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# ON A HIGH-ORDER ITERATIVE SCHEME FOR A NONLINEAR LOVE EQUATION

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*Abstract.* In this paper, a high-order iterative scheme is established for a nonlinear Love equation associated with homogeneous Dirichlet boundary conditions. This is a development based on recent results (L. T. P. Ngoc, N. T. Long (2011); L. X. Truong, L. T. P. Ngoc, N. T. Long (2009)) to get a convergent sequence at a rate of order $N \geqslant 2$ to a local unique weak solution of the above mentioned equation.

*Keywords*: nonlinear Love equation; Faedo-Galerkin method; convergence of order $N$

*MSC 2010*: 35L20, 35L70

## 1. Introduction

In this paper, we consider the following Dirichlet problem for a nonlinear Love equation

$$u_{tt} - u_{xx} - u_{xxtt} = f(x,t,u), \quad 0 < x < 1,\ 0 < t < T, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = \tilde{u}_0(x), \quad u_t(x,0) = \tilde{u}_1(x), \tag{1.3}$$

where $\tilde{u}_0, \tilde{u}_1, f$ are given functions.

When $f = 0$, (1.1) is related to the Love equation

$$u_{tt} - \frac{E}{\varrho} u_{xx} - 2\mu^2 k^2 u_{xxtt} = 0, \tag{1.4}$$

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presented by V. Radochová in 1978 (see [17]). This equation describes the vertical oscillations of a rod, which was established from Euler's variational equation of an energy functional

$$\int_0^T \mathrm{d}t \int_0^L \Big[\frac{1}{2}F\varrho(u_t^2+\mu^2k^2u_{tx}^2)-\frac{1}{2}F(Eu_x^2+\varrho\mu^2k^2u_xu_{xtt})\Big]\,\mathrm{d}x. \tag{1.5}$$

The parameters in (1.5) have the following meaning: $u$ is the displacement, $L$ is the length of the rod, $F$ is the area of cross-section, $k$ is the cross-section radius, $E$ is the Young modulus of the material, and $\varrho$ is the mass density. By using the Fourier method, Radochová [17] obtained a classical solution of equation (1.4) associated with the initial condition (1.3) and boundary conditions

$$u(0,t)=u(L,t)=0,$$

or

$$\begin{cases} u(0,t)=0, \\ \varepsilon u_{xtt}(L,t)+c^2u_x(L,t)=0, \end{cases} \tag{1.6}$$

where $c^2=E/\varrho$, $\varepsilon=2\mu^2k^2$.

Equations of Love waves or Love-type waves have been studied by many authors, we refer to [3], [6], [11], [10], [16], and references therein.

In [10], by combining the linearization method for the nonlinear term, the Faedo-Galerkin method and the weak compactness method, the existence of a unique weak solution of a Dirichlet problem for the nonlinear Love equation $u_{tt}-u_{xx}-u_{xxtt}=f(x,t,u,u_x,u_t,u_{xt})$ is proved. We note, however, the recurrent sequence obtained here converges only at a rate of order 1.

It is well known that Newton's method and its variants are used to solve nonlinear operator equations or systems of nonlinear equations, see [15] and references therein. In case $\lim\limits_{n\to\infty} u_n=u$, one speaks of *convergence of order* $N$ if $|u_{n+1}-u|\leqslant C|u_n-u|^N$ for some $C>0$ and all large $N$. In the special cases $N=1$ with $C<1$ and $N=2$ one also speaks of linear and quadratic convergence, respectively, see [5]. Based on the ideas about recurrence relations of these methods, a high-order iterative scheme can be constructed for solving the nonlinear operator equation, see [13], [12], [19], [20].

In [18], a symmetric version of the regularized long wave equation (SRLWE)

$$\begin{cases} u_{xxt}-u_t=\varrho_x+uu_x, \\ \varrho_t+u_x=0, \end{cases} \tag{1.7}$$

has been proposed as a model for propagation of weakly nonlinear ion acoustic and space-charge waves. Obviously, eliminating $\varrho$ from (1.7), we get

$$u_{tt} - u_{xx} - u_{xxtt} = -uu_{xt} - u_x u_t. \tag{1.8}$$

The SRLWE (1.8) is explicitly symmetric in the $x$ and $t$ derivatives and it is very similar to the regularized long wave equation which describes shallow water waves and plasma drift waves [1], [2]. The SRLWE also arises in many other areas of mathematical physics [4], [9], [14]. Note that (1.8) is a special form of the equation discussed in [10].

Motivated by results for Love equations in [11], [10], and based on the use of a high-order iterative scheme in [13], [12], [19], [20], in this note, we will establish a similar scheme to get the convergence of order $N$ for problem (1.1)–(1.3). To achieve this purpose, we define a recurrent sequence $\{u_m\}$ associated with equation (1.1) as follows:

$$\frac{\partial^2 u_m}{\partial t^2} - \frac{\partial^2 u_m}{\partial x^2} - \frac{\partial^4 u_m}{\partial t^2 \partial x^2} = \sum_{i=0}^{N-1} \frac{1}{i!} \frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})(u_m - u_{m-1})^i, \ \ 0 < x < 1, \ 0 < t < T,$$

with $u_m$ satisfying (1.2), (1.3) and $u_0 \equiv 0$. If $f \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, we prove that the sequence $\{u_m\}$ converges at a rate of order $N$ to a weak unique solution of problem (1.1)–(1.3).

