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ON BLOCK TRIANGULAR MATRICES WITH SIGNED DRAZIN INVERSE

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Abstract. The sign pattern of a real matrix A, denoted by $\operatorname{sgn} A$, is the $(+,-,0)$-matrix obtained from A by replacing each entry by its sign. Let $\mathcal{Q}(A)$ denote the set of all real matrices B such that $\operatorname{sgn} B = \operatorname{sgn} A$. For a square real matrix A, the Drazin inverse of A is the unique real matrix X such that $A^{k+1}X = A^k$, $XAX = X$ and $AX = XA$, where k is the Drazin index of A. We say that A has signed Drazin inverse if $\operatorname{sgn} \tilde{A}^{\rm d} = \operatorname{sgn} A^{\rm d}$ for any $\tilde{A} \in \mathcal{Q}(A)$, where $A^{\rm d}$ denotes the Drazin inverse of A. In this paper, we give necessary conditions for some block triangular matrices to have signed Drazin inverse.

Keywords: sign pattern matrix; signed Drazin inverse; strong sign nonsingular matrix

MSC 2010: 15B35, 15A09

1. Introduction

A matrix $\mathcal{A}$ whose entries consist of $\{+,-,0\}$ is called a *sign pattern matrix*. The *sign pattern* of a real matrix A, denoted by $\operatorname{sgn} A$, is the sign pattern matrix obtained from A by replacing each entry by its sign. Let $\mathcal{Q}(A)$ denote the set of all real matrices B such that $\operatorname{sgn} B = \operatorname{sgn} A$.

A square real matrix A is called a *sign nonsingular matrix* (abbreviated SNS matrix) if each matrix in $\mathcal{Q}(A)$ is nonsingular. An SNS matrix A is called a *strong sign nonsingular matrix* (abbreviated $\mathrm{S^2NS}$ matrix), if the inverses of the matrices in $\mathcal{Q}(A)$ all have the same sign pattern.

For a square real matrix A, the *Drazin index* of A is the smallest nonnegative integer k such that $\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$, denoted by $\operatorname{ind}(A)$. The *Drazin inverse* of A is the real matrix X satisfying three matrix equations: $A^{k+1}X = A^k$, $XAX = X$

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and $AX = XA$, where $k = \operatorname{ind}(A)$. Let A^{d} denote the Drazin inverse of A. It is well known that A^{d} exists and is unique (see [4], [13]). A is nonsingular if and only if $\operatorname{ind}(A) = 0$, and $A^{\mathrm{d}} = A^{-1}$ when A is nonsingular. We say that A has *signed Drazin inverse* (or simply say "A^{d} is signed"), if $\operatorname{sgn} \tilde{A}^{\mathrm{d}} = \operatorname{sgn} A^{\mathrm{d}}$ for any $\tilde{A} \in \mathcal{Q}(A)$.

S^2NS matrices and related digraph characterizations have been extensively studied in combinatorial matrix theory (see [1], [3], [8], [9], [11]). Matrices with signed generalized inverse are generalizations of S^2NS matrices. Some results on matrices with signed generalized inverse can be found in [5], [7], [8], [10], [12], [13].

For a reducible square real matrix M, there exists a permutation matrix P such that $M = P \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right) P^{\top}$, where A and C are square. Then $M^{\mathrm{d}} = P \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)^{\mathrm{d}} P^{\top}$. Clearly M^{d} is signed if and only if the block triangular matrix $\left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$ has signed Drazin inverse. In this paper, we give some results on block triangular matrices with signed Drazin inverse.

2. Preliminaries

In this section we give some auxiliary lemmas.

