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UNITAL EXTENSIONS OF AF-ALGEBRAS BY PURELY INFINITE SIMPLE ALGEBRAS

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Abstract. In this paper, we consider the classification of unital extensions of AF-algebras by their six-term exact sequences in K-theory. Using the classification theory of C^*-algebras and the universal coefficient theorem for unital extensions, we give a complete characterization of isomorphisms between unital extensions of AF-algebras by stable Cuntz algebras. Moreover, we also prove a classification theorem for certain unital extensions of AF-algebras by stable purely infinite simple C^*-algebras with nontrivial K_1-groups up to isomorphism.

Keywords: AF-algebra; extension; purely infinite simple algebra

MSC 2010: 46L05, 46L35

1. Introduction

Great progress has been made in classifying simple C^*-algebras till now (see [4], [5], [7], [14], [13], [10], [9], [11], [16], etc.). But there are still many non-simple C^*-algebras in need of classification. Among these algebras, extension algebras are an important class. The existing results for classification of such algebras mainly focus on classification of non-unital extensions up to stable isomorphism, for example, [3], [17], [21].

Naturally, isomorphisms of unital extensions should also be considered. As we know, classification for unital extensions up to isomorphism is very different from the non-unital case. In [23], the second-named author considered unital extensions of $A\mathbb{T}$-algebras and proved that the six-term exact sequence in K-theory together with the Elliott invariants of the ideal and quotient is a complete invariant.

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As the succeeding work of [22], [21], [20], [23], [24], the purpose of this note is to classify unital essential extensions of AF-algebras by stable purely infinite simple algebras. Using the classification theory of C^*-algebras and the universal coefficient theorem for unital extensions obtained by the second author ([22], [23]), we give a complete characterization of isomorphisms between unital extensions of AF-algebras by stable Cuntz algebras. We also prove a classification theorem for certain unital extensions of AF-algebras by stable purely infinite simple C^*-algebras with nontrivial K_1 groups up to isomorphism.

2. Preliminaries

First, we recall some notations for C^*-algebra extensions and their K-theory. One can see [1], [17], [18], [19], [22] for more details.

Let A and B be C^*-algebras. Recall that an extension of A by B is a short exact sequence $0 \to B \xrightarrow{\alpha} E \xrightarrow{\beta} A \to 0$ of C^*-algebras. Denote this extension by e or (E, α, β) and denote by $\mathrm{Ext}(A, B)$ the set of essential extensions of A by B.

Given an extension (E, α, β), $\alpha(B)$ is an ideal of E. Hence there is a homomorphism σ from E into the multiplier algebra $\mathcal{M}(B)$ of B. Let π be the quotient map from $\mathcal{M}(B)$ into the corona algebra $\mathcal{Q}(B)$. The Busby invariant of (E, α, β) is a homomorphism τ from A into $\mathcal{Q}(B)$ such that $\tau(a) = \pi(\sigma(x))$ for a in A, where x is in E and $\beta(x) = a$. Then τ is the only homomorphism making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & B & \longrightarrow & \mathcal{M}(B) & \xrightarrow{\pi} & \mathcal{Q}(B) & \longrightarrow & 0. \end{array}$$

The extension (E, α, β) is essential if and only if τ is injective. It is called trivial if there is a homomorphism $\gamma\colon A \to E$ such that $\beta \circ \gamma = \mathrm{id}_A$. The extension (E, α, β) is called unital if A is unital and τ is a unital homomorphism.

Definition 2.1. Suppose that $e_i\colon 0 \to B \xrightarrow{\alpha_i} E_i \xrightarrow{\beta_i} A \to 0$ for $i = 1, 2$ are two extensions of A by B, with associated Busby invariants τ_i. We say that (E_1, α_1, β_1) and (E_2, α_2, β_2) are unitarily equivalent (denoted by $e_1 \overset{s}{\sim} e_2$), if there is a unitary $u \in \mathcal{M}(B)$ such that $\tau_2(a) = \pi(u)\tau_1(a)\pi(u)^*$ for every $a \in A$.

Denote by $\mathbf{Ext}_s(A, B)$ the set of unitary equivalence classes of extensions of A by B.