Note that, if equation (1.1) does not contain the term $u_{xxtt}$, a solution $u$ of problem (1.1)–(1.3) can be found in the space $S_1 = \{u \in L^\infty(0,T; H_0^1 \cap H^2)\colon u_t \in L^\infty(0,T; H_0^1),\ u_{tt} \in L^\infty(0,T; L^2)\}$, whereas adding the term $u_{xxtt}$ yields $u \in S = \{u \in L^\infty(0,T; H_0^1 \cap H^2)\colon u_t, u_{tt} \in L^\infty(0,T; H_0^1 \cap H^2)\}$. Since $S \subset S_1$, it means that the regularity of solutions improves.

## 2. A high-order iterative scheme

First, we put $\Omega = (0,1)$ and denote the usual function spaces used in this paper by $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in $L^2$ or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in $L^2$ and we denote by $\|\cdot\|_X$ the norm in the Banach space $X$. We call $X'$ the dual space of $X$.

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \nabla u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $(\partial u/\partial t)(x,t)$, $(\partial^2 u/\partial t^2)(x,t)$, $(\partial u/\partial x)(x,t)$, $(\partial^2 u/\partial x^2)(x,t)$, respectively.

Next, we will define the following norms on appropriate spaces. This functional setting allows us to make precise the concept of a weak solution of problem (1.1)–(1.3) used in this note. We will use the norm $\|v\|_{H^1} = (\|v\|^2 + \|v_x\|^2)^{1/2}$ on $H^1$. It is known that the imbedding $H^1 \hookrightarrow C^0([0,1])$ is compact and $\|v\|_{C^0([0,1])} \leqslant \sqrt{2}\|v\|_{H^1}$, for all $v \in H^1$. Furthermore, on $H_0^1 = \{u \in H^1\colon\ u(0) = u(1) = 0\}$, the two norms $v \mapsto \|v\|_{H^1}$ and $v \mapsto \|v_x\|$ are equivalent and $\|v\|_{C^0([0,1])} \leqslant \|v_x\|$ for all $v \in H_0^1$. Finally, on $H_0^1 \cap H^2 = \{v \in H^2\colon\ v(0) = v(1) = 0\}$, we will use the norm $\|v\|_{H_0^1\cap H^2} = \sqrt{\|v_x\|^2 + \|v_{xx}\|^2}$.

**Definition.** We say that $u$ is a weak solution of problem (1.1)–(1.3) if

$$u \in L^\infty(0,T; H_0^1 \cap H^2), \quad \dot{u},\ \ddot{u} \in L^\infty(0,T; H_0^1 \cap H^2),$$

and $u$ satisfies the following variational equation:

$$\langle \ddot{u}(t), w\rangle + \langle u_x(t) + \ddot{u}_x(t), w_x\rangle = \langle f(x,t,u), w\rangle$$

for all $w \in H_0^1$ and a.e. $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0, \quad \dot{u}(0) = \tilde{u}_1.$$

Now, we make the following assumptions:

($A_1$) $\tilde{u}_0,\ \tilde{u}_1 \in H_0^1 \cap H^2$,

($A_2$) $f \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R})$ such that

(i) $\partial^i f/\partial u^i \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, $0 \leqslant i \leqslant N-1$,

(ii) $\partial^N f/\partial u^N \in C^0([0,1] \times \mathbb{R}_+ \times \mathbb{R})$,

(iii) $f(0,t,0) = f(1,t,0) = 0$ for all $t \geqslant 0$.

Fix $T^* > 0$. For each $M > 0$ given, we set the constants $K_0(M,f)$, $K_1(M,f)$, $K_M(f)$ as follows:

$$\begin{cases} K_0(M,f) = \sup\{|f(x,t,u)|\colon\ 0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant T^*,\ |u| \leqslant M\}, \\ K_1(M,f) = K_0(M,f) + K_0\Big(M, \dfrac{\partial f}{\partial x}\Big) + K_0\Big(M, \dfrac{\partial f}{\partial t}\Big) + K_0\Big(M, \dfrac{\partial f}{\partial u}\Big), \\ \quad K_M(f) = \displaystyle\sum_{i=0}^{N-1} K_1\Big(M, \frac{\partial^i f}{\partial u^i}\Big) + K_0\Big(M, \frac{\partial^N f}{\partial u^N}\Big). \end{cases}$$

For every $T \in (0,T^*]$ and $M > 0$, we put

$$\begin{cases} W(M,T) = \{v \in L^\infty(0,T; H_0^1 \cap H^2)\colon\ v_t \in L^\infty(0,T; H_0^1 \cap H^2), \\ \qquad\qquad v_{tt} \in L^\infty(0,T; H_0^1),\ \text{with } \|v\|_{L^\infty(0,T;H_0^1\cap H^2)}, \\ \qquad\qquad \|v_t\|_{L^\infty(0,T;H_0^1\cap H^2)},\ \|v_{tt}\|_{L^\infty(0,T;H_0^1)} \leqslant M\}, \\ W_1(M,T) = \{v \in W(M,T)\colon\ v_{tt} \in L^\infty(0,T; H_0^1 \cap H^2)\}. \end{cases}$$

In the following, we will establish the recurrent sequence $\{u_m\}$ via a high-order iterative scheme.

