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TRACKING THROUGH SINGULARITIES USING SLIDING MODE DIFFERENTIATORS

Bernardino Castillo–Toledo, Stefano Di Gennaro, Armando López–Cuevas

In this work, an alternative solution to the tracking problem for a SISO nonlinear dynamical system exhibiting points of singularity is given. An inversion–based controller is synthesized using the Fliess generalized observability canonical form associated to the system. This form depends on the input and its derivatives. For this purpose, a robust exact differentiator is used for estimating the control derivatives signals with the aim of defining a control law depending on such control derivative estimates and on the system state variables. This control law is such that, when applied to the system, bounded tracking error near the singularities is guaranteed.

Keywords: singularities, sliding mode differentiator, tracking

Classification: 93C10, 41A30

1. INTRODUCTION

As is well–known, the solution of the problem of tracking a given reference signal is closely related to the invertibility properties of the input–output map of the system [5]. For linear systems, depending on the approach considered, a number of results have been obtained [2]. On their basis, some extensions have been made in the nonlinear setting [4, 24].

In general, the invertibility of a system is given in terms of certain regularity properties. In particular, for nonlinear SISO systems, this is given in terms of the existence of a well defined relative degree in a neighborhood of a point x_0 of interest [12]. However, if the relative degree is not well defined in this point, viz. x_0 is a point of singularity, a realization of a (local) inverse of the system in a neighborhood of x_0 for tracking purposes is not directly implementable, and other approaches must be tried. Several physical systems present this characteristic, as for example that in [9] where a singularity appears when trying to control the velocity of a sailboat. The authors deal partially with this problem by changing the operation point and thus avoiding the singular points. To do so, however a pre–compensator that adjusts the desired output to one that the controller can manage needs to be introduced. In [1] an image–based approach to perform visual control for differential–drive robots is performed. The system presents singularities which are treated as in [11]. In [18], for underactuated robots, singular points are neglected, resulting in an inaccurate tracking.

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Several methods have been proposed to deal with this problem, such as those in [11, 14] for exact tracking through singularities, where the concepts of degree and rank of singularity are introduced. In [7, 8], by extending some results of Krener [13] the problem is studied by showing that if an approximated system is found, then, under certain conditions on the approximations and the zero dynamics of the system, the control law calculated for the approximated system guarantees a bounded tracking error and bounded internal variables when applied to the original system. Such an approximation is obtained by neglecting the term that cause the singularity. In [3], an approximating system is obtained by an appropriate extension of the relative degree, based on the generalized observability canonical form [6]. In [22] an approach to track a desired reference is proposed by switching the control law when the term involved with singularity is within a neighborhood of the singular point. The approach proposed in this work is basically a modification of the method proposed in [8], where the main difference consists of changing the control law depending on how far the state is from the singularity. In [10] the authors achieved tracking using switched controllers, which move the state along integral curves at discrete times. In the method proposed in this work, less error derivatives have to be considered to avoid the term with the singularity. As a result, a direct control over the desired surface is not guaranteed.

In the cited works, the main feature is that the singularity–free approximated system is obtained neglecting some terms involving the singularity terms. Obviously, by doing this, it is possible that some important nonlinear dynamics are neglected in the approximated system, and the closed loop behavior could be unsatisfactory. To overcome this difficulty, in this work a different approximated system is proposed: instead of neglecting these terms generating the singularity, they are estimated. It is shown that under certain assumptions on the system, the application of this method results in good tracking performance even in the presence of singular points. This is because, unlike other schemes [3, 22], this approach does not neglect terms involving singularities. On the contrary, this terms are taken into account in the control law. For, a robust sliding mode differentiator is used to get the desired estimations [15, 16], showing that the closed loop behaviour has a better performance. A further advantage of the proposed approach is that it will ensure smooth and bounded control signals near the singular points, since it is not necessary to switch control near these singular points.

The paper is organized as follows. In Section 2, the sliding mode differentiator, used in the following of this work, is presented along with the coordinate transformation applied to the system to obtain the Fliess Generalized Observability Canonical Form (GOCF). In Section 3 the main results are presented, and it is shown that tracking through the singular points can be achieved, even in the presence of bounded perturbations. In Section 4, two examples are shown and discussed and a comparison with other methods proposed in the literature is done. Finally, some concluding remarks are given.

2. A ROBUST SLIDING MODE DIFFERENTIATOR AND THE APPROXIMATED SYSTEM

In this work, the robust sliding mode differentiator proposed in [15, 16, 20] is used to obtain an estimate of the terms appearing in the system dynamics, determining the singularities. This differentiator is based on the homogeneity principle, which ensures its finite–time convergence [17], and can be seen as a limit case of a more general finite–time homogeneous observer [19]. To briefly introduce the robust sliding mode differentiator, let $f(t)$ be an unknown function on $[0,\infty)$, with the kth derivatives having a Lipschitz constant L. The problem of high–order sliding–mode differentiator design is to find real–time robust estimations of $f(t), f^{(1)}(t), \dots, f^{(k)}(t)$. An answer to this problem has been given by the following result.