It is known that $e_1 \overset{s}{\sim} e_2$ if and only if there exist a unitary element $u \in \mathcal{M}(B)$ and homomorphism $\varphi\colon E_1 \to E_2$ such that the following diagram commutes:

$$\begin{CD}
0 @>>> B @>\alpha_1>> E_1 @>\beta_1>> A @>>> 0\\
@. @V\mathrm{Ad}uVV @V\varphi VV @V\mathrm{id}VV @.\\
0 @>>> B @>\alpha_2>> E_2 @>\beta_2>> A @>>> 0.
\end{CD}$$

We say (E_1, α_1, β_1) and (E_2, α_2, β_2) are weakly unitarily equivalent (denoted by $e_1 \overset{w}{\sim} e_2$), if there is a unitary $v \in \mathcal{Q}(B)$ such that $\tau_2(a) = v\tau_1(a)v^*$ for every $a \in A$. Denote by $\mathbf{Ext}_w(A, B)$ the set of weakly unitary equivalence classes of extensions of A by B.

Denote by $\mathbf{Ext}_*^u(A, B)$ the equivalence classes of unital essential extensions of A by B for $* = s, w$.

Definition 2.2. Two extensions (E_1, α_1, β_1) and (E_2, α_2, β_2) are called congruent (denoted by $e_1 \equiv e_2$), if there is an isomorphism $\varphi\colon E_1 \to E_2$ such that the following diagram commutes:

$$\begin{CD}
0 @>>> B @>\alpha_1>> E_1 @>\beta_1>> A @>>> 0\\
@. @V\mathrm{id}_BVV @V\varphi VV @V\mathrm{id}_AVV @.\\
0 @>>> B @>\alpha_2>> E_2 @>\beta_2>> A @>>> 0.
\end{CD}$$

Definition 2.3. Two extensions (E_1, α_1, β_1) and (E_2, α_2, β_2) are called isomorphic (denoted by $e_1 \cong e_2$), if there are isomorphisms φ, η, ψ such that the following diagram commutes:

$$\begin{CD}
0 @>>> B @>\alpha_1>> E_1 @>\beta_1>> A @>>> 0\\
@. @V\varphi VV @V\eta VV @V\psi VV @.\\
0 @>>> B @>\alpha_2>> E_2 @>\beta_2>> A @>>> 0.
\end{CD}$$

If B is a stable C^*-algebra, then the sum of two extensions τ_1 and τ_2 is the extension whose Busby invariant is $\tau_1 \oplus \tau_2\colon A \to \mathcal{Q}(B) \oplus \mathcal{Q}(B) \subseteq \mathcal{M}_2(\mathcal{Q}(B)) \cong \mathcal{Q}(B)$, where the last isomorphism is a standard isomorphism. So $\mathbf{Ext}_*(A, B)$ is a commutative semigroup with respect to the above addition, and the set of equivalence classes of essential trivial extensions of A by B is a subsemigroup of $\mathbf{Ext}_*(A, B)$.

Define $\mathrm{Ext}_*(A, B)$ as the quotient of $\mathbf{Ext}_*(A, B)$ by the subsemigroup of essential trivial extensions of A by B for $* = s, w$. By ([1], Proposition 15.6.4), $\mathrm{Ext}_s(A, B) \cong$

$\operatorname{Ext}_w(A, B)$. So we write them as $\operatorname{Ext}(A, B)$. When $e \in \operatorname{Ext}(A, B)$, we write $[e]$ for the equivalence class of e in $\operatorname{Ext}(A, B)$.

Let $e \in \operatorname{Ext}(A, B)$ and let C, D be C^*-algebras. Assume that $\beta\colon B \to C$ is a surjective homomorphism and $\alpha \in \operatorname{Hom}(D, A)$. Then there are two induced extensions unique up to congruence, such that the following diagrams commute:

$$\begin{array}{rcccccccc} e\alpha\colon & 0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & D & \longrightarrow & 0 \\ & & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \alpha} & & \\ e\colon & 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{rcccccccc} e\colon & 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\ \beta e\colon & 0 & \longrightarrow & C & \longrightarrow & E'' & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

One can see [8], [17] and [15] for details.

Let $e\colon 0 \to B \to E \to A \to 0$ be an extension of A by B. Denote by $K(e)$ the six-term exact sequence of e in K-theory:

$$\begin{array}{ccccc} K_0(B) & \longrightarrow & K_0(E) & \longrightarrow & K_0(A) \\ {\scriptstyle \delta_1}\Big\uparrow & & & & \Big\downarrow{\scriptstyle \delta_0} \\ K_1(A) & \longleftarrow & K_1(E) & \longleftarrow & K_1(B). \end{array}$$

Let $e_i\colon 0 \to B_i \to E_i \to A_i \to 0$, $i = 1, 2$, be two extensions. We say $(\alpha_*, \beta_*, \eta_*)\colon K(e_1) \to K(e_2)$ a morphism if there are homomorphisms $\alpha_*\colon K_*(A_1) \to K_*(A_2)$, $\beta_*\colon K_*(B_1) \to K_*(B_2)$ and $\eta_*\colon K_*(E_1) \to K_*(E_2)$ such that the obvious diagram is commutative.