Lemma 2.1 ([4]). *Let* $M = \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$ *be a square matrix, where* A *and* C *are square,* $\operatorname{ind}(A) = k$, $\operatorname{ind}(C) = j$. *Then*

$$M^{\mathrm{d}} = \begin{pmatrix} A^{\mathrm{d}} & X \\ 0 & C^{\mathrm{d}} \end{pmatrix},$$

where

$$X = \left[\sum_{i=0}^{j-1} (A^{\mathrm{d}})^{i+2} B C^i\right](I - CC^{\mathrm{d}}) + (I - AA^{\mathrm{d}})\left[\sum_{i=0}^{k-1} A^i B (C^{\mathrm{d}})^{i+2}\right] - A^{\mathrm{d}} B C^{\mathrm{d}}.$$

Lemma 2.2 ([4]). *Let* $M = \left(\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right)$ *be a square matrix, where* A *is square. Then*

$$M^{\mathrm{d}} = \begin{pmatrix} A^{\mathrm{d}} & (A^{\mathrm{d}})^2 B \\ 0 & 0 \end{pmatrix}.$$

Lemma 2.3 ([13]). *Let* $M = \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$ *be a square matrix with signed Drazin inverse, where* A *and* C *are square. Then* A^{d} *and* C^{d} *are signed.*

The *term rank* of a matrix A is the maximal cardinality of the sets of nonzero entries of A no two of which lie in the same row or the same column. For a square matrix A of order n, we say that A has *full term rank* if the term rank of A is n.

Lemma 2.4 ([13]). *Let A be a square real matrix with full term rank. Then A^{d} is signed if and only if A is an* $\mathrm{S^2NS}$ *matrix.*

For a nilpotent matrix A, the *nilpotent index* of A is the smallest integer k such that $A^k = 0$. A real matrix A is *sign nilpotent* if each matrix in $\mathcal{Q}(A)$ is nilpotent. For a sign nilpotent matrix A, the *nilpotent index* of $\operatorname{sgn} A$ is the maximum nilpotent index of all matrices in $\mathcal{Q}(A)$.

Two square matrices B, C are *permutation similar* if there exists a permutation matrix P such that $PBP^{\top} = C$.

Lemma 2.5 ([6]). *A square matrix A is sign nilpotent if and only if A is permutation similar to a strictly upper triangular matrix.*

A signed digraph S is a digraph in which each of its arcs is assigned a sign $+$ or $-$. The sign of a subdigraph S_1 of S is defined to be the product of the signs of all arcs of S_1.

Let $A = (a_{ij})$ be a square real matrix of order n. The *associated digraph* $D(A)$ of A is defined to be the digraph with the vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, j);\ a_{ij} \neq 0\}$. The *associated signed digraph* $S(A)$ of A is obtained from $D(A)$ by assigning the sign of a_{ij} to each arc (i, j) in $D(A)$. A is *fully indecomposable* if A does not contain a nonvacuous zero submatrix whose number of rows and number of columns sum to n (see [8]).

Lemma 2.6 ([2]). *Let A be a square real matrix such that all diagonal entries of A are nonzero. Then its associated digraph $D(A)$ is strongly connected if and only if A is fully indecomposable.*

A signed digraph S is called an $\mathrm{S^2NS}$ *signed digraph* if S satisfies the following two conditions:

(1) The sign of each cycle of S is negative.
(2) Each pair of paths in S with the same initial vertex and the same terminal vertex has the same sign.

Lemma 2.7 ([8]). *Let A be a square real matrix such that all diagonal entries of A are negative. Then A is an* $\mathrm{S^2NS}$ *matrix if and only if its associated signed digraph $S(A)$ is an* $\mathrm{S^2NS}$ *signed digraph.*

A matrix is said to be *totally nonzero* if it has no zero entries.

Lemma 2.8 ([8]). *Let A be an* $\mathrm{S^2NS}$ *matrix. Then A^{-1} is totally nonzero if and only if A is fully indecomposable.*

For a matrix M, let $M[i, j]$ denote the (i, j)-entry of M.

Lemma 2.9 ([1]). *Let A be a fully indecomposable* $\mathrm{S^2NS}$ *matrix and $A[p, q] \neq 0$. Then* $\operatorname{sgn} A^{-1}[q, p] = \operatorname{sgn} A[p, q]$.