**Theorem 2.1.** *Suppose that the assumptions* $(\mathrm{A}_1)$, $(\mathrm{A}_2)$ *are fulfilled. Then there exist positive constants* $M, T$ *and a sequence* $\{u_m\} \subset W_1(M,T)$ *defined as follows:*

(i) *the first term is* $u_0 = 0$;

(ii) *with each given term*

$$u_{m-1} \in W_1(M,T), \tag{2.1}$$

*there exists* $u_m \in W_1(M,T)$ $(m \geqslant 1)$ *satisfying*

$$\begin{cases} \langle \ddot{u}_m(t), w\rangle + \langle u_{mx}(t) + \ddot{u}_{mx}(t), w_x\rangle = \langle F_m(t), w\rangle \quad \forall\, w \in H_0^1, \\ u_m(0) = \tilde{u}_0, \dot{u}_m(0) = \tilde{u}_1, \end{cases} \tag{2.2}$$

*in which*

$$F_m(x,t) = \sum_{i=0}^{N-1} \frac{1}{i!}\frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})(u_m - u_{m-1})^i. \tag{2.3}$$

P r o o f. *Approximating solutions.* To prove this theorem, we use the Faedo-Galerkin method.

Consider a special orthonormal basis $\{w_j\}$ on $H_0^1$: $w_j(x) = \sqrt{2}\sin(j\pi x)$, $j = 1,2,\ldots$, formed by the eigenfunctions of the Laplacian $-\Delta = -\partial^2/\partial x^2$. It is clear that $w_j$ satisfies

$$-\Delta w_j = \lambda_j w_j,\ w_j \in H_0^1 \cap H^2, \quad \lambda_j = (j\pi)^2,\ j = 1,2,\ldots$$

If

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t) w_j$$

is a solution of the system

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + \langle u_{mx}^{(k)}(t) + \ddot{u}_{mx}^{(k)}(t), w_{jx}\rangle = \langle F_m^{(k)}(t), w_j\rangle, \quad j = 1,2,\ldots,k, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{2.4}$$

with

$$\begin{cases} \tilde{u}_{0k} = \displaystyle\sum_{j=1}^{k} \alpha_j^{(k)} w_j \longrightarrow \tilde{u}_0 \quad \text{strongly in } H_0^1 \cap H^2, \\ \tilde{u}_{1k} = \displaystyle\sum_{j=1}^{k} \beta_j^{(k)} w_j \longrightarrow \tilde{u}_1 \quad \text{strongly in } H_0^1 \cap H^2, \end{cases} \tag{2.5}$$

and

$$
(2.6)\qquad \begin{cases} F_m^{(k)}(x,t) = \displaystyle\sum_{i=0}^{N-1} \frac{1}{i!}\frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})(u_m^{(k)} - u_{m-1})^i \\ \qquad\qquad\; = \displaystyle\sum_{j=0}^{N-1} \Psi_j(x,t,u_{m-1})(u_m^{(k)})^j, \\ \Psi_j(x,t,u_{m-1}) = \displaystyle\sum_{i=j}^{N-1} \frac{(-1)^{i-j}}{j!(i-j)!}\frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})u_{m-1}^{i-j}, \end{cases}
$$

then $c_{mj}^{(k)}$ satisfies the following system of nonlinear ordinary differential equations:

$$
(2.7)\qquad \begin{cases} \ddot{c}_{mj}^{(k)}(t) + \mu_j^2 c_{mj}^{(k)}(t) = f_{mj}^{(k)}(t), \\ c_{mj}^{(k)}(0) = \alpha_j^{(k)}, \quad \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)}, \quad 1 \leqslant j \leqslant k, \end{cases}
$$

where

$$
f_{mj}^{(k)}(t) = \frac{1}{1+\lambda_j}\langle F_m^{(k)}(t), w_j\rangle, \quad \mu_j^2 = \frac{\lambda_j}{1+\lambda_j}, \quad \lambda_j = (j\pi)^2, \quad 1 \leqslant j \leqslant k.
$$

Using Banach's contraction principle, it is not difficult to show that (2.7) has a unique solution $c_{mj}^{(k)}(t)$ in $[0, T_m^{(k)}]$, with certain $T_m^{(k)} \in (0,T]$ (see [12]). Therefore, (2.4) has a unique solution $u_m^{(k)}(t)$ in $[0, T_m^{(k)}]$.