Theorem 2.1. (Saif et al. [20]) Given an unknown function $f(t)$ on $[0,\infty)$, with the kth derivative having a Lipschitz constant L, the solution of the dynamic system

$$\dot z = g(z, f(t)) = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{k-1} \\ v_k \end{pmatrix} = \begin{pmatrix} z_1 - \lambda_0 |z_0 - f(t)|^{k/(k+1)} \operatorname{sign}(z_0 - f(t)) \\ z_2 - \lambda_1 |z_1 - v_0|^{k-1/k} \operatorname{sign}(z_1 - v_0) \\ \vdots \\ z_k - \lambda_{k-1} |z_{k-1} - v_{k-2}|^{1/2} \operatorname{sign}(z_{k-1} - v_{k-2}) \\ -\lambda_k \operatorname{sign}(z_k - v_{k-1}) \end{pmatrix} \tag{1}$$

with $z = \begin{pmatrix} z_0 & z_1 & \cdots & z_k \end{pmatrix}^T$ and $\lambda_1, \dots, \lambda_k > 0$, is Lyapunov stable, namely there exist $\delta_{t_0}, T_{t_0} > 0$ such that any solution of (1) satisfying

$$\left| z_i(t_0) - f^{(i)}(t_0) \right| \le \delta_{t_0}, \qquad i = 0, \dots, k$$

at the initial time t_0, satisfies

$$\begin{aligned} z_0 &= f(t) \\ z_i &= v_{i-1} = f^{(i)}(t), \qquad i = 1, \dots, k \end{aligned}$$

for any $t \ge t_0 + T_{t_0}$.

In the following we consider the class of nonlinear system affine in the input, described by the equations

$$\begin{aligned} \dot x &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{2}$$

where $x \in \mathcal{D}_x \subset \mathbb{R}^n$, $u \in \mathcal{D}_u \subset \mathbb{R}$, $y \in \mathcal{D}_y \subset \mathbb{R}$ are the state, input and output of the system. In order to guarantee the existence of a regular control signal, we will introduce the following instrumental assumption [7, 12], necessary for the existence of at least one approximated system, the simplest being the linear one.

Assumption 1. The nonlinear system (2) is strong regular, i.e the *linearization* of the system at an equilibrium x_0 has a well defined relative degree.

Choosing a change of coordinate $q = \begin{pmatrix} q_1 & \cdots & q_n \end{pmatrix}^T = P(x, u, \dots, u^{(r)})$ by defining

$$q_i = y^{(i-1)}, \qquad i = 1, \dots, n$$

system (2) is described by the so–called *generalized observability canonical form*

$$
\begin{aligned}
\dot q_1 &= p_1(x)\big|_{x=P^{-1}(q)} = q_2\\
\dot q_2 &= p_2(x)\big|_{x=P^{-1}(q)} = q_3\\
&\vdots\\
\dot q_{r-1} &= p_{r-1}(x)\big|_{x=P^{-1}(q)} = q_r\\
\dot q_r &= p_r(x,u)\big|_{x=P^{-1}(q)} = q_{r+1}\\
\dot q_{r+1} &= p_{r+1}(x,u,\dot u)\big|_{x=P^{-1}(q)} = q_{r+2}\\
&\vdots\\
\dot q_n &= p_n(x,u,\dots,u^{(n-r)})\big|_{x=P^{-1}(q)}\\
y &= q_1
\end{aligned}
\tag{3}
$$

where r is the smallest integer such that in the derivative of the output of (2) the input appears. A trivial calculation shows that

$$\frac{\partial p_r}{\partial u}(x,u) = \frac{\partial p_n}{\partial u^{(n-r)}}(x,u,\dots,u^{(n-r)}) = L_g L_f^{r-1} h(x).$$

If $L_g L_f^{(r-1)} h(x) \neq 0$ for all x in a neighborhood of the point of interest x_0, then r is the relative degree of the system, and coincides with the relative degree of its linearized part. An inversion based control scheme can be used, such as the exact linearization or output tracking [12], by means of a coordinate transformation of the form

$$p_n(x,u,\dots,u^{(n-r)}) = v. \tag{4}$$

If x_0 is a singular point, i.e. $L_g L_f^{r-1} h(x_0) = 0$, then equation (4) can not be applied directly, since it would lead to unbounded input signals near the point x_0.

3. TRACKING THROUGH SINGULARITIES VIA ROBUST DIFFERENTIATION

In this section the robust exact differentiator recalled in the previous section is used to achieve tracking through singularities. The differentiator estimates the control signal derivatives, and the actual controller takes as inputs these estimates, as is shown in Figure 1.

To introduce the method, let us first define the minimum integer $\beta \geq 0$ such that

$$\frac{\partial p_n}{\partial u^{(\beta)}}(x,u,\dots,u^{(\beta)},u^{(\beta+1)},\dots,u^{(n-r)}) \neq 0. \tag{5}$$

We also consider the following assumption.

Assumption 2. Let us assume that (5) is valid in a neighborhood I_u corresponding to the point of interest x_0.

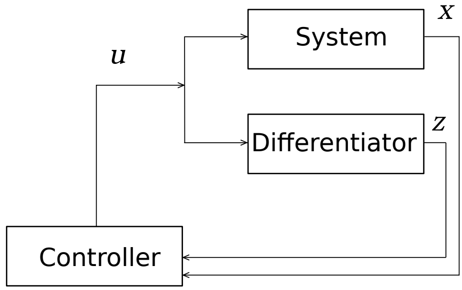

Fig. 1. Schematic representation of the method.