A real matrix A of order n has a *signed determinant* if the determinants of the matrices in $\mathcal{Q}(A)$ all have the same sign. The standard determinant expansion of $A = (a_{ij})$ is

$$\det A = \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)},$$

where $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ.

Lemma 2.10 ([3]). *Let A be a square real matrix. Then the following statements are equivalent:*

(1) *A is an* SNS *matrix.*
(2) $\det A \neq 0$ *and A has a signed determinant.*
(3) *There is a nonzero term in the standard determinant expansion of A and every nonzero term has the same sign.*

A real matrix A of order n has an *identically zero determinant* if $\det \tilde{A} = 0$ for any $\tilde{A} \in \mathcal{Q}(A)$. Clearly A has an identically zero determinant if and only if the term rank of A is less than n.

Lemma 2.11 ([3]). *Every square submatrix of a fully indecomposable* $\mathrm{S^2NS}$ *matrix is an* SNS *matrix or has an identically zero determinant.*

3. Main results

For a square real matrix A, its sign pattern $\operatorname{sgn} A$ is called *potentially nilpotent* if there exists a nilpotent matrix $\tilde{A} \in \mathcal{Q}(A)$ (see [6]). It is known that the Drazin inverse of a nilpotent matrix A is always a zero matrix, and the nilpotent index of A equals its Drazin index $\operatorname{ind}(A)$.

We use $M[p, :]$ and $M[:, r]$ to denote the p-th row and the r-th column of a matrix M, respectively.

Theorem 3.1. *Let* $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ *be a square matrix with signed Drazin inverse, where* A *is square,* $\operatorname{sgn} C$ *is potentially nilpotent. Then there exists a permutation matrix* P *such that*

$$\begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & P^\top \end{pmatrix} = \begin{pmatrix} A & F \\ 0 & N \end{pmatrix},$$

where $N = PCP^\top$ *is a strictly upper triangular matrix. Moreover, we have:*

(1) $\operatorname{sgn}\{(\tilde{A}^{\mathrm{d}})^2 \tilde{F}[:,1]\} = \operatorname{sgn}\{(A^{\mathrm{d}})^2 F[:,1]\}$ *for any* $\tilde{A} \in \mathcal{Q}(A)$, $\tilde{F} \in \mathcal{Q}(F)$.

(2) *For any* $\tilde{A} \in \mathcal{Q}(A)$, $\tilde{F} \in \mathcal{Q}(F)$, $\tilde{N} \in \mathcal{Q}(N)$, *the Hadamard product of* $(\tilde{A}^{\mathrm{d}})^2 \tilde{F}$ *and* $\sum_{i=1}^{j-1} (\tilde{A}^{\mathrm{d}})^{i+2} \tilde{F} \tilde{N}^i$ *is nonnegative, where* j *is the nilpotent index of* $\operatorname{sgn} C$.

Proof. Since M^{d} is signed, by Lemma 2.3, C^{d} is signed. Since $\operatorname{sgn} C$ is potentially nilpotent, C is sign nilpotent. By Lemma 2.5, there exists a permutation matrix P such that $PCP^\top = N$ is a strictly upper triangular matrix.

Let $L = \begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix} M \begin{pmatrix} I & 0 \\ 0 & P^\top \end{pmatrix} = \begin{pmatrix} A & F \\ 0 & N \end{pmatrix}$. Then L^{d} is signed. For any $\tilde{L} = \begin{pmatrix} \tilde{A} & \tilde{F} \\ 0 & \tilde{N} \end{pmatrix} \in \mathcal{Q}(L)$, by Lemma 2.1 we have $\tilde{L}^{\mathrm{d}} = \begin{pmatrix} \tilde{A}^{\mathrm{d}} & X \\ 0 & 0 \end{pmatrix}$, where $X = \sum_{i=0}^{j-1} (\tilde{A}^{\mathrm{d}})^{i+2} \tilde{F} \tilde{N}^i$, j being the nilpotent index of $\operatorname{sgn} C$. Since $\tilde{N}$ is strictly upper triangular, the first column of $\tilde{N}$ is a zero vector. So the first column of X is $X[:,1] = (\tilde{A}^{\mathrm{d}})^2 \tilde{F}[:,1]$. Hence we have

