as in Theorem 3.

Proof. Let $c \in (a, b)$, $\lambda < \nu$. From the hypotheses that a is LC (R) and b is LP, and $S_{\min}$ is bounded below with lower bound ν and from Lemma 5, Theorem 4.1 in [3], it follows that

(1) The deficiency index of (1.1) on (a, c) is $d_1 = 2$, the deficiency index of (1.1) on (c, b) is $d_2 = 1$.

(2) The Friedrichs extension S_F exists and has discrete spectrum with lower bound ν.

(3) There exist $d_1 = 2$ solutions of equation (1.1) on (a, c) which lie in $L^2((a, c), w)$, denoted as z_1, z_2, determined by the initial conditions

$$z_1(c) = 0, \quad z_1^{[1]}(c) = 1; \quad z_2(c) = 1, \quad z_2^{[1]}(c) = 0.$$

(4) There exists a $d_2 = 1$ solution of equation (1.1) on (c, b) which lies in $L^2((c, b), w)$, denoted as z_3, determined by the initial conditions

$$z_3(c) = 0, \quad z_3^{[1]}(c) = 1.$$

(5) There exist $m_1 = 2d_1 - 2k = 2$ Limit-Circle solutions at a, $d_1 - m_1 = 0$ Limit-Point solutions at a.

(6) There exist $m_2 = 2d_2 - 2k = 0$ Limit-Circle solutions at b, $d_2 - m_2 = 1$ Limit-Point solutions at b.

(7) The solutions z_1, z_2, z_3 can be extended to (a, b) so that the extended functions, denoted by z_1, z_2, z_3, satisfy $z_j \in D_{\max}(a, b)$, $j = 1, 2, 3$, and z_1, z_2 are identically zero in a left neighborhood of b, z_3 is identically zero in a right neighborhood of a, and

$$D_{\max} = D_{\min} \dotplus \operatorname{span}\{z_1, z_2, z_3\}.$$

Define z_a as in Definition 8, Define Z_a as in Theorem 3. We know that only the LC solutions contribute to the determination of the self-adjoint boundary conditions [3], [14]. Since M has no LC solutions at b, there is no restricting condition at b. Therefore,

$$D(S_F) = D_{\min} \dotplus \operatorname{span}\{z_a\} \quad \text{or} \quad D(S_F) = D_{\min} \dotplus \operatorname{span}\{Z_a\}.$$

□

5. Comments

Niessen and Zettl [11], building on the work of Friedrichs [1], [2], Rellich [12], Kalf [5], Rosenberger [13], and others, characterized the Friedrichs extension in terms of boundary conditions determined by the principal and nonprincipal solutions. In this section we summarize these results and compare them with our results which are obtained by a completely different method. Our method is based on the construction of a Hilbert space H_s whose elements are the functions of the domain $D(S_{\min})$ of the minimal symmetric operator $S_{\min}$. The method used in [11] is based on 'regularizing' singular S-L problems and then applying the regular results from [10]; it relies heavily on the oscillatory properties of (1.1) in contrast to our approach which relies on the construction of a Hilbert space whose elements are the domain of $S_{\min}$. Of course, the Friedrichs domain is unique so our characterization is equivalent to that of Niessen-Zettl given in [11]. As we will see below the NZ approach is a 'top down' approach while ours is a 'bottom up' approach. This can be seen from the perspective of the well known von Neumann formula for the domain of the adjoint of a symmetric operator A in an abstract Hilbert space H:

$$D(A^*) = D(A) \dotplus N_\lambda \dotplus N_{\overline{\lambda}}, \quad \operatorname{Im}(\lambda) \neq 0,$$

where N_λ and $N_{\overline{\lambda}}$ are the deficiency spaces. So A is self-adjoint if and only if both the deficiency spaces are $\{0\}$. In our case we have

$$D_{\max} = D_{\min} \dotplus N_\lambda \dotplus N_{\overline{\lambda}}, \quad \operatorname{Im}(\lambda) \neq 0.$$

So to get self-adjointness one needs to empty both the deficiency spaces. The NZ approach can be described as making $D_{\max}$ smaller while our approach here can be described as making $D_{\min}$ bigger; in both cases until equality is achieved. The details follow below.