Referring to the definition of β, equations (3) rewrite

$$\begin{aligned} \dot q_i &= p_i(P^{-1}(q)) = q_{i+1}, && i=1,\dots,r-1\\ \dot q_j &= p_j(P^{-1}(q),u,\dots,u^{(j-r)}) = q_{j+1}, && j=r,\dots,r+\beta\\ \dot q_k &= p_k(P^{-1}(q),u,\dots,u^{(k-r)}) = q_{k+1}, && k=r+\beta+1,\dots,n-1\\ \dot q_n &= p_n(P^{-1}(q),u,\dots,u^{(n-r)})\\ y &= q_1. \end{aligned}$$

Let us consider the output tracking error $e_1 = q_1 - y_{\rm ref}$, with $y_{\rm ref}(t) \in C^n$ a reference signal, and let us define

$$e = q - Y_{\rm ref}(t) = \begin{pmatrix} e_1\\ e_2\\ \vdots\\ e_n \end{pmatrix}, \qquad Y_{\rm ref}(t) = \begin{pmatrix} y_{\rm ref}(t)\\ \dot y_{\rm ref}(t)\\ \vdots\\ y_{\rm ref}^{(n-1)}(t) \end{pmatrix}$$

with $e_j = q_j - y_{\rm ref}^{(j-1)}(t)$. One works out

$$\begin{aligned} \dot e_i &= p_i(P^{-1}(e+Y_{\rm ref})) - y_{\rm ref}^{(i)} =: e_{i+1}, && i=1,\dots,r-1\\ \dot e_j &= p_j(P^{-1}(e+Y_{\rm ref}),u,\dots,u^{(j-r)}) - y_{\rm ref}^{(j)} =: e_{j+1}, && j=r,\dots,r+\beta \end{aligned}$$

while for the successive derivatives

$$\begin{aligned} \dot e_{r+\beta+1} &= e_{r+\beta+2} + \psi_{r+\beta+1}(t,e,u,\dots,u^{(\beta)},u^{(\beta+1)},\widehat{u^{(\beta+1)}})\\ &\;\;\vdots\\ \dot e_{n-1} &= e_n + \psi_{n-1}(t,e,u,\dots,u^{(\beta)},u^{(\beta+1)},\widehat{u^{(\beta+1)}},\dots,u^{(n-r-1)},\widehat{u^{(n-r-1)}})\\ \dot e_n &= p_n(P^{-1}(e+Y_{\rm ref}),u,\dots,u^{(\beta)},\widehat{u^{(\beta+1)}},\dots,\widehat{u^{(n-r)}}) - y_{\rm ref}^{(n)}\\ &\quad + \psi_n(t,e,u,\dots,u^{(\beta)},u^{(\beta+1)},\widehat{u^{(\beta+1)}},\dots,u^{(n-r)},\widehat{u^{(n-r)}}) \end{aligned} \tag{6}$$

with $\psi_{r+\beta+1}, \dots, \psi_n$ appropriate functions

$$\begin{aligned}
\psi_{r+\beta+1} &= p_{r+\beta+1}(P^{-1}(e+Y_{\mathrm{ref}}), u, \dots, u^{(\beta)}, u^{(\beta+1)}) \\
&\qquad - p_{r+\beta+1}(P^{-1}(e+Y_{\mathrm{ref}}), u, \dots, u^{(\beta)}, \widehat{u^{(\beta+1)}}) \\
\psi_{r+\beta+2} &= p_{r+\beta+1}(P^{-1}(e+Y_{\mathrm{ref}}), u, \dots, u^{(\beta)}, u^{(\beta+1)}, u^{(\beta+2)}) \\
&\qquad - p_{r+\beta+1}(P^{-1}(e+Y_{\mathrm{ref}}), u, \dots, u^{(\beta)}, \widehat{u^{(\beta+1)}}, \widehat{u^{(\beta+2)}}) \\
&\ \vdots \\
\psi_n &= p_{n-1}(P^{-1}(e+Y_{\mathrm{ref}}), u, \dots, u^{(\beta)}, u^{(\beta+1)}, \dots, u^{(n-r)}) \\
&\qquad - p_{n-1}(P^{-1}(e+Y_{\mathrm{ref}}), u, \dots, u^{(\beta)}, \widehat{u^{(\beta+1)}}, \dots, \widehat{u^{(n-r)}})
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
e_{r+\beta+2} &= p_{r+\beta+1}(P^{-1}(e+Y_{\mathrm{ref}}), u, \dots, u^{(\beta)}, \widehat{u^{(\beta+1)}}) - y_{\mathrm{ref}}^{(r+\beta+1)} \\
e_{r+\beta+3} &= p_{r+\beta+2}(P^{-1}(e+Y_{\mathrm{ref}}), u, \dots, u^{(\beta)}, \widehat{u^{(\beta+1)}}, \widehat{u^{(\beta+2)}}) - y_{\mathrm{ref}}^{(r+\beta+2)} \\
&\ \vdots \\
e_n &= p_{n-1}(P^{-1}(e+Y_{\mathrm{ref}}), u, \dots, u^{(\beta)}, \widehat{u^{(\beta+1)}}, \dots, \widehat{u^{(n-r-1)}}) - y_{\mathrm{ref}}^{(n-1)}
\end{aligned}$$