$$\operatorname{sgn}\{(\tilde{A}^{\mathrm{d}})^2 \tilde{F}[:,1]\} = \operatorname{sgn}\{(A^{\mathrm{d}})^2 F[:,1]\},$$

and part (1) holds.

The r-th column of X is

$$X[:,r] = (\tilde{A}^{\mathrm{d}})^2 \tilde{F}[:,r] + \sum_{i=1}^{j-1} (\tilde{A}^{\mathrm{d}})^{i+2} \tilde{F} \tilde{N}^i[:,r].$$

Since $\tilde{N}$ is strictly upper triangular, the column vector $\tilde{F}\tilde{N}^i[:,r]$ is a linear combination of $\tilde{F}[:,1], \tilde{F}[:,2], \ldots, \tilde{F}[:,r-1]$. If the Hadamard product of $(\tilde{A}^{\mathrm{d}})^2 \tilde{F}[:,r]$ and $\sum_{i=1}^{j-1} (\tilde{A}^{\mathrm{d}})^{i+2} \tilde{F} \tilde{N}^i[:,r]$ is not nonnegative, then there exists an integer q such that

$$\operatorname{sgn}\{(\tilde{A}^{\mathrm{d}})^2[q,:] \tilde{F}[:,r]\} = -\operatorname{sgn}\left\{\sum_{i=1}^{j-1} (\tilde{A}^{\mathrm{d}})^{i+2}[q,:] \tilde{F} \tilde{N}^i[:,r]\right\} \neq 0.$$

Then we can choose $\tilde{F}[:,1], \tilde{F}[:,2], \ldots, \tilde{F}[:,r]$ such that $X[q,r] > 0$, and we can also choose $\tilde{F}[:,1], \tilde{F}[:,2], \ldots, \tilde{F}[:,r]$ such that $X[q,r] < 0$, a contradiction to L^{d} being signed. Hence the Hadamard product of $(\tilde{A}^{\mathrm{d}})^2 \tilde{F}$ and $\sum_{i=1}^{j-1} (\tilde{A}^{\mathrm{d}})^{i+2} \tilde{F} \tilde{N}^i$ is nonnegative, and part (2) holds. $\square$

For two sign pattern matrices $\mathcal{A}$ and $\mathcal{B}$, we say that the (i,j)-entry of $\mathcal{AB}$ is an *uncertain entry* if there are two nonzero terms in the sum $\sum\limits_k \mathcal{A}[i,k]\mathcal{B}[k,j]$ that have opposite signs.

Lemma 3.2. *Let A be a nonsingular real matrix, and let $\mathcal{A} = \operatorname{sgn} A^{-1}$. If the (i,j)-entry of $\mathcal{A}^2$ is an uncertain entry, then there exist nonsingular matrices $A_1, A_2 \in \mathcal{Q}(A)$ such that $A_1^{-2}[i,j] > 0$, $A_2^{-2}[i,j] < 0$.*

P r o o f. If the (i,j)-entry of $\mathcal{A}^2$ is an uncertain entry, then there exist integers p, q ($p \neq q$) such that $A^{-1}[i,p]A^{-1}[p,j] > 0$, $A^{-1}[i,q]A^{-1}[q,j] < 0$. Let $D_r(m)$ be the diagonal matrix obtained from an identity matrix I by replacing the r-th entry of I by a positive number m. Let $A_1 = D_p(m)A$, $A_2 = D_q(m)A$, then $A_1, A_2 \in \mathcal{Q}(A)$ and $A_1^{-1} = A^{-1}D_p(1/m)$, $A_2^{-1} = A^{-1}D_q(1/m)$. By computation, we have