If α_*, β_*, η_* are isomorphisms, $K(e_1)$ and $K(e_2)$ are called isomorphic, written $K(e_1) \cong K(e_2)$. Furthermore, if $[p]_0 \in K_0(E_1)$ and $[q]_0 \in K_0(E_2)$ such that $\eta_0([p]_0) = [q]_0$, then the isomorphism is written by $(K(e_1), [p]_0) \cong (K(e_2), [q]_0)$. When $A_1 = A_2 = A$, $B_1 = B_2 = B$ and there is an isomorphism $(\mathrm{id}_{K_*(A)}, \mathrm{id}_{K_*(B)}, \eta_*)\colon K(e_1) \to K(e_2)$, then they are called congruent, written $K(e_1) \equiv K(e_2)$. Similarly, $(K(e_1), [p]_0) \equiv (K(e_2), [q]_0)$.

In this paper, we only consider essential extensions.

3. Extensions by Cuntz algebras

First, we calculate the K-theory of extensions of the AF-algebras by stable Cuntz algebras. Let A be an AF-algebra. Recall that the Elliott invariant of A is the tuple

$$(K_0(A), K_0(A)^+, D(A)),$$

where $D(A)$ is the scale consisting of the images in $K_0(A)$ of projections of A. We denote it by $\mathrm{Ell}(A)$. When A is a unital AF-algebra, we set $\mathrm{Ell}(A) = (K_0(A), K_0(A)^+, [1_A]_0)$.

In this section, we write $B = \mathcal{O}_\infty \otimes \mathcal{K}$ or $B = \mathcal{O}_n \otimes \mathcal{K}$. It is known that $K_0(\mathcal{O}_\infty \otimes \mathcal{K}) = \mathbb{Z}$, $K_0(\mathcal{O}_n \otimes \mathcal{K}) = \mathbb{Z}_{n-1}$, $K_1(\mathcal{O}_\infty \otimes \mathcal{K}) = 0$ and $K_1(\mathcal{O}_n \otimes \mathcal{K}) = 0$.

Assume that A is a unital separable AF-algebra and $0 \to B \to E \to A \to 0$ is an extension of A by B. Then there exists a six-term exact sequence of K-groups:

$$\begin{array}{ccccc} K_1(B) & \longrightarrow & K_1(E) & \longrightarrow & K_1(A) \\ \uparrow{\scriptstyle \delta_0} & & & & \downarrow{\scriptstyle \delta_1} \\ K_0(A) & \longleftarrow & K_0(E) & \longleftarrow & K_0(B). \end{array}$$

From $K_1(A) = 0$, we have $\delta_1 = \delta_0 = 0$ and $K_1(E) = 0$ too. Moreover, we also get a short exact sequence

$$0 \longrightarrow K_0(B) \longrightarrow K_0(E) \longrightarrow K_0(A) \longrightarrow 0\ .$$

Suppose that A is a unital C^*-algebra. Recall that (see [1]) a trivial unital extension τ is called strongly unital if τ can lift to a unital homomorphism from A to $M(B)$. Denote by $Ext_s^u(A,B)$ $[Ext_w^u(A,B)]$ the quotient of $\mathbf{Ext}_s^u(A,B)$ $[\mathbf{Ext}_w^u(A,B)]$ by strongly unital trivial extensions. Let e be an extension of A by B, e is called absorbing [unital-absorbing] if e is unitarily equivalent to $e \oplus \sigma$ for any trivial [strongly unital trivial] extension σ of A by B.