where $\widehat{u^{(j)}}$ stands for estimate of the derivative of order $j = \beta+1, \dots, n-r-1$. It is worth noticing that once some estimates of these derivatives $\widehat{u^{(j)}}$ are obtained by means of a robust sliding mode differentiator of the type of Theorem 2.1, the functions ψ_i in (7) will be zero in a finite time. With this idea in mind, using Assumption 2 and the implicit function theorem, there exists a neighborhood I_w in $[0,\infty) \times \mathbb{R}^n \times \mathbb{R}^\beta \times \mathbb{R}^{(n-r)}$ such that for each $w = (t, e, \gamma, \hat{\gamma}_d) \in I_w$ the equation

$$p_n(P^{-1}(e+Y_{\mathrm{ref}}), u, \dots, u^{(\beta)}, \widehat{u^{(\beta+1)}}, \dots, \widehat{u^{(n-r)}}) - y_{\mathrm{ref}}^{(n)} = v(e)$$

for $v(e)$ an appropriate function defined later on, has a unique solution $u^{(\beta)} \in I_u$ given by

$$u^{(\beta)} = \vartheta(t, e, u, \dots, u^{(\beta-1)}, \widehat{u^{(\beta+1)}}, \dots, \widehat{u^{(n-r)}}) = \vartheta(t, e, \gamma, \hat{\gamma}_d) \tag{8}$$

for a certain function ϑ. In (8), we have denoted, for the sake of compactness,

$$\gamma = \begin{pmatrix} u \\ \dot{u} \\ \vdots \\ u^{(\beta-1)} \end{pmatrix}, \quad \gamma_d = \begin{pmatrix} u^{(\beta+1)} \\ u^{(\beta+2)} \\ \vdots \\ u^{(n-r)} \end{pmatrix}, \quad \hat{\gamma}_d = \begin{pmatrix} \widehat{u^{(\beta+1)}} \\ \widehat{u^{(\beta+2)}} \\ \vdots \\ \widehat{u^{(n-r)}} \end{pmatrix}.$$

The function $v(e)$ can be chosen as follows

$$v(e) = -\sum_{i=1}^{n} \alpha_{i-1} e_i$$

with $p(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$ a Hurwitz polynomial. With $u^{(\beta)}$ chosen as in (8), the last equation of (6) becomes

$$\dot{e}_n = -\sum_{i=1}^{n} \alpha_{i-1} e_i + \psi_n(t, e, \gamma, \hat{\gamma}_d)$$

and the closed loop dynamics write

$$\begin{aligned} \dot{e} &= Ae + \varPsi(t, e, \gamma, \vartheta(e, \gamma, \hat{\gamma}_d), \gamma_d, \hat{\gamma}_d) \\ &= Ae + \varPsi_0(t, e, \gamma) + \varPsi_1(t, e, \gamma, \gamma_d, \hat{\gamma}_d) \end{aligned} \tag{9}$$

with

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{pmatrix} \\ \varPsi(t, e, \gamma, \gamma_d, \hat{\gamma}_d) &= \begin{pmatrix} \psi_1 = 0 \\ \vdots \\ \psi_{r+\beta} = 0 \\ \psi_{r+\beta+1}(t, e, \gamma, \vartheta(t, e, \gamma, \gamma_d), \gamma_d, \hat{\gamma}_d) \\ \vdots \\ \psi_{n-1}(t, e, \gamma, \vartheta(t, e, \gamma, \gamma_d), \gamma_d, \hat{\gamma}_d) \\ \psi_n(t, e, \gamma, \vartheta(t, e, \gamma, \gamma_d), \gamma_d, \hat{\gamma}_d) \end{pmatrix} \end{aligned} \tag{10}$$

and with

$$\begin{aligned} \varPsi_0(e, \gamma) &= \varPsi(t, e, \gamma, \vartheta(t, e, \gamma, \gamma_d), \gamma_d, \hat{\gamma}_d)\big|_{\hat{\gamma}_d = \gamma_d} \\ \varPsi_1(e, \gamma, \gamma_d, \hat{\gamma}_d) &= \varPsi(t, e, \gamma, \vartheta(t, e, \gamma, \hat{\gamma}_d), \gamma_d, \hat{\gamma}_d) - \varPsi_0(t, e, \gamma, \hat{\gamma}_d). \end{aligned} \tag{11}$$

It is worth noticing that $\varPsi_1(e, \gamma, \gamma_d, \gamma_d) = 0$. Making use of Theorem 2.1 [15], in what follows we will work out an estimation of the $\hat{\gamma}_d$. For, we set

$$f(t) = u^{(\beta)}, \quad z_0 = \widehat{u^{(\beta)}}, \quad z_1 = \widehat{u^{(\beta+1)}}, \quad \ldots, \quad z_m = \widehat{u^{(\beta+m)}} \tag{12}$$

where $m \geq 1$ in the order to obtain a sufficiently regular estimate of $u^{(\beta)}$, and where for notation convenience we define

$$u_k = u^{(k)}, \qquad k = 0, \ldots, \beta - 1. \tag{13}$$

We can now state the main result of the paper.