$$\begin{aligned} A_1^{-2}[i,j] &= \sum_k A_1^{-1}[i,k]A_1^{-1}[k,j] = A_1^{-1}[i,p]A_1^{-1}[p,j] + \sum_{k\neq p} A_1^{-1}[i,k]A_1^{-1}[k,j] \\ &= \frac{A^{-1}[i,p]A_1^{-1}[p,j]}{m} + \sum_{k\neq p} A^{-1}[i,k]A_1^{-1}[k,j]. \end{aligned}$$

Similarly we also have

$$A_2^{-2}[i,j] = \frac{A^{-1}[i,q]A_2^{-1}[q,j]}{m} + \sum_{k\neq q} A^{-1}[i,k]A_2^{-1}[k,j].$$

If $j \neq p$ and $j \neq q$, then

$$\begin{aligned} A_1^{-2}[i,j] &= \frac{A^{-1}[i,p]A^{-1}[p,j]}{m} + \sum_{k\neq p} A^{-1}[i,k]A^{-1}[k,j], \\ A_2^{-2}[i,j] &= \frac{A^{-1}[i,q]A^{-1}[q,j]}{m} + \sum_{k\neq q} A^{-1}[i,k]A^{-1}[k,j]. \end{aligned}$$

We can choose m such that $\operatorname{sgn} A_1^{-2}[i,j] = \operatorname{sgn}(A^{-1}[i,p]A^{-1}[p,j])/m > 0$, and $\operatorname{sgn} A_2^{-2}[i,j] = \operatorname{sgn}(A^{-1}[i,q]A^{-1}[q,j])/m < 0$.

If $j = p \neq q$, then

$$\begin{aligned} A_1^{-2}[i,j] &= \frac{A^{-1}[i,p]A^{-1}[p,j]}{m^2} + \frac{1}{m}\sum_{k\neq p} A^{-1}[i,k]A^{-1}[k,j], \\ A_2^{-2}[i,j] &= \frac{A^{-1}[i,q]A^{-1}[q,j]}{m} + \sum_{k\neq q} A^{-1}[i,k]A^{-1}[k,j]. \end{aligned}$$

Let $a = A^{-1}[i,p]A^{-1}[p,j] > 0$, $b = \sum\limits_{k\neq p} A^{-1}[i,k]A^{-1}[k,j]$, then $A_1^{-2}[i,j] = a/m^2 + b/m$. Consider the limit

$$\lim_{m\to 0}\frac{b/m}{a/m^2} = \lim_{m\to 0}\frac{bm}{a} = 0.$$

So there exists $m > 0$ such that $|b|/m < a/m^2$. Hence we can choose m such that $\operatorname{sgn} A_1^{-2}[i,j] = \operatorname{sgn}(A^{-1}[i,p]A^{-1}[p,j])/m > 0$, and $\operatorname{sgn} A_2^{-2}[i,j] = \operatorname{sgn}(A^{-1}[i,q]A^{-1}[q,j])/m < 0$.

If $j = q \neq p$, similarly to the above arguments we can choose m such that $A_1^{-2}[i,j] > 0$, $A_2^{-2}[i,j] < 0$. □

Theorem 3.3. *Let $M = \begin{pmatrix} A & B\\ 0 & C\end{pmatrix}$ be a square matrix with signed Drazin inverse, where A is a square matrix with full term rank, B has at least one column without zero entries, $\operatorname{sgn} C$ is potentially nilpotent. Then A is an $\mathrm{S^2NS}$ matrix, and neither $\mathcal{A}^2$ nor $\mathcal{A}^2\mathcal{B}$ have uncertain entries, where $\mathcal{A} = \operatorname{sgn} A^{-1}$, $\mathcal{B} = \operatorname{sgn} B$.*

