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OSCILLATION OF THE THIRD ORDER EULER DIFFERENTIAL EQUATION WITH DELAY

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Abstract. In the paper we offer criteria for oscillation of the third order Euler differential equation with delay

$$y'''(t) + \frac{k^2}{t^3} y(ct) = 0.$$

We provide detail analysis of the properties of this equation, we fill the gap in the oscillation theory and provide necessary and sufficient conditions for oscillation of equation considered.

Keywords: third-order functional differential equation; Euler equation; oscillation; nonoscillation

MSC 2010: 34K11, 34C10

1. Introduction

The object of this paper is to present sufficient conditions for the oscillation of the third-order functional differential equation

$$(\mathrm{E_D}) \qquad y'''(t) + \frac{k^2}{t^3} y(ct) = 0, \quad t \geqslant t_0 > 0,$$

where $0 < c < 1$ and $k \neq 0$. By a solution of $(\mathrm{E_D})$ we mean a function defined on the initial interval $[ct_0, t_0]$ which satisfies $(\mathrm{E_D})$ for every $t \geqslant t_0$. A solution of $(\mathrm{E_D})$ is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is called nonoscillatory. Equation $(\mathrm{E_D})$ is said to be oscillatory if all its solutions are oscillatory.

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It follows from the familiar lemma of Kiguradze [5], [6], [10] that the set of all nonoscillatory (say, positive) solutions $\mathcal{N}$ of (E_D) has the decomposition

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2,$$

where

$$\begin{aligned} y(t) \in \mathcal{N}_0 \quad &\Longleftrightarrow \quad y(t) > 0,\ y'(t) < 0,\ y''(t) > 0,\ y'''(t) < 0,\\ y(t) \in \mathcal{N}_2 \quad &\Longleftrightarrow \quad y(t) > 0,\ y'(t) > 0,\ y''(t) > 0,\ y'''(t) < 0. \end{aligned}$$

Equation (E_D) is a natural generalization of the well-known Euler differential equation

$$\text{(E)} \qquad y'''(t) + \frac{k^2}{t^3} y(t) = 0$$

and will be called the Euler differential equation with delay. Following [3], [8], [9], we say that equation (E_D) has property (A) if $\mathcal{N} = \mathcal{N}_0$. Property (A) of various third order differential equations has been studied by many authors [5], [6], [8]–[10].

Both the equations (E) and (E_D) play a very important role in the oscillation theory, especially in the comparison theory, where these equations serve as comparative equations from which we deduce properties of more general equations, see e.g. [1]–[10]. Therefore it is desirable to have strong criteria for oscillation and/or property (A) of (E_D). It is well known, see [4] and [6], that

$$\text{equation (E) has property (A)} \iff k > \frac{2}{3\sqrt{3}}.$$

Applying the existing comparison theorems, see [4] and [8], we can extend this result to equation (E_D), as follows:

$$\begin{aligned} k^2 > \frac{2}{c^2 3\sqrt{3}} \quad &\Longrightarrow \quad \text{equation } (E_D) \text{ has property (A)},\\ k^2 \leqslant \frac{2}{3\sqrt{3}} \quad &\Longrightarrow \quad \text{equtaion } (E_D) \text{ has not property (A)}. \end{aligned}$$

However, these results do not apply to the remaining case $2/(3\sqrt{3}) < k^2 \leqslant 2/(c^2 3\sqrt{3})$. On the other hand, these criteria say nothing about oscillation of (E_D). In this paper, we will fill this gap and get an efficient necessary and sufficient condition for oscillation of (E_D), and what is more, we provide also an interesting connection between oscillation of (E_D) and the classes $\mathcal{N}_0$ and $\mathcal{N}_2$.

2. Main results

First we transform equation $(\mathrm{E_D})$ into the form of a delay differential equation with constant coefficients and constant delay. We set

$$t = \mathrm{e}^s, \quad y(t) = x(s), \quad r = -\ln c, \tag{2.1}$$

with $r > 0$. Then as usual

$$t\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}s}, \quad t^2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{\mathrm{d}^2x}{\mathrm{d}s^2} - \frac{\mathrm{d}x}{\mathrm{d}s}, \quad t^3\frac{\mathrm{d}^3y}{\mathrm{d}t^3} = \frac{\mathrm{d}^3x}{\mathrm{d}s^3} - 3\frac{\mathrm{d}^2x}{\mathrm{d}s^2} + 2\frac{\mathrm{d}x}{\mathrm{d}s}$$

and $(\mathrm{E_D})$ becomes

$$\frac{\mathrm{d}^3x}{\mathrm{d}s^3} - 3\frac{\mathrm{d}^2x}{\mathrm{d}s^2} + 2\frac{\mathrm{d}x}{\mathrm{d}s} + k^2x(s-r) = 0. \tag{$\mathrm{E_C}$}$$

Obviously equation $(\mathrm{E_D})$ oscillates if and only if equation $(\mathrm{E_C})$ does. With equation $(\mathrm{E_D})$ we associate the characteristic equation

$$\lambda(\lambda-1)(\lambda-2) + k^2c^\lambda \equiv \lambda(\lambda-1)(\lambda-2) + k^2\mathrm{e}^{-r\lambda} = 0 \tag{CH}$$

which is obtained by assuming the solution of $(\mathrm{E_D})$ of the form $y(t) = t^\lambda$ or looking for the solution of $(\mathrm{E_C})$ of the type $x(s) = \mathrm{e}^{\lambda s}$.