Theorem 3.1. Assume that Assumptions 1 and 2 are verified. Consider the closed–loop dynamic system

$$\begin{aligned}
\dot{e} &= Ae + \Psi_0(e,\gamma,z) + \Psi_1(e,\gamma,\gamma_d,z) \\
\dot{u}_0 &= u_1 \\
\dot{u}_1 &= u_2 \\
&\ \vdots \\
\dot{u}_{\beta-2} &= u_{\beta-1} \\
\dot{u}_{\beta-1} &= \vartheta(e,\gamma,z) \\
\dot{z} &= g(z, u^{(\beta)})
\end{aligned} \tag{14}$$

with A as in (10), Ψ_0, Ψ_1 as in (11), $u_0, \ldots, u_{\beta-1}$ as in (13), $z = \begin{pmatrix} z_0 & z_1 & \cdots & z_k \end{pmatrix}^T$ as in (12), g given by (1). Assume that the coefficients α_i in A are chosen so that the matrix

$$\tilde{A} = \begin{pmatrix} A & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with

$$A_{12} = \frac{\partial \Psi_0(e,\gamma,z)}{\partial \gamma}|_{(0,0,0)}, \qquad A_{21} = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ -\alpha_0 & \cdots & -\alpha_{n-1} \end{pmatrix}$$

$$A_{22} = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ \frac{\partial \alpha(0,\gamma,0)}{\partial u_0} & \frac{\partial \alpha(0,\gamma,0)}{\partial u_1} & \cdots & \cdots & \frac{\partial \alpha(0,\gamma,0)}{\partial u_{\beta-1}} \end{pmatrix}$$

is Hurwitz. Then the system

$$\begin{aligned}
\dot{E} &= \tilde{A}E \\
\dot{z} &= g(z, \alpha(E,z))
\end{aligned}$$

with $E = (e,\gamma)^T$, is asymptotically stable in the origin. Moreover, for a small positive η, the solution of the perturbed system (14) guarantees that $\|e\| \leq \eta$.

To prove this statement first we recall the following result.

Lemma 3.1. (Isidori [12]) Let us consider the system

$$\dot{E} = \bar{A}E + \psi(E,z) \tag{15}$$
$$\dot{z} = \bar{g}(E,z). \tag{16}$$

If $\psi(0,z)=0$ and $\frac{\partial}{\partial z}\psi(0,0)=0$ and $\bar{g}(0,0)$ is a stable equilibrium point of (16), and the origin of $\dot{E}=\tilde{A}E$ is asymptotically stable, then the equilibrium point $(0,0)$ of system (15) and (16) is asymptotically stable.

Proof of Theorem 3.1. The closed–loop error dynamics and the robust differentiator can be written as the composite system

$$
\begin{aligned}
\dot{E} &= \tilde{A}E + \tilde{\Psi}_1(E,z,\gamma_d)\\
\dot{z} &= g(z,\alpha(E,z))
\end{aligned}
$$

with $\tilde{\Psi}_1(E,z,\gamma_d)=(0,\Psi_1(E,z,\gamma_d))^T$. Let us consider first the case in which the perturbation term $\tilde{\Psi}_1$ is zero. Since $\tilde{A}$ is Hurwitz, and $g(0,0)=0$ is asymptotically stable thanks to Theorem 2.1, by Lemma 3.1 one has that $(E,z)=(0,0)$ is an asymptotically stable equilibrium point. Now, if the perturbation term is bounded, then by the total stability Theorem [12], the output tracking error is bounded. □

4. ILLUSTRATIVE EXAMPLES

Example 1. Let us consider the well–known ball and beam system described by the following equations

$$
\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= x_1x_4^2 - \mathrm{g}\sin x_3\\
\dot{x}_3 &= x_4\\
\dot{x}_4 &= u\\
y &= x_1
\end{aligned}
\tag{17}
$$

with g the gravity acceleration. Applying the proposed method we obtain

$$
\begin{aligned}
e_1 &= x_1 - y_{\mathrm{ref}}\\
e_2 &= x_2 - \dot{y}_{\mathrm{ref}}\\
e_3 &= x_1x_4^2 - \mathrm{g}\sin x_3 - \ddot{y}_{\mathrm{ref}}\\
e_4 &= x_2x_4^2 + 2x_1x_4u - \mathrm{g}x_4\cos x_3 - y_{\mathrm{ref}}^{(3)}
\end{aligned}
$$

where the error dynamics are given by

$$
\begin{aligned}
\dot{e}_1 &= e_2\\
\dot{e}_2 &= e_3\\
\dot{e}_3 &= e_4\\
\dot{e}_4 &= x_1x_4^4 + (4x_2x_4 - \mathrm{g}\cos x_3)u + 2x_1u^2 + 2x_1x_4(\dot{u} + \widehat{u^{(1)}} - \widehat{u^{(1)}}) - y_{\mathrm{ref}}^{(4)}\\
&= p_4(t,e,u,\widehat{u^{(1)}}) - y_{\mathrm{ref}}^{(4)} + \psi_4(t,e,z)
\end{aligned}
$$

with $\beta=0$ and

$$
\psi_4(t,e,z) = 2x_1x_4(\dot{u} - \widehat{u^{(1)}})\Big|_{\substack{x_1=e_1+y_{\mathrm{ref}}\\ x_4=P^{-1}(e+Y_{\mathrm{ref}})\\ \widehat{u^{(1)}}=z.}}
$$