P r o o f. It follows from Lemma 2.3 that $A^{\mathrm d}$ is signed. Since A has full term rank, by Lemma 2.4, A is an $\mathrm{S^2NS}$ matrix. By Theorem 3.1, there exists a permutation matrix P such that

$$\begin{pmatrix} I & 0\\ 0 & P\end{pmatrix}\begin{pmatrix} A & B\\ 0 & C\end{pmatrix}\begin{pmatrix} I & 0\\ 0 & P^{\top}\end{pmatrix} = \begin{pmatrix} A & F\\ 0 & N\end{pmatrix},$$

where $N = PCP^{\top}$ is a strictly upper triangular matrix. Let $L = \begin{pmatrix} A & F\\ 0 & N\end{pmatrix}$, then $L^{\mathrm d}$ is signed. For any $\tilde L = \begin{pmatrix} \tilde A & \tilde F\\ 0 & \tilde N\end{pmatrix} \in \mathcal{Q}(L)$, by Lemma 2.1 we have $\tilde L^{\mathrm d} = \begin{pmatrix} \tilde A^{-1} & X\\ 0 & 0\end{pmatrix}$, where $X = \sum\limits_{i=0}^{j-1}(\tilde A^{-1})^{i+2}\tilde F\tilde N^{i}$, j being the nilpotent index of $\operatorname{sgn} N$. Since $F = BP^{\top}$ and B has at least one column without zero entries, $\tilde F$ has at least one column without zero entries. Suppose that the r-th column of $\tilde F$ has no zero entries. The r-th column of X is

$$X[:,r] = \tilde A^{-2}\tilde F[:,r] + \sum_{i=1}^{j-1}(\tilde A^{-1})^{i+2}\tilde F\tilde N^{i}[:,r].$$

Let $\mathcal{A} = \operatorname{sgn} A^{-1}$. If $\mathcal{A}^2$ has at least one uncertain entry, by Lemma 3.2 there exist integers p, q and $A_1, A_2 \in \mathcal{Q}(A)$ such that $A_1^{-2}[p,q] > 0$, $A_2^{-2}[p,q] < 0$. By Theorem 3.1, the Hadamard product of $\tilde A^{-2}\tilde F$ and $\sum\limits_{i=1}^{j-1}(\tilde A^{-1})^{i+2}\tilde F\tilde N^{i}$ is nonnegative. Since $\tilde F[:,r]$ has no zero entries, we can choose $\tilde A$ and $\tilde F[:,r]$ such that $X[p,r] > 0$, and we can also choose $\tilde A$ and $\tilde F[:,r]$ such that $X[p,r] < 0$, a contradiction to $L^{\mathrm d}$ being signed. Hence $\mathcal{A}^2$ has no uncertain entries.

Let $\mathcal{B} = \operatorname{sgn} B$. Since $F = BP^\top$, $\mathcal{A}^2\mathcal{B}$ has no uncertain entries if and only if $\mathcal{A}^2\mathcal{F}$ has no uncertain entries, where $\mathcal{F} = \operatorname{sgn} F$. Next we will show that $\mathcal{A}^2\mathcal{F}_r$ has no uncertain entries for any r, where $\mathcal{F}_r = \operatorname{sgn} F[:, r]$. If the p-th entry of $\mathcal{A}^2\mathcal{F}_r$ is an uncertain entry, then there exist $F_1, F_2 \in \mathcal{Q}(F)$ such that $A^{-2}[p, :]F_1[:, r] > 0$, $A^{-2}[p, :]F_2[:, r] < 0$. Recall that the Hadamard product of $\tilde{A}^{-2}\tilde{F}$ and $\sum_{i=1}^{j-1}(\tilde{A}^{-1})^{i+2}\tilde{F}\tilde{N}^i$ is nonnegative. So we can choose $\tilde{F}[:, r]$ such that $X[p, r] > 0$, and we can also choose $\tilde{F}[:, r]$ such that $X[p, r] < 0$, a contradiction to L^{d} being signed. Hence $\mathcal{A}^2\mathcal{F}$ has no uncertain entries, i.e., $\mathcal{A}^2\mathcal{B}$ has no uncertain entries. □