The following estimates allow one to take $T_m^{(k)} = T$ independent of $m$ and $k$. By such a priori estimates of $u_m^{(k)}(t)$, it can be extended outside $[0, T_m^{(k)}]$ and then, a solution defined in $[0,T]$ will be obtained.

*Estimates.* Multiply $(2.4)_1$ by $\dot{c}_{mj}^{(k)}(t)$ and sum over $j$. After that, integrating with respect to the time variable from 0 to $t$, we have

$$
(2.8)\qquad \begin{aligned} p_m^{(k)}(t) &\equiv \|\dot{u}_m^{(k)}(t)\|^2 + \|u_{mx}^{(k)}(t)\|^2 + \|\dot{u}_{mx}^{(k)}(t)\|^2 \\ &= p_m^{(k)}(0) + 2\int_0^t \langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\rangle \,\mathrm{d}s. \end{aligned}
$$

Replacing $w_j$ in $(2.4)_1$ by $-w_{jxx}/\lambda_j$, and integrating by parts, we obtain

$$
\langle \ddot{u}_{mx}^{(k)}(t), w_{jx}\rangle + \langle u_{mxx}^{(k)}(t) + \ddot{u}_{mxx}^{(k)}(t), w_{jxx}\rangle = \langle F_{mx}^{(k)}(t), w_{jx}\rangle, \quad 1 \leqslant j \leqslant k,
$$

therefore, in the same way as (2.8),

$$
(2.9)\qquad \begin{aligned} q_m^{(k)}(t) &\equiv \|\dot{u}_{mx}^{(k)}(t)\|^2 + \|u_{mxx}^{(k)}(t)\|^2 + \|\dot{u}_{mxx}^{(k)}(t)\|^2 \\ &= q_m^{(k)}(0) + 2\int_0^t \langle F_{mx}^{(k)}(s), \dot{u}_{mx}^{(k)}(s)\rangle \,\mathrm{d}s. \end{aligned}
$$

Furthermore, because $c_{mj}^{(k)}(t)$ is a solution of the system (2.7), both $\dddot{c}_{mj}^{(k)}(t)$ and $\dddot{u}_m^{(k)}(t) = \sum_{j=1}^{k} \dddot{c}_{mj}^{(k)}(t)w_j$ are defined. Hence, we can take the derivative with respect to $t$ of $(2.4)_1$ and then

$$\langle \dddot{u}_m^{(k)}(t), w_j\rangle + \langle \dot{u}_{mx}^{(k)}(t) + \ddot{u}_{mx}^{(k)}(t), w_{jx}\rangle = \langle \dot{F}_m^{(k)}(t), w_j\rangle, \tag{2.10}$$

for all $1 \leqslant j \leqslant m$. Multiplying (2.10) by $\ddot{c}_{mj}(t)$, summing over $j$ and integrating from 0 to $t$ implies

$$\begin{aligned} r_m^{(k)}(t) &= \|\ddot{u}_m^{(k)}(t)\|^2 + \|\dot{u}_{mx}^{(k)}(t)\|^2 + \|\ddot{u}_{mx}^{(k)}(t)\|^2 \\ &= r_m^{(k)}(0) + 2\int_0^t \langle \dot{F}_m^{(k)}(s), \ddot{u}_m^{(k)}(s)\rangle \,\mathrm{d}s. \end{aligned} \tag{2.11}$$

Combining (2.8), (2.9), and (2.11) leads to

$$\begin{aligned} S_m^{(k)}(t) &= p_m^{(k)}(t) + q_m^{(k)}(t) + r_m^{(k)}(t) \\ &= S_m^{(k)}(0) + 2\int_0^t \langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\rangle \,\mathrm{d}s \\ &\quad + 2\int_0^t \langle F_{mx}^{(k)}(s), \dot{u}_{mx}^{(k)}(s)\rangle \,\mathrm{d}s + 2\int_0^t \langle \dot{F}_m^{(k)}(s), \ddot{u}_m^{(k)}(s)\rangle \,\mathrm{d}s \\ &\equiv S_m^{(k)}(0) + \sum_{j=1}^{3} I_j. \end{aligned} \tag{2.12}$$

Letting $t \to 0_+$ in $(2.4)_1$ and multiplying the result obtained by $\ddot{c}_{mj}^{(k)}(0)$, we get

$$\|\ddot{u}_m^{(k)}(0)\|^2 + \|\ddot{u}_{mx}^{(k)}(0)\|^2 + \langle u_{mx}^{(k)}(0), \ddot{u}_{mx}^{(k)}(0)\rangle = \langle F_m^{(k)}(0), \ddot{u}_m^{(k)}(0)\rangle.$$