The control law is obtained as

$$u = u^{(\beta)} = \frac{v(e) - x_1 x_4^4 - 2x_1^2(\hat{u})^2 - 2x_1 x_4 \widehat{u^{(1)}} + \mathrm{g} x_4^2 \sin x_3 + y_{\mathrm{ref}}^{(4)}}{4x_2 x_4 - \mathrm{g}\cos x_3}$$

with

$$v(e) = -\alpha_0 e_1 - \alpha_1 e_2 - \alpha_2 e_3 - \alpha_3 e_4$$

and for

$$\{x \in \mathcal{D}_x \subset \mathbb{R}^n \,||4x_2x_4 - \mathrm{g}\cos x_3| > 0\}\big|_{x=P^{-1}(e+Y_{\mathrm{ref}})}$$

where $p(\lambda) = \lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ is a Hurwitz polynomial. The differentiator which gives an estimate of $\dot{u}$ has the following form

$$\begin{aligned} \dot{z}_0 &= v_0, \qquad\qquad v_0 = -\lambda_1 |z_0 - u|^{1/2}\,\mathrm{sign}(z_0 - u) + z_1 \\ \dot{z}_1 &= -\lambda_0\,\mathrm{sign}(z_1 - v_0). \end{aligned} \tag{18}$$

The simulations results are shown in Figures 2 – 4. Figure 2 shows a comparison of the output and the reference signals. Figure 3 shows that the control signal is smooth and bounded. For the sake of comparison, the output tracking errors given by the approximated method proposed in [8, 22] (SAM) and the method proposed in this work (RDAM) are shown in Figure 4. We may note that the method based on the robust differentiator performs better, even if the differentiator parameters were not finely tuned.

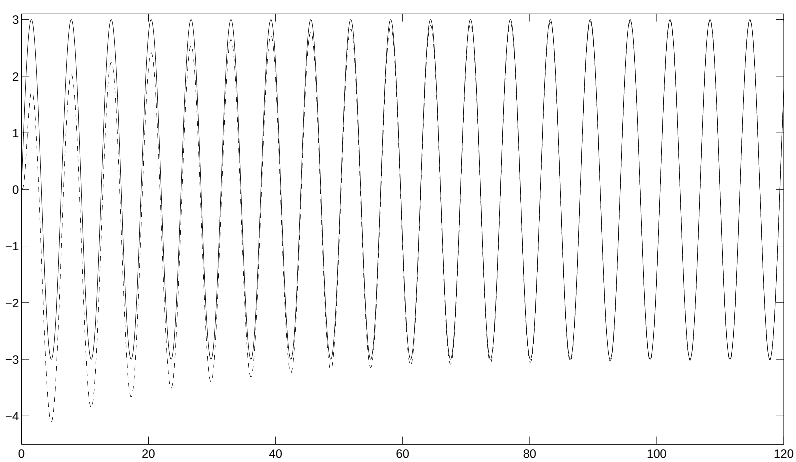

Fig. 2. Output signal (dashed) and reference (solid) [m vs s].

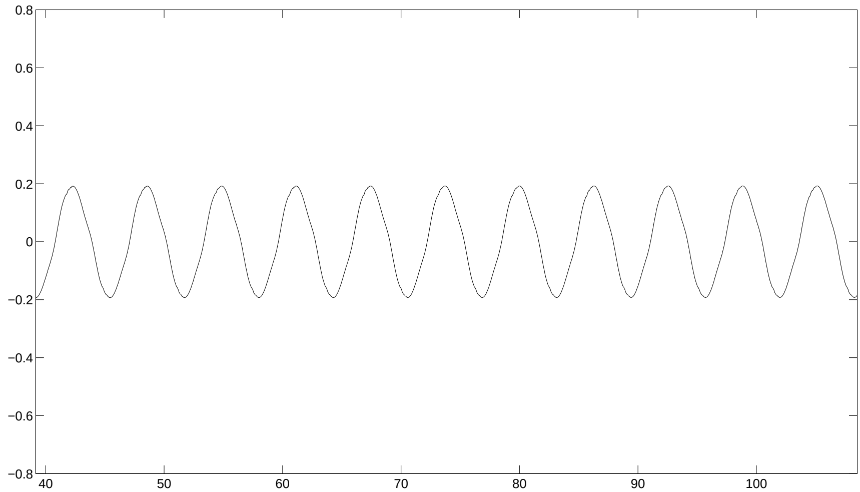

Fig. 3. Input signal in the RDAM method [N m vs s].