Theorem 3.4. *Let $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ be a square matrix with signed Drazin inverse, where A is a square matrix such that all diagonal entries of A are nonzero, B has at least one column without zero entries,* $\operatorname{sgn} C$ *is potentially nilpotent. Then there exist permutation matrices P, Q such that*

$$\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\begin{pmatrix} P^\top & 0 \\ 0 & Q^\top \end{pmatrix} = \begin{pmatrix} A_1 & 0 & F_1 \\ 0 & A_2 & F_2 \\ 0 & 0 & N \end{pmatrix},$$

where $N = QCQ^\top$ is a strictly upper triangular matrix. Moreover, we have:

(1) *$-A_1$ and A_2 are upper triangular matrices with negative diagonal entries, and their associated signed digraphs $S(-A_1)$ and $S(A_2)$ are* $\mathrm{S^2NS}$ *signed digraphs (A_1 or A_2 can be vacuous).*

(2) *Neither $\mathcal{A}_1^2\mathcal{F}_1$ nor $\mathcal{A}_2^2\mathcal{F}_2$ have uncertain entries, where $\mathcal{A}_i = \operatorname{sgn} A_i^{-1}$, $\mathcal{F}_i =$* $\operatorname{sgn} F_i$ $(i = 1, 2)$.

(3) *For any $\tilde{M} \in \mathcal{Q}(M)$, the eigenvalues of $\tilde{M}$ consist of all diagonal entries of $\tilde{M}$.*

Proof. Theorem 3.1 implies that there exists a permutation matrix Q such that $N = QCQ^\top$ is a strictly upper triangular matrix. Since all diagonal entries of A are nonzero, A has full term rank. By Theorem 3.3, A is an $\mathrm{S^2NS}$ matrix, and neither $\mathcal{A}^2$ nor $\mathcal{A}^2\mathcal{B}$ have uncertain entries, where $\mathcal{A} = \operatorname{sgn} A^{-1}$, $\mathcal{B} = \operatorname{sgn} B$.

If A is irreducible, then its associated digraph $D(A)$ is strongly connected. By Lemma 2.6, A is fully indecomposable. From Lemma 2.8 we know that A^{-1} is totally nonzero. If A has both positive and negative diagonal entries, i.e., there exist integers p, q such that $A[p, p] > 0$, $A[q, q] < 0$, then $A^{-1}[p, p] > 0$, $A^{-1}[q, q] < 0$ by Lemma 2.9. Since $\mathcal{A}^2$ has no uncertain entries, we have $A^{-1}[p, q] = A^{-1}[q, p] = 0$, a contradiction to A^{-1} being totally nonzero. So all diagonal entries of A have the same sign. If A has order $n \geqslant 2$, since A is irreducible, there exist i, j $(i < j)$ such that $A[i, j] \neq 0$. By Lemma 2.9, $\mathcal{A}[j, i] = \operatorname{sgn} A[i, j]$. Moreover, we have

$$A^{-1}[i, j] = (-1)^{i+j}\frac{\det A(j, i)}{\det A},$$

where $A(j,i)$ denotes the submatrix of A obtained by deleting the j-th row and the i-th column of A. Since all diagonal entries of A have the same sign, by Lemma 2.10 we get

$$\operatorname{sgn}(\det A) = \operatorname{sgn}(A[1,1])^n.$$

Since the term rank of $A(j,i)$ is $n-1$, by Lemma 2.11, $A(j,i)$ is an SNS matrix. Lemma 2.10 implies that

$$\operatorname{sgn}(\det A(j,i)) = (-1)^{j-i-1}\operatorname{sgn}((A[1,1])^{n-2}A[i,j]).$$

Hence we have

$$\operatorname{sgn} A^{-1}[i,j] = (-1)^{i+j}\frac{\operatorname{sgn}(\det A(j,i))}{\operatorname{sgn}(\det A)} = -\operatorname{sgn} A[i,j] = -\operatorname{sgn} A^{-1}[j,i] \neq 0.$$

Recall that A^{-1} is totally nonzero. Since $A^{-1}[i,j]$ and $A^{-1}[j,i]$ have opposite signs, $\mathcal{A}^2[i,i]$ is an uncertain entry, a contradiction. Hence A is an $\mathrm{S}^2\mathrm{NS}$ matrix of order 1 if A is irreducible.