Consequently,

$$\begin{aligned} \xi_m^{(k)} &= \|\ddot{u}_m^{(k)}(0)\|^2 + \|\ddot{u}_{mx}^{(k)}(0)\|^2 \\ &\leqslant \|u_{mx}^{(k)}(0)\|\|\ddot{u}_{mx}^{(k)}(0)\| + \|F_m^{(k)}(0)\|\|\ddot{u}_m^{(k)}(0)\| \\ &\leqslant \|u_{mx}^{(k)}(0)\|\sqrt{\xi_m^{(k)}} + \|F_m^{(k)}(0)\|\sqrt{\xi_m^{(k)}} \\ &\leqslant \frac{1}{2}\xi_m^{(k)} + \frac{1}{2}(\|u_{mx}^{(k)}(0)\| + \|F_m^{(k)}(0)\|)^2 \\ &= \frac{1}{2}\xi_m^{(k)} + \frac{1}{2}\biggl(\|\tilde{u}_{0kx}\| + \biggl\|\sum_{i=0}^{N-1} \frac{1}{i!}\frac{\partial^i f}{\partial u^i}(x,t,\tilde{u}_0)(\tilde{u}_{0k} - \tilde{u}_0)^i\biggr\|\biggr)^2 \\ &= \frac{1}{2}\xi_m^{(k)} + \frac{1}{2}\biggl(\|\tilde{u}_{0kx}\| + \sum_{i=0}^{N-1} \frac{(\|\tilde{u}_{0kx}\| + \|\tilde{u}_{0x}\|)^i}{i!} \sup_{\substack{0\leqslant x\leqslant 1,\, 0\leqslant t\leqslant T^*,\\ |z|\leqslant \|\tilde{u}_{0x}\|}} \biggl|\frac{\partial^i f}{\partial u^i}(x,t,z)\biggr|\biggr)^2, \end{aligned}$$

which gives that for all $m, k \in \mathbb{N}$,

$$\xi_m^{(k)} \leqslant \Big(\|\tilde{u}_{0kx}\| + \sum_{i=0}^{N-1} \frac{(\|\tilde{u}_{0kx}\| + \|\tilde{u}_{0x}\|)^i}{i!} \sup_{\substack{0\leqslant x\leqslant 1,\, 0\leqslant t\leqslant T^*,\\ |z|\leqslant\|\tilde{u}_{0x}\|}} \Big|\frac{\partial^i f}{\partial u^i} f(x,t,z)\Big|\Big)^2. \tag{2.13}$$

By (2.5) and (2.13), we can deduce that there exists a constant $M > 0$, independent of $k$ and $m$, such that

$$\begin{aligned} S_m^{(k)}(0) &= \|\tilde{u}_{1k}\|^2 + \|\tilde{u}_{0k}\|^2 + 3\|\tilde{u}_{1kx}\|^2 + \|\tilde{u}_{0kxx}\|^2 + \|\tilde{u}_{1kxx}\|^2 + \xi_m^{(k)} \\ &\leqslant \frac{M^2}{4} \quad \forall\, m, k \in \mathbb{N}. \end{aligned} \tag{2.14}$$

In order to continue the proof, we will state the following properties of $F_m^{(k)}(t)$, $F_{mx}^{(k)}(t)$, $\dot{F}_m^{(k)}(t)$. Their proof is analogous to [12], Lemma 3.3.

$$\begin{aligned} &\text{(i)} \quad \|F_m^{(k)}(t)\| \leqslant \tilde{b}_M\Big[1 + \Big(\sqrt{S_m^{(k)}(t)}\Big)^{N-1}\Big], \\ &\text{(ii)} \quad \|F_{mx}^{(k)}(t)\| \leqslant \tilde{b}_M\Big[1 + \Big(\sqrt{S_m^{(k)}(t)}\Big)^{N-1}\Big], \\ &\text{(iii)} \quad \|\dot{F}_m^{(k)}(t)\| \leqslant \tilde{b}_M\Big[1 + \Big(\sqrt{S_m^{(k)}(t)}\Big)^{N-1}\Big], \end{aligned} \tag{2.15}$$

where $\tilde{b}_M = (M+N)K_M(f)\sum\limits_{i=0}^{N-1}\tilde{a}_i$ and $\tilde{a}_0 = 1 + \sum\limits_{i=1}^{N-1} 2^{i-1}M^i/i!$, $\tilde{a}_i = 2^{i-1}/i!$, $i = 1, 2, \ldots, N-1$.