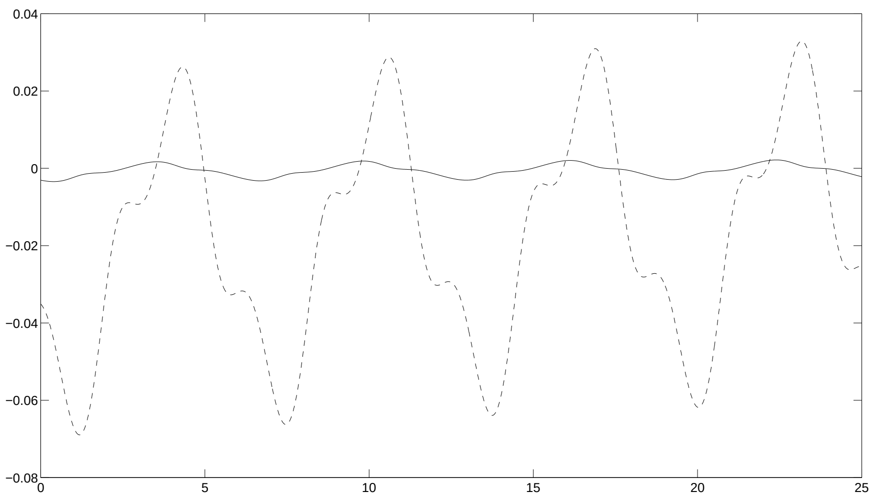

Fig. 4. Comparison of the errors resulting from the RDAM (solid) and SAM (dashed) methods [m vs s].

Example 2. Let us consider the same system presented in [10]

$$\begin{aligned}\dot{x}_1 &= {x_3}^2 - x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= u\\ y &= x_1.\end{aligned} \tag{19}$$

Applying the method proposed in this work, it follows that

$$\begin{aligned}e_1 &= x_1 - y_{\text{ref}}\\ e_2 &= {x_3}^2 - x_2 - \dot{y}_{\text{ref}}\\ e_3 &= 2x_3u - x_3 - \ddot{y}_{\text{ref}}.\end{aligned}$$

The error system is given by

$$\begin{aligned}\dot{e}_1 &= e_2\\ \dot{e}_2 &= e_3\\ \dot{e}_3 &= 2u^2 - u + 2x_3(\dot{u} + \widehat{u^{(1)}} - \widehat{u^{(1)}}) - \dddot{y}_{\text{ref}}\\ &= p_3(t, e, u, \widehat{u^{(1)}}) - y_{\text{ref}}^{(3)} + \psi_3(t, e, z)\end{aligned}$$

with $\beta = 0$ and

$$\psi_4(t, e, z) = 2x_3(\dot{u} - \widehat{u^{(1)}}).$$

The control is given by

$$u = v(e) + 2\widehat{u^2} - 2x_3\widehat{u^{(1)}} - \dddot{y}_{\text{ref}}$$

where $v(e) = \alpha_0 e_1 - \alpha_1 e_2 - \alpha_2 e_3$ and the differentiator taken as in (18). Figure 5 shows the tracking results when $y_{\text{ref}} = \sin t$ for the method proposed here (upper subplot) and the method presented in [10] (lower subplot) referred to as *HM*. A detail of the result is shown in Figure 6, where it can be noted that the tracking in *HM* is accurate only at discrete times due to the discrete nature of the controller. The tracking errors signals for both methods are shown in Figure 7. A comparison between the two control laws is shown in Figure 8. It can be noticed that the *HM* method gives a discontinuous and high–amplitude control law, while the control given by the method proposed in this work is smooth and of relative low amplitude.

CONCLUSIONS

A new method for tracking a reference signal through singularities has been presented. This method is based on the use of a robust differentiator to estimate the derivatives of the input signal. With respect to existing techniques, the proposed procedure avoids neglecting nonlinear terms determining the singularity condition. Some simulations show that this method gives better results than existing methods available in the literature. This effectiveness of the proposed method is due to the estimation of derivatives of the control involved in the singularity, instead of neglecting those terms or avoiding the singularity region.

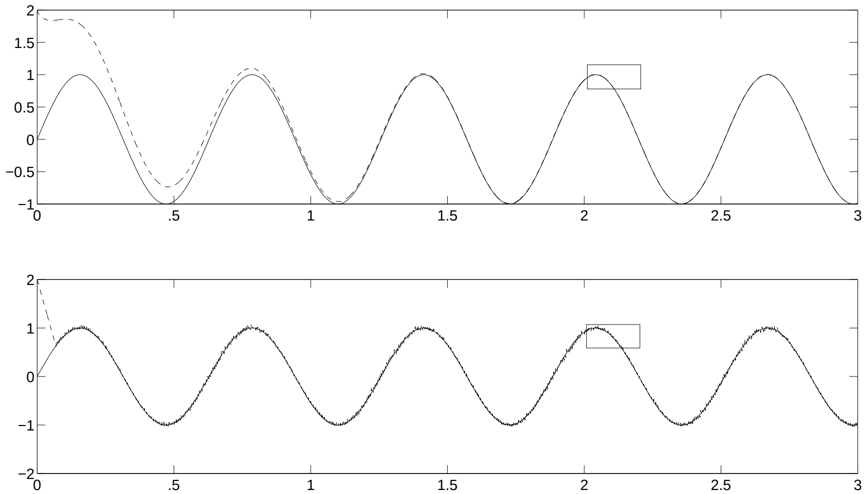

Fig. 5. Output signal (dashed) and reference (solid) [m vs s]. Upper subplot represent the tracking using the method proposed here (RDAM), the lower subplot is the tracking with HM.