If A is reducible, then according to the above arguments, each irreducible component of A is an $\mathrm{S}^2\mathrm{NS}$ matrix of order 1. So A is permutation similar to an upper triangular $\mathrm{S}^2\mathrm{NS}$ matrix. If $A[i,i] > 0$ and $A[j,j] < 0$, then $A^{-1}[i,i] > 0$ and $A^{-1}[j,j] < 0$. Since $\mathcal{A}^2$ has no uncertain entries, we have $A^{-1}[i,j] = A^{-1}[j,i] = 0$. Hence $\mathcal{A} = \operatorname{sgn} A^{-1}$ is permutation similar to $\begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix}$, where $-\mathcal{A}_1$ and $\mathcal{A}_2$ are upper triangular sign patterns with negative diagonal entries. Hence there exists a permutation matrix P such that $PAP^\top = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where $-A_1$ and A_2 are upper triangular $\mathrm{S}^2\mathrm{NS}$ matrices with negative diagonal entries, and $\operatorname{sgn} A_i^{-1} = \mathcal{A}_i$ $(i=1,2)$. By Lemma 2.7, their associated signed digraphs $S(-A_1)$ and $S(A_2)$ are $\mathrm{S}^2\mathrm{NS}$ signed digraphs. Hence part (1) holds.

Let $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = PBQ^\top$, where F_1 has the same number of rows as A_1. Since $\mathcal{A}^2\mathcal{B}$ has no uncertain entries, both $\mathcal{A}_1^2\mathcal{F}_1$ and $\mathcal{A}_2^2\mathcal{F}_2$ have no uncertain entries, where $\mathcal{F}_i = \operatorname{sgn} F_i$ $(i=1,2)$. Hence part (2) holds.

Note that $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ is permutation similar to $\begin{pmatrix} A_1 & 0 & F_1 \\ 0 & A_2 & F_2 \\ 0 & 0 & N \end{pmatrix}$. Hence part (3) holds. $\square$

Theorem 3.5. *Let $M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$ be a square real matrix, where A is a square matrix with full term rank, and B has at least one column without zero entries. The M^{d} is signed if and only if A is an* $\mathrm{S}^2\mathrm{NS}$ *matrix, and neither $\mathcal{A}^2$ nor $\mathcal{A}^2\mathcal{B}$ have uncertain entries, where $\mathcal{A} = \operatorname{sgn} A^{-1}$, $\mathcal{B} = \operatorname{sgn} B$.*

P r o o f. If M^{d} is signed, by Theorem 3.3, A is an $\mathrm{S}^2\mathrm{NS}$ matrix, and neither $\mathcal{A}^2$ nor $\mathcal{A}^2\mathcal{B}$ have uncertain entries.

If A is an S^2NS matrix, then by Lemma 2.2, $\begin{pmatrix} \tilde{A} & \tilde{B} \\ 0 & 0 \end{pmatrix}^{\mathrm{d}} = \begin{pmatrix} \tilde{A}^{-1} & \tilde{A}^{-2}\tilde{B} \\ 0 & 0 \end{pmatrix}$ for any $\tilde{A} \in \mathcal{Q}(A)$, $\tilde{B} \in \mathcal{Q}(B)$. If neither $\mathcal{A}^2$ nor $\mathcal{A}^2\mathcal{B}$ have uncertain entries, then M^{d} is signed. □

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