Using (2.15)(i), we have

$$\begin{aligned} I_1 &= 2\int_0^t \langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\rangle \,\mathrm{d}s \leqslant 2\int_0^t \|F_m^{(k)}(s)\|\|\dot{u}_m^{(k)}(s)\| \,\mathrm{d}s \\ &\leqslant 2\tilde{b}_M \int_0^t \Big[1 + \Big(\sqrt{S_m^{(k)}(s)}\Big)^{N-1}\Big]\sqrt{S_m^{(k)}(s)} \,\mathrm{d}s \leqslant 4\tilde{b}_M\Big[T + \int_0^t (S_m^{(k)}(s))^N \,\mathrm{d}s\Big], \end{aligned} \tag{2.16}$$

and, similarly,

$$I_2 \leqslant 4\tilde{b}_M\Big[T + \int_0^t (S_m^{(k)}(s))^N \,\mathrm{d}s\Big], \tag{2.17}$$

$$I_3 \leqslant 4\tilde{b}_M\Big[T + \int_0^t (S_m^{(k)}(s))^N \,\mathrm{d}s\Big]. \tag{2.18}$$

Combining (2.12), (2.14), (2.16)–(2.18), it follows that

$$S_m^{(k)}(t) \leqslant \frac{M^2}{4} + 12T\tilde{b}_M + 12\tilde{b}_M\int_0^t (S_m^{(k)}(s))^N \,\mathrm{d}s, \quad 0 \leqslant t \leqslant T. \tag{2.19}$$

Then, by solving a nonlinear Volterra integral equation (based on the methods in [7]), it follows that there exists a constant $T > 0$ independent of $k$ and $m$ such that

$$S_m^{(k)}(t) \leqslant M^2 \quad \forall\, t \in [0,T], \quad \forall\, k, m \in \mathbb{N}. \tag{2.20}$$

Therefore, we can take constant $T_m^{(k)} = T$ for all $m$ and $k$. Thus,

$$u_m^{(k)} \in W(M,T) \quad \text{for all } m \text{ and } k. \tag{2.21}$$

*Convergence.* Thanks to (2.21), there exists a subsequence of $\{u_m^{(k)}\}$, denoted by the same symbol, such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in } L^\infty(0,T;H_0^1 \cap H^2) \text{ weakly*}, \\ \dot{u}_m^{(k)} \to \dot{u}_m & \text{in } L^\infty(0,T;H_0^1 \cap H^2) \text{ weakly*}, \\ \ddot{u}_m^{(k)} \to \ddot{u}_m & \text{in } L^\infty(0,T;H_0^1) \text{ weakly*}, \\ u_m \in W(M,T). \end{cases} \tag{2.22}$$

Applying the compactness lemma of Lions ([8], page 57) and the Riesz-Fischer theorem, from (2.22), there exists a subsequence of $\{u_m^{(k)}\}$, also denoted by the same symbol, satisfying

$$\begin{cases} u_m^{(k)} \to u_m & \text{strongly in } L^2(0,T;H_0^1) \text{ and a.e. in } Q_T, \\ \dot{u}_m^{(k)} \to \dot{u}_m & \text{strongly in } L^2(0,T;H_0^1) \text{ and a.e. in } Q_T. \end{cases} \tag{2.23}$$

On the other hand, by $L^\infty(0,T;H_0^1 \cap H^2) \hookrightarrow L^\infty(Q_T)$ and the inequality

$$|a^j - b^j| \leqslant jM^{j-1}|a-b| \quad \forall\, a,b \in [-M,M],\ \forall\, M > 0,\ \forall\, j \in \mathbb{N},$$

we deduce from (2.20) that

$$|(u_m^{(k)})^j - (u_m)^j| \leqslant jM^{j-1}|u_m^{(k)} - u_m|, \quad j = 0,\dots,N-1. \tag{2.24}$$

Therefore, (2.23) and (2.24) give

$$(u_m^{(k)})^j \to (u_m)^j \quad \text{strongly in } L^2(Q_T). \tag{2.25}$$

Note that

$$\begin{aligned} \|F_m^{(k)} - F_m\|_{L^2(Q_T)} &\leqslant \sum_{j=0}^{N-1} \|\Psi_j(\cdot,\cdot,u_{m-1})\|_{L^\infty(Q_T)} \|(u_m^{(k)})^j - (u_m)^j\|_{L^2(Q_T)} \\ &\leqslant K_M(f) \sum_{j=0}^{N-1} \sum_{i=j}^{N-1} \frac{M^{i-j}}{j!\,(i-j)!} \|(u_m^{(k)})^j - (u_m)^j\|_{L^2(Q_T)}, \end{aligned} \tag{2.26}$$

so (2.25) leads to

$$F_m^{(k)} \to F_m \quad \text{strongly in } L^2(Q_T). \tag{2.27}$$

Passing to limit in (2.4), (2.5), we have $u_m$ satisfying (2.2), (2.3) in $L^2(0,T)$. On the other hand, it follows from $(2.2)_1$ and $(2.22)_4$ that

$$\frac{\partial^2}{\partial x^2}(\ddot{u}_m(t)+u_m(t)) = \ddot{u}_m(t) - F_m(t) \in L^\infty(0,T;H_0^1).$$

Consequently,

$$\ddot{u}_m(t)+u_m(t) = \Phi \in L^\infty(0,T;H_0^1\cap H^2),$$

and then

$$\ddot{u}_m(t) = \Phi - u_m(t) \in L^\infty(0,T;H_0^1\cap H^2).$$

Hence, $u_m \in W_1(M,T)$ and Theorem 2.1 is proved. □

Next, we set

$$W_1(T) = \{v\in L^\infty(0,T;H_0^1)\colon\ \dot{v}\in L^\infty(0,T;H_0^1)\}.$$

Then $W_1(T)$ is a Banach space with respect to the norm

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;H_0^1)} + \|\dot{v}\|_{L^\infty(0,T;H_0^1)}.$$

**Theorem 2.2.** *Suppose that the assumptions* ($A_1$), ($A_2$) *are fulfilled. Then*

(i) *problem* (1.1)–(1.3) *has a unique weak solution* $u\in W_1(M,T)$, *where the constants* $M>0$ *and* $T>0$ *are chosen as in* (2.14), (2.20).