First we list some well known facts in the next proposition for the convenience of the reader. Recall that an endpoint of (1.1) is oscillatory if there is a nontrivial solution which has an infinite number of zeros in any neighborhood of that endpoint; an endpoint is nonoscillatory if it is not oscillatory. No interior point is oscillatory. At a regular or LC endpoint the oscillation is independent of real λ. At an LP endpoint the oscillation depends on λ. So we say that an LP endpoint is LPO if equation (1.1) is oscillatory for every $\lambda \in \mathbb{R}$.

Proposition 1. *Let* (1.1) *and* (1.2) *hold. Then*

(1) *The equation* (1.1) *is oscillatory at one endpoint of J for some $\lambda \in \mathbb{R}$ if and only if $S_{\min}$ is not bounded below.*

(2) *If an endpoint is LP then no boundary condition is required or allowed at that endpoint in order to determine a self-adjoint extension of the minimal operator $S_{\min}$ in the Hilbert space $H = L^2(J, w)$.*

(3) *Assume a is LC. Then equation* (1.1) *is oscillatory for some $\lambda \in \mathbb{R}$ if and only if it is oscillatory for all $\lambda \in \mathbb{R}$. The same result holds at b.*

Proof. These are well known [16]. □

By combining a number of results from [11] and [10], the following theorem can be obtained.

Theorem 5 (Niessen-Zettl)**.** *Let* (1.1) *and* (1.2) *hold and let $[y, z] = y(p\overline{z}') - z(p\overline{y}')$ for $y, z \in D_{\max}$ denote the Lagrange bracket. Assume there is a $\lambda_a \in \mathbb{R}$ such that* (1.1) *is nonoscillatory for $\lambda = \lambda_a$ and let u_a be the principal solution at a; assume there is a $\lambda_b \in \mathbb{R}$ such that* (1.1) *is nonoscillatory for $\lambda = \lambda_b$ and let u_b be the principal solution at b. Recall that the principal solution is unique up to constant real multiples. Then:*

(1) *If a and b are regular, then the Friedrichs extension S_F exists and its domain is given by*

$$D(S_F) = \{y \in D_{\max} \colon\ y(a) = 0 = y(b)\}.$$

(2) *If a is regular and b is LCNO, then the Friedrichs extension S_F exists and its domain is given by*

$$D(S_F) = \{y \in D_{\max} \colon\ y(a) = 0 = [y, u_b](b)\}.$$

(3) *If b is regular and a is LCNO, then the Friedrichs extension S_F exists and its domain is given by*

$$D(S_F) = \{y \in D_{\max}\colon\ [y, u_a](a) = 0 = y(b)\}.$$

(4) *If each of a and b is LCNO, then the Friedrichs extension S_F exists and its domain is given by*

$$D(S_F) = \{y \in D_{\max}\colon\ [y, u_a](a) = 0 = [y, u_b](b)\}.$$

(5) *If a is regular and b is LP but not LPO, then the Friedrichs extension S_F exists and its domain is given by*

$$D(S_F) = \{y \in D_{\max}\colon\ y(a) = 0\}.$$

(6) *If b is regular and a is LP but not LPO, then the Friedrichs extension S_F exists and its domain is given by*

$$D(S_F) = \{y \in D_{\max}\colon\ y(b) = 0\}.$$

(7) *If both the endpoints are LP but not LPO, then the minimal operator is bounded below, self-adjoint and has no proper self-adjoint extension. In this case*

$$D(S_F) = D_{\min}.$$

(8) *If b is LCNO and a is LP but not LPO, then the Friedrichs extension S_F exists and its domain is given by*

$$D(S_F) = \{y \in D_{\max}\colon\ [y, u_b](b) = 0\}.$$

(9) *If a is LCNO and b is LP but not LPO, then the Friedrichs extension S_F exists and its domain is given by*

$$D(S_F) = \{y \in D_{\max}\colon\ [y, u_a](a) = 0\}.$$

Remark 5. In the NZ Theorem there is no explicit assumption about the regular, LC or LP classification of the endpoints. In contrast, our approach here makes no explicit mention of oscillation or nonoscillation.

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Authors' addresses: *Siqin Yao*, *Jiong Sun*, Math. Dept., Inner Mongolia University, Hohhot, 010021, China, e-mail: `siqin@imu.edu.cn`, `masun@imu.edu.cn`; *Anton Zettl*, Math. Dept., Northern Illinois University, DeKalb, Il. 60115, U.S.A., e-mail: `zettl@math.niu.edu`.