Lemma 3.1 ([22]). *Let A be a unital separable amenable C^*-algebra with $A \in \mathcal{N}$, let B be a purely infinite stable C^*-algebra. Then there is a short exact sequence of groups*

$$0 \longrightarrow \Sigma \longrightarrow \mathrm{Ext}_s^u(A,B) \longrightarrow \Gamma \oplus \mathrm{Hom}(K_1(A), K_0(B)) \longrightarrow 0,$$

where

$$\Sigma = \mathrm{Ext}(K_0(A), [1_A]_0, K_0(B)) \oplus \mathrm{Ext}(K_1(A), K_1(B))$$

and

$$\Gamma = \{f \in \mathrm{Hom}(K_0(A), K_1(B));\ f([1_A]_0) = 0\}.$$

Lemma 3.2. *Assume that A is a unital separable AF-algebra and $B = \mathcal{O}_\infty \otimes \mathcal{K}$ or $B = \mathcal{O}_n \otimes \mathcal{K}$. Then one has*

$$\mathrm{Ext}_s^u(A, B) \cong \mathrm{Ext}(K_0(A), [1_A]_0, K_0(B)).$$

Proof. It is known that $K_1(B) = 0$, $K_0(B) = \mathbb{Z}$ or $\mathbb{Z}_{n-1}$ and hence $\Gamma = 0$. Since $K_1(A) = 0$, it shows from Lemma 3.1 that

$$\mathrm{Ext}_s^u(A, B) \cong \mathrm{Ext}(K_0(A), [1_A]_0, K_0(B)).$$

□

Lemma 3.3 (see [3], [17])**.** *Let A and B be separable nuclear C^*-algebras in $\mathcal{N}$ with B stable. Suppose x_1 and x_2 are elements of $\mathrm{Ext}(A, B)$. Then $K(x_1) = K(x_2)$ in $\mathrm{Ext}(A, B)$ if and only if there exist elements a of $KK(A, A)$ and b of $KK(B, B)$ with $K_*(a) = \mathrm{id}_{K_*(A)}$ and $K_*(b) = \mathrm{id}_{K_*(B)}$ such that $x_1 b = a x_2$.*

Note that a C^*-algebra B has the corona factorization property if and only if every full projection p in $M(B)$ is Murray-von Neumann equivalent to $1_{M(B)}$. By [6], [12], stable purely infinite simple C^*-algebras have the corona factorization property.

Theorem 3.4. *Assume that A is a unital separable AF-algebra and $B = \mathcal{O}_\infty \otimes \mathcal{K}$ or $B = \mathcal{O}_n \otimes \mathcal{K}$. If e_1 and e_2 in $\mathrm{Ext}(A, B)$ are unital essential extensions, then the following are equivalent:*

(1) *$E_1 \cong E_2$;*

(2) *$e_1 \cong e_2$;*

(3) *there exist isomorphisms $\alpha\colon (K_0(A), K_0(A)^+, [1_A]_0) \to (K_0(A), K_0(A)^+, [1_A]_0)$, $\beta\colon K_0(B) \to K_0(B)$ and $\eta\colon K_0(E_1) \to K_0(E_2)$ with $\eta([1_{E_1}]_0) = [1_{E_2}]_0$ such that the following diagram is commutative:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0(B) & \longrightarrow & K_0(E_1) & \longrightarrow & K_0(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & K_0(B) & \longrightarrow & K_0(E_2) & \longrightarrow & K_0(A) & \longrightarrow & 0.
\end{array}$$

Proof. It is trivial to see that (2) $\Rightarrow$ (1) by the definition of isomorphism of extensions. For (1) $\Rightarrow$ (2), there is an isomorphism $\eta\colon E_1 \to E_2$. It holds that $\eta(B) = B$ since e_1, e_2 are essential extensions and B is simple. One thus get $e_1 \cong e_2$. It is obvious that (2) $\Rightarrow$ (3), so we only need to show (3) $\Rightarrow$ (2).

By the classification theorems for AF-algebras and purely infinite simple C^*-algebras, there are automorphisms $\varphi\colon A \to A$ and $\psi\colon B \to B$ such that $K_0(\varphi) = \alpha$ and $K_0(\psi) = \beta$. We have that $e_1\varphi^{-1} \cong e_1$ and $\psi^{-1}e_2 \cong e_2$ from ([17], Proposition 1.2) and the following commutative diagram:

$$
\begin{array}{rccccccccc}
K(e_1\varphi^{-1})\colon & 0 & \longrightarrow & K_0(B) & \longrightarrow & K_0(E_1) & \longrightarrow & K_0(A) & \longrightarrow & 0 \\
& & & \Big\downarrow{\scriptstyle \mathrm{id}_{K_0(B)}} & & \Big\downarrow{\scriptstyle \mathrm{id}_{K_0(E_1)}} & & \Big\downarrow{\scriptstyle K_0(\varphi^{-1})} & & \\
K(e_1)\colon & 0 & \longrightarrow & K_0(B) & \longrightarrow & K_0(E_1) & \longrightarrow & K_0(A) & \longrightarrow & 0 \\
& & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \eta} & & \Big\downarrow{\scriptstyle \alpha} & & \\
K(e_2)\colon & 0 & \longrightarrow & K_0(B) & \longrightarrow & K_0(E_2) & \longrightarrow & K_0(A) & \longrightarrow & 0 \\
& & & \Big\downarrow{\scriptstyle K_0(\psi^{-1})} & & \Big\downarrow{\scriptstyle \mathrm{id}_{K_0(E_2)}} & & \Big\downarrow{\scriptstyle \mathrm{id}_{K_0(A)}} & & \\
K(\psi^{-1}e_2)\colon & 0 & \longrightarrow & K_0(B) & \longrightarrow & K_0(E_2) & \longrightarrow & K_0(A) & \longrightarrow & 0.
\end{array}
$$