*Furthermore,*

(ii) *the recurrent sequence* $\{u_m\}$, *defined by* (2.1)–(2.3), *converges at a rate of order* $N$ *to the solution* $u$ *strongly in the space* $W_1(T)$ *in the sense*

$$\|u_m-u\|_{W_1(T)} \leqslant C\|u_{m-1}-u\|^N_{W_1(T)}, \tag{2.28}$$

*for all* $m\geqslant 1$, *where* $C$ *is a suitable constant. On the other hand, the estimate is fulfilled*

$$\|u_m-u\|_{W_1(T)} \leqslant C_T(\beta_T)^{N^m} \quad \forall\, m\in\mathbb{N}, \tag{2.29}$$

*where* $C_T$ *and* $\beta_T<1$ *are constants depending only on* $\tilde{u}_0$, $\tilde{u}_1$, $f$, *and* $T$.

P r o o f. In the sequel, we will prove Theorem 2.2 only with $N\geqslant 2$.

*Existence.* We can prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$.

Indeed, let $w_m = u_{m+1}-u_m$. Then $w_m$ satisfies the variational problem

$$\begin{cases} \langle \ddot{w}_m(t), w\rangle + \langle w_{mx}(t)+\ddot{w}_{mx}(t), w_x\rangle = \langle F_{m+1}(t)-F_m(t), w\rangle \quad \forall\, w\in H_0^1,\\ w_m(0)=\dot{w}_m(0)=0. \end{cases} \tag{2.30}$$

Taking $w = \dot{w}_m$ in (2.30), after integrating in $t$, we get

$$Z_m(t) = 2\int_0^t \langle F_{m+1}(s) - F_m(s), \dot{w}_m(s)\rangle \,\mathrm{d}s, \tag{2.31}$$

where

$$Z_m(t) = \|\dot{w}_m(t)\|^2 + \|w_{mx}(t)\|^2 + \|\dot{w}_{mx}(t)\|^2. \tag{2.32}$$

Using Taylor's expansion of the function $f(x,t,u_m)$ around the point $u_{m-1}$ up to order $N$, we obtain

$$\begin{aligned} f(x,t,u_m) &- f(x,t,u_{m-1}) \\ &= \sum_{i=1}^{N-1} \frac{1}{i!}\frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})w_{m-1}^i + \frac{1}{N!}\frac{\partial^N f}{\partial u^N}(x,t,\bar{\lambda}_m)w_{m-1}^N, \end{aligned} \tag{2.33}$$

where $\bar{\lambda}_m = \bar{\lambda}_m(x,t) = u_{m-1} + \theta_1(u_m - u_{m-1})$, $0 < \theta_1 < 1$.

Hence, it follows from (2.3) and (2.33) that

$$\begin{aligned} F_{m+1}(x,t) - F_m(x,t) &= f(x,t,u_m) - f(x,t,u_{m-1}) \\ &\quad + \sum_{i=1}^{N-1} \frac{1}{i!}\frac{\partial^i f}{\partial u^i}(x,t,u_m)w_m^i - \sum_{i=1}^{N-1} \frac{1}{i!}\frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})w_{m-1}^i \\ &= \sum_{i=1}^{N-1} \frac{1}{i!}\frac{\partial^i f}{\partial u^i}(x,t,u_m)w_m^i + \frac{1}{N!}\frac{\partial^N f}{\partial u^N}(x,t,\bar{\lambda}_m)w_{m-1}^N. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|F_{m+1}(t) - F_m(t)\| &\leqslant K_M(f)\sum_{i=1}^{N} \frac{1}{i!}\|w_{mx}(t)\|^i + \frac{1}{N!}K_M(f)\|w_{m-1\ x}(t)\|^N \\ &\leqslant \gamma_T^{(1)}\sqrt{Z_m(t)} + \gamma_T^{(2)}\big(\sqrt{Z_{m-1}(t)}\big)^N, \end{aligned} \tag{2.34}$$

where

$$\gamma_T^{(1)} = K_M(f)\sum_{i=1}^{N}\frac{1}{i!}M^{i-1}, \quad \gamma_T^{(2)} = \frac{1}{N!}K_M(f).$$