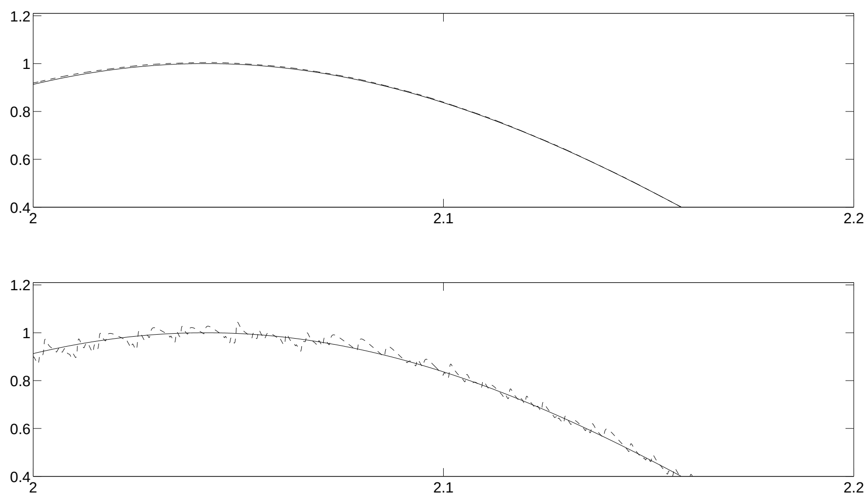

Fig. 6. A selected segment of the tracking.

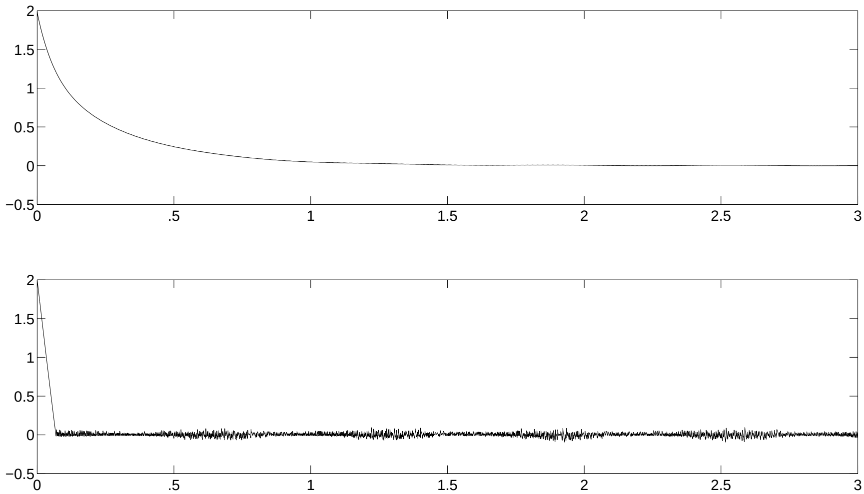

Fig. 7. Error signals due to both methods. Upper plot correspond to the RDAM method and lower plot to HM method.

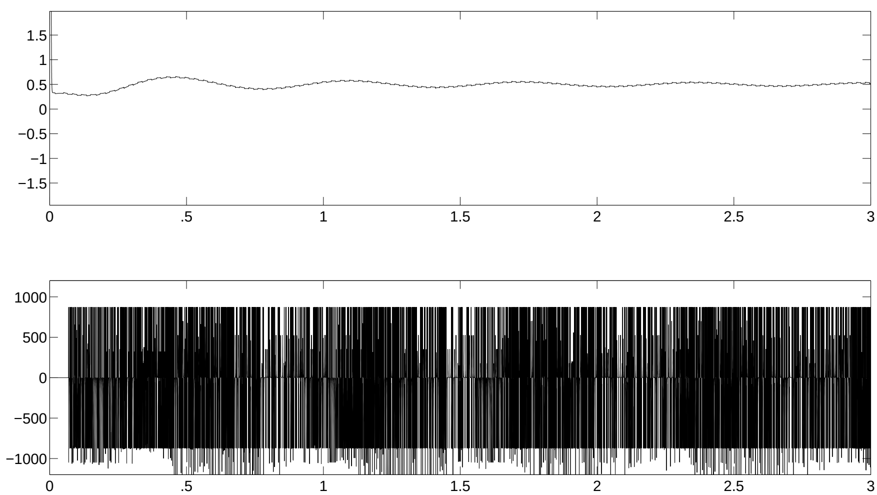

Fig. 8. Control signal. Upper plot RDAM, lower plot HM.

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Bernardino Castillo–Toledo, CINVESTAV, Unidad Guadalajara, Av. del Bosque 1145, Col. El Bajío, Zapopan, 45015, Jalisco. México.
e-mail: toledo@gdl.cinvestav.mx

Stefano Di Gennaro, Department of Information Engineering, Computer Science and Mathematics Center of Excellence DEWS, University of L'Aquila, Via Vetoio, Loc. Coppito, 67100 L'Aquila. Italy.
e-mail: stefano.digennaro@univaq.it

Armando López–Cuevas, CINVESTAV, Unidad Guadalajara, Av. del Bosque 1145, Col. El Bajío, Zapopan, 45015, Jalisco. México.
e-mail: acuevas@gdl.cinvestav.mx