Therefore, one has

$$(K(e_1\varphi^{-1}), [1_{E_1}]_0) \equiv (K(\psi^{-1}e_2), [1_{E_2}]_0)$$

from $K_0(\varphi) = \alpha$ and $K_0(\psi) = \beta$. According to Lemma 3.2, it follows that $[\tau_{e_1\varphi^{-1}}] = [\tau_{\psi^{-1}e_2}]$ in $\mathrm{Ext}_s^u(A, B)$. Since B has the corona factorization property, every unital full extension by B is unital-absorbing. It follows that $\tau_{e_1\varphi^{-1}}$ and $\tau_{\psi^{-1}e_2}$ are unitarily equivalent. Therefore, $e_1\varphi^{-1}$ and $\psi^{-1}e_2$ are isomorphic and so $e_1 \cong e_2$. □

Theorem 3.5. *Assume that A is a unital separable AF-algebra and $B = \mathcal{O}_\infty \otimes \mathcal{K}$ or $B = \mathcal{O}_n \otimes \mathcal{K}$. Suppose $e_i\colon 0 \to B \overset{\varphi_i}{\to} E_i \overset{\psi_i}{\to} A \to 0$ are two unital essential extensions. Then $E_1 \cong E_2$ if and only if there exists an isomorphism*

$$\eta\colon (K_0(E_1), K_0(E_1)^+, [1_{E_1}]_0) \to (K_0(E_2), K_0(E_2)^+, [1_{E_2}]_0).$$

Proof. We only need to prove the "if" part. Because $K_0(A)$ is torsion-free group, the two extensions in K-theory

$$0 \longrightarrow K_0(B) \xrightarrow{K_0(\varphi_i)} K_0(E_i) \xrightarrow{K_0(\psi_i)} K_0(A) \longrightarrow 0, \quad i = 1, 2$$

are pure extensions. It is well-known that $K_0(B) = K_0(B)^+$ since B is purely infinite simple C^*-algebra. Moreover, one has $K_0(B) \subset K_0(E_i)^+$, $i = 1, 2$, and $K_0(\psi_i)(K_0(E_i)^+) \subseteq K_0(A)^+$, $i = 1, 2$.

Suppose that

$$\eta\colon (K_0(E_1), K_0(E_1)^+, [1_{E_1}]_0) \to (K_0(E_2), K_0(E_2)^+, [1_{E_2}]_0)$$

is an isomorphism. For any $x \in K_0(\varphi_1)(K_0(B))$, $K_0(\psi_2)(\eta(x))$ and $K_0(\psi_2)(\eta(-x))$ are all in $K_0(A)^+$ since $K_0(B)$ is a group. As A is unital and finite, $(K_0(A), K_0(A)^+)$ is an ordered abelian group. Hence $K_0(\psi_2)(\eta(x)) = 0$ and $\eta(x) \in K_0(\varphi_2)(K_0(B))$. It thus follows that $\eta(K_0(\varphi_1)(K_0(B))) \subset K_0(\varphi_2)(K_0(B))$. Conversely, considering the isomorphism η^{-1} and arguing similarly, one can get $\eta^{-1}(K_0(\varphi_2)(K_0(B))) \subset K_0(\varphi_1)(K_0(B))$. Hence $\eta(K_0(\varphi_1)(K_0(B))) = K_0(\varphi_2)(K_0(B))$. Therefore, there exist two isomorphisms $\alpha\colon K_0(A) \to K_0(A)$ and $\beta\colon K_0(B) \to K_0(B)$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(B) & \xrightarrow{K_0(\varphi_1)} & K_0(E_1) & \xrightarrow{K_0(\psi_1)} & K_0(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\alpha} & & \\ 0 & \longrightarrow & K_0(B) & \xrightarrow{K_0(\varphi_2)} & K_0(E_2) & \xrightarrow{K_0(\psi_2)} & K_0(A) & \longrightarrow & 0. \end{array}$$