Then we deduce from (2.31), (2.32), and (2.34) that

$$\begin{aligned} Z_m(t) &\leqslant 2\int_0^t \|F_{m+1}(s) - F_m(s)\|\|\dot{w}_m(s)\|\,\mathrm{d}s \\ &\leqslant 2\int_0^t [\gamma_T^{(1)}\sqrt{Z_m(s)} + \gamma_T^{(2)}(\sqrt{Z_{m-1}(s)})^N]\sqrt{Z_m(s)}\,\mathrm{d}s \end{aligned} \tag{2.35}$$

$$\leqslant \gamma_T^{(2)} \int_0^T Z_{m-1}^N(s)\,\mathrm{d}s + (2\gamma_T^{(1)} + \gamma_T^{(2)}) \int_0^t Z_m(s)\,\mathrm{d}s$$
$$\leqslant T\gamma_T^{(2)} \|w_{m-1}\|_{W_1(T)}^{2N} + (2\gamma_T^{(1)} + \gamma_T^{(2)}) \int_0^t Z_m(s)\,\mathrm{d}s.$$

Using Gronwall's lemma, (2.35) leads to

$$\|w_m\|_{W_1(T)} \leqslant \mu_T \|w_{m-1}\|_{W_1(T)}^N, \tag{2.36}$$

where $\mu_T = 2\sqrt{\gamma_T^{(2)} T \exp((2\gamma_T^{(1)} + \gamma_T^{(2)})T)}$.

Choosing $T$ small enough such that

$$\|u_1 - u_0\|_{W_1(T)} \mu_T^{1/(N-1)} = \|u_1\|_{W_1(T)} \mu_T^{1/(N-1)} \leqslant M \mu_T^{1/(N-1)} \equiv \beta_T < 1,$$

it follows from (2.36) that for all $m$ and $p$,

$$\begin{aligned} \|u_m - u_{m+p}\|_{W_1(T)} &\leqslant (1 - \|u_1 - u_0\|_{W_1(T)} \mu_T^{1/(N-1)})^{-1} (\mu_T)^{-1/(N-1)} \\ &\quad \times (\|u_1 - u_0\|_{W_1(T)} \mu_T^{1/(N-1)})^{N^m} \\ &\leqslant (1 - \beta_T)^{-1} (\mu_T)^{-1/(N-1)} (\beta_T)^{N^m}. \end{aligned} \tag{2.37}$$

Hence, $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \to u \quad \text{strongly in } W_1(T). \tag{2.38}$$

Note that since $u_m \in W_1(M,T)$, there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in } L^\infty(0,T; H_0^1 \cap H^2) \text{ weakly*}, \\ \dot u_{m_j} \to \dot u & \text{in } L^\infty(0,T; H_0^1 \cap H^2) \text{ weakly*}, \\ \ddot u_{m_j} \to \ddot u & \text{in } L^\infty(0,T; H_0^1) \text{ weakly*}, \\ u \in W_1(M,T). \end{cases} \tag{2.39}$$

We have

$$\begin{aligned} \|F_m(\cdot,t) - f(\cdot,t,u(t))\| &= \left\| \sum_{i=1}^{N-1} \frac{1}{i!} \frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})(u_m - u_{m-1})^i \right\| \\ &\leqslant K_M(f) \sum_{i=1}^{N-1} \frac{1}{i!} \|u_m - u_{m-1}\|_{W_1(T)}^i. \end{aligned} \tag{2.40}$$

Hence, (2.38) and (2.40) imply that

$$F_m(t) \to f(\cdot, t, u(t)) \quad \text{strongly in } L^\infty(0, T; L^2).$$

Finally, passing to limit in (2.2) and (2.3) as $m = m_j \to \infty$, there exists $u \in W(M, T)$ satisfying the equation

$$\langle \ddot{u}(t), w\rangle + \langle u_x(t) + \ddot{u}_x(t), w_x\rangle = \langle f(\cdot, t, u(t)), w\rangle,$$

for all $w \in H_0^1$ and the initial condition

$$u(0) = \tilde{u}_0, \quad \dot{u}(0) = \tilde{u}_1.$$

*Uniqueness.* Applying a similar argument as used in the proof of Theorem 2.1, $u \in W_1(M, T)$ is the local unique weak solution of problem (1.1)–(1.3).

Passing to the limit in (2.37) as $p \to \infty$ for fixed $m$, we get (2.29). In the same way as (2.29), (2.28) follows. Theorem 2.2 is proved completely. □

Remark. (i) If the convergence of $\{u_m\}$ is only at a rate of order 1, it follows from (2.29) that the error at the $m$-th step is $C_T(\beta_T)^m$ with $0 < \beta_T = \mu_T < 1$ ($T$ is small enough). If the convergence of $\{u_m\}$ is at a rate of order $N \geqslant 2$, this error is $C_T(\beta_T)^{N^m}$ and thus converges more rapidly, where $0 < \beta_T = M\mu_T^{1/(N-1)} < 1$ and $T$ is also small enough.

(ii) In constructing a $N$-order iterative scheme, the function $f$ has to satisfy (A2). This condition can be relaxed if we only consider the existence of a solution, see [10]–[13], [14], [17], [18].

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