We next prove that α is an order isomorphism and $\alpha([1_A]_0) = [1_A]_0$. It is easy to check that $\alpha([1_A]_0) = [1_A]_0$ since e_i $(i = 1, 2)$ are unital extensions and $\eta([1_{E_1}]_0) = [1_{E_2}]_0$. As A and B are of real rank zero and $K_1(B) = 0$, this shows that $K_0(\psi_i)(K_0(E_i)^+) = K_0(A)^+$, $i = 1, 2$. So we have $\alpha(K_0(A)^+) = K_0(A)^+$ and it thus follows that $E_1 \cong E_2$ by Theorem 3.4. □

When A is non-unital separable AF-algebra, by using the UCT instead of Lemma 3.2 one can similarly obtain the following corollary which is contained in [3].

Corollary 3.6. *Assume that A is a separable non-unital AF-algebra and $B = \mathcal{O}_\infty \otimes \mathcal{K}$ or $B = \mathcal{O}_n \otimes \mathcal{K}$. If e_1 and e_2 in $\mathrm{Ext}(A, B)$ are non-unital essential extensions, then the following are equivalent:*

(1) $E_1 \cong E_2$;

(2) $e_1 \cong e_2$;

(3) *there exist isomorphisms* $\alpha\colon (K_0(A), K_0(A)^+, D(A)) \to (K_0(A), K_0(A)^+, D(A))$, $\beta\colon K_0(B) \to K_0(B)$ *and* $\eta\colon K_0(E_1) \to K_0(E_2)$ *such that the following diagram is commutative:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(B) & \longrightarrow & K_0(E_1) & \longrightarrow & K_0(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\alpha} & & \\ 0 & \longrightarrow & K_0(B) & \longrightarrow & K_0(E_2) & \longrightarrow & K_0(A) & \longrightarrow & 0; \end{array}$$

(4) $(K_0(E_1), K_0(E_1)^+, D(E_1)) \cong (K_0(E_2), K_0(E_2)^+, D(E_2))$.

Remark 3.7. We mention that Corollary 3.6 also holds whenever A is unital separable AF-algebra and e_i, $i = 1, 2$, are non-unital essential extensions. Moreover, it is clear that Theorem 3.4, Theorem 3.5 and Corollary 3.6 also hold whenever the Cuntz algebras are replaced by separable purely infinite simple nuclear C^*-algebras satisfying the UCT and having trivial K_1-groups.

4. Extensions by purely infinite simple C^*-algebras

Next we consider the case of separable purely infinite simple nuclear C^*-algebras (which are also called Kirchberg algebras) with nontrivial K_1-group.

For Kirchberg algebras B satisfying the UCT, the Elliott invariant $\mathrm{Ell}(B)$ is $(K_0(B), K_1(B))$ [or $(K_0(B), [1_B]_0, K_1(B))$ when B is unital].

The following result (Theorem 4.1) concerning non-unital extensions is contained in [3], so we omit the proof.

Theorem 4.1. *Assume that A is a separable AF-algebra, B is a non-unital separable purely infinite simple nuclear C^*-algebra satisfying the UCT and $e_i\colon 0 \to B \to E_i \to A \to 0$ are non-unital essential extensions of A by B. Then the following are equivalent:*

(1) $E_1 \cong E_2$;

(2) $e_1 \cong e_2$;

(3) *there exist isomorphisms* $\alpha_\sharp\colon \mathrm{Ell}(A) \to \mathrm{Ell}(A)$, $\beta_\sharp\colon \mathrm{Ell}(B) \to \mathrm{Ell}(B)$ *and* $\eta_\sharp\colon \mathrm{Ell}(E_1) \to \mathrm{Ell}(E_2)$ *such that the following diagram is commutative:*

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & K_0(B) & \longrightarrow & K_0(E_1) & \longrightarrow & K_0(A) & \longrightarrow & K_1(B) & \longrightarrow & K_1(E_1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \eta_0} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \eta_1} & & \\ 0 & \longrightarrow & K_0(B) & \longrightarrow & K_0(E_2) & \longrightarrow & K_0(A) & \longrightarrow & K_1(B) & \longrightarrow & K_1(E_2) & \longrightarrow & 0. \end{array}$$

Lemma 4.2 ([23])**.** *Let A be a separable nuclear C^*-algebra with unit. Then the natural homomorphism $\mathrm{Ext}_w^u(A, B) \to \mathrm{Ext}(A, B)$ is injective.*

Lemma 4.3 ([23])**.** *Let $e_i\colon 0 \to B \to E_i \to A \to 0$ be an essential unital extension with Busby invariant τ_i for $i = 1, 2$. Suppose e_1 is weakly unitarily equivalent to e_2 by a unitary $u \in \mathcal{Q}(B)$. Then*

$$(K(e_1), [1]_0) \equiv (K(e_2), [1]_0)$$

if and only if $\pi([u]_1)$ *is in* $G' = \{f([1_A]_0);\ f \in \operatorname{Hom}(\operatorname{Ker}\delta_0, \operatorname{Coker}\delta_1)\}$, *where* $\pi\colon K_1(\mathcal{Q}(B)) \cong K_0(B) \to \operatorname{Coker}\delta_1$ *is the quotient map and* δ_i *is the index map from* $K_i(A)$ *to* $K_{1-i}(B)$.

Lemma 4.4 ([22], Theorem 3.10). *Let* A *be a unital separable nuclear* C^**-algebra with* $A \in \mathcal{N}$. *Then there is a short exact sequence of groups*

$$0 \to K_1(\mathcal{Q}(B))/G \to \operatorname{Ext}_s^u(A,B) \to \operatorname{Ext}_w^u(A,B) \to 0,$$

where $G = \{f([1]_0);\ f \in \operatorname{Hom}(K_0(A), K_0(B))\}$.

Theorem 4.5. *Assume that* A *is a unital separable AF-algebra,* B *is a non-unital separable purely infinite simple nuclear* C^**-algebra satisfying the UCT and* $e_i\colon 0 \to B \to E_i \to A \to 0$ *is a unital essential extension of* A *by* B *such that* $\operatorname{Ker}\delta^0_{e_i}$ *is a direct summand of* $K_0(A)$, *where* $\delta^0_{e_i}$ *is the exponential map of* e_i *for* $i = 1, 2$. *Then the following are equivalent:*

(1) $E_1 \cong E_2$;
(2) $e_1 \cong e_2$;
(3) *there are isomorphisms* $\beta_*\colon K_*(B) \to K_*(B)$, $\eta_*\colon (K_*(E), [1]_0) \to (K_*(E), [1]_0)$ *and* $\alpha_*\colon (K_0(A), K_0(A)^+, [1]_0) \to (K_0(A), K_0(A)^+, [1]_0)$ *such that*

$$(\beta_*, \eta_*, \alpha_*)\colon (K(e_1), [1_{E_1}]_0) \to (K(e_2), [1_{E_2}]_0)$$

is an isomorphism.

Proof. We only need to show (3) $\Rightarrow$ (2). Similarly to Theorem 3.4, there are isomorphisms $\varphi\colon A \to A$ and $\psi\colon B \to B$ such that $\varphi_* = \alpha_*$ and $\psi_* = \beta_*$. Hence, we have an extension isomorphism

$$(\mathrm{id}_{K_*(B_2)}, \eta_*, \mathrm{id}_{K_*(A_1)})\colon (K(\psi e_1), [1]_0) \longrightarrow (K(e_2\varphi), [1]_0).$$

Therefore, $(K(\psi e_1), [1]_0) \equiv (K(e_2\varphi), [1]_0)$.

Similarly, there are isomorphisms $h\colon A \to A$ and $g\colon B \to B$ such that $K_*(h) = \mathrm{id}_{K_*(A)}$, $K_*(g) = \mathrm{id}_{K_*(B)}$ and

$$[e_1]KK(\psi)KK(g) = KK(h)KK(\varphi)[e_2]$$

in $\operatorname{Ext}(A,B)$. Set $\sigma_1 = (g\psi)e_1$ and $\sigma_2 = e_2(\varphi h)$ with Busby invariants τ_1 and τ_2, respectively. So there are two commutative diagrams

$$\begin{array}{cccccccccc}
e_1\colon & 0 & \longrightarrow & B & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\
& & & \downarrow{\scriptstyle g\psi} & & \| & & \| & & \\
\sigma_1\colon & 0 & \longrightarrow & B & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

and

$$
\begin{array}{ccccccccccc}
\sigma_2\colon & 0 & \longrightarrow & B & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0 \\
 & & & \Vert & & \Vert & & \downarrow{\scriptstyle \varphi h} & & \\
e_2\colon & 0 & \longrightarrow & B & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0.
\end{array}
$$

Furthermore,

$$(K(\sigma_1), [1]_0) \equiv (K(\psi e_1), [1]_0), \ (K(\sigma_2), [1]_0) \equiv (K(e_1\varphi), [1]_0).$$

Then $(K(\sigma_1), [1]_0) \equiv (K(\sigma_2), [1]_0)$. So we have the following commutative diagram in K-theory

$$
\begin{array}{ccccccccccc}
K(\sigma_1)\colon & \longrightarrow & K_0(B) & \longrightarrow & K_0(E_1) & \longrightarrow & K_0(A) & \xrightarrow{\delta^0_{\sigma_1}} & K_1(B) & \longrightarrow \\
 & & \downarrow{\scriptstyle \beta_0^{-1}} & & \Vert & & \Vert & & \downarrow{\scriptstyle \beta_1^{-1}} & \\
K(e_1)\colon & \longrightarrow & K_0(B) & \longrightarrow & K_0(E_1) & \longrightarrow & K_0(A) & \xrightarrow{\delta^0_{e_1}} & K_1(B) & \longrightarrow \\
 & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \eta_0} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \beta_1} & \\
K(e_2)\colon & \longrightarrow & K_0(B) & \longrightarrow & K_0(E_2) & \longrightarrow & K_0(A) & \xrightarrow{\delta^0_{e_2}} & K_1(B) & \longrightarrow \\
 & & \Vert & & \Vert & & \downarrow{\scriptstyle \alpha_0^{-1}} & & \Vert & \\
K(\sigma_2)\colon & \longrightarrow & K_0(B) & \longrightarrow & K_0(E_2) & \longrightarrow & K_0(A) & \xrightarrow{\delta^0_{\sigma_2}} & K_1(B) & \longrightarrow
\end{array}
$$

Note that $\delta^0_{\sigma_1} = K_1(g\psi)\delta^0_{e_1} = \beta_1\delta^0_{e_1}$, $\delta^0_{\sigma_2} = \delta^0_{e_2}K_0(\varphi h) = \delta^0_{e_2}\alpha_0$ and $\delta^0_{\sigma_1} = \delta^0_{\sigma_2}$. Since β_1 is an isomorphism, it follows that $\operatorname{Ker}\delta^0_{\sigma_1} = \operatorname{Ker}\delta^0_{e_1}$. Hence $\operatorname{Ker}\delta^0_{\sigma_1}$ is a direct summand of $K_0(A)$. Since $\delta^1_{\sigma_1} = 0 = \delta^1_{\sigma_2}$, we have $\operatorname{Coker}\delta^1_{\sigma_1} = \operatorname{Coker}\delta^1_{\sigma_2} = K_0(B)$ and the quotient map π from $K_0(B)$ to $\operatorname{Coker}\delta^1_{\sigma_1}$ is the identity map.

By the fact $[\sigma_1] = [\sigma_2]$ in $\operatorname{Ext}(A, B)$ and Lemma 4.2, we have $[\sigma_1] = [\sigma_2]$ in $\operatorname{Ext}^u_w(A, B)$. Since B has corona factorization property, every unital full extension by B is unital-absorbing. Hence there is a unitary $u \in \mathcal{Q}(B)$ such that $\tau_2 = \operatorname{Ad} u \circ \tau_1$. By Lemma 4.3 we have $[u]_1 \in G'$. Hence there exists a homomorphism ϱ from $\operatorname{Ker}\delta^0_{\sigma_1}$ to $K_0(B)$ such that $\varrho([1_A]_0) = [u]_1$. Since $\operatorname{Ker}\delta^0_{\sigma_1}$ is a direct summand of $K_0(A)$, there exists a homomorphism $\tilde{\varrho}$ from $K_0(A)$ to $K_0(B)$ such that $\tilde{\varrho}|_{\operatorname{Ker}\delta^0_{\sigma_1}} = \varrho$. Therefore, $[u]_1$ is in G. By Lemma 4.4, σ_1 is strongly unitarily equivalent to σ_2. It follows that

$$e_1 \cong \sigma_1 \sim \sigma_2 \cong e_2.$$

□

Remark 4.6. If E_i $(i = 1, 2)$ in Theorem 4.5 are of real rank zero, the exponential maps $\delta^0_{e_i}$, $i = 1, 2$, will be trivial and so the kernel observation holds true automatically. This special case is contained in [2] and [3] which deal with the class of extensions with real rank zero where the exponential maps $\delta^0_{e_i}$, $i = 1, 2$, are trivial. This is very different from the case we consider here.

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