In the next lemma which is due to Arino and Győri [1] we connect the properties of $(\mathrm{E_C})$ and (CH).

Lemma 2.1. *Equation* $(\mathrm{E_C})$ *oscillates if and only if* (CH) *has no real roots.*

Now we will explore the properties of the characteristic equation (CH). Let us denote $f(\lambda) = -\lambda(\lambda-1)(\lambda-2)c^{-\lambda}$, then the characteristic equation can be written in the form

$$f(\lambda) = k^2. \tag{CH}$$

It is easy to verify that for any $c \in (0,1)$ the function $f(\lambda)$ has three zero points and moreover

$$\begin{aligned} f(\lambda) &\to -\infty \quad \text{for } \lambda \to \infty,\\ f(\lambda) &\to 0 \quad \text{for } \lambda \to -\infty. \end{aligned}$$

Consequently, we can conclude that there exists the maximum $f_{\max}$ of $f(\lambda)$, $\lambda \in \mathbb{R}$. The information obtained permits us to sketch the graph of the function $f(\lambda)$ (see Figures 1 and 2).

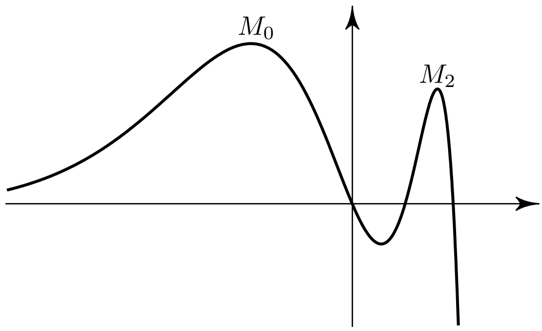

Figure 1. Graph of $f(\lambda)$ with $c = 0.35$.

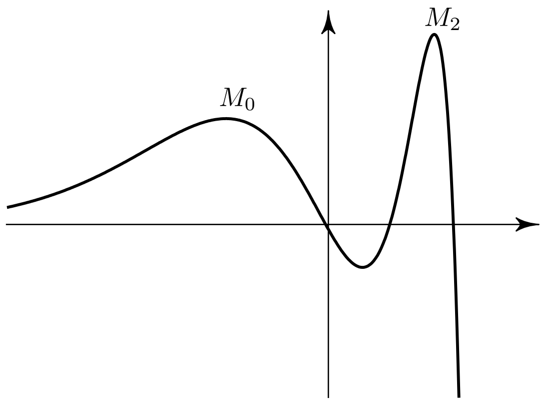

Figure 2. Graph of $f(\lambda)$ with $c = 0.27$.

Obviously, the characteristic equation (CH) has no real root if and only if holds $k^2 > f_{\max}$. Let us denote

$$M_0(c) = \max_{\lambda \in (-\infty, 0)} f(\lambda), \quad M_2(c) = \max_{\lambda \in (1,2)} f(\lambda),$$

then in view of our previous observation the next result is obvious.

Theorem 2.1. *Equation* ($\mathrm{E_D}$) *is oscillatory iff* $k^2 > \max\{M_0(c), M_2(c)\}$.

Remark 2.1. Evidently $M = \max\{M_0(c), M_2(c)\} \in (2/(3\sqrt{3}), 2/(c^2 3\sqrt{3}))$ and due to Theorem 2.1 if $k^2 > M$ equation ($\mathrm{E_D}$) is oscillatory, while for $k^2 < M$ equation ($\mathrm{E_D}$) possesses a nonoscillatory solution. Therefore there is no gap for oscillation of equation ($\mathrm{E_D}$).

Now we explore which of the local maxima $M_0(c)$ and $M_2(c)$ will be dominated and becomes maximum $f_{\max}$ and how this process depends on value of parameter c. We consider $M_0(c)$ and $M_2(c)$ as functions defined on $(0,1)$.

Lemma 2.2. *Function $M_0(c)$ is increasing on $(0,1)$ and moreover,*

$$M_0(0) = \lim_{c\to 0+} M_0(c) = 0, \quad M_0(1) = \lim_{c\to 1-} M_0(c) = \infty.$$

Lemma 2.3. *Function $M_2(c)$ is decreasing on $(0,1)$ and moreover,*

$$M_2(0) = \lim_{c\to 0+} M_2(c) = \infty, \quad M_2(1) = \lim_{c\to 1-} M_2(c) = \frac{2}{3\sqrt{3}}.$$

Proof. The proofs of both lemmas follow immediately from the definition of the functions $M_0(c)$ and $M_2(c)$ and so they can be omitted. □

Combining the last two results, we obtain the following theoretical result dealing with the relationship between the dominance of $M_0(c)$ and $M_2(c)$.

Theorem 2.2. *There exists a unique $c^* \in (0,1)$ such that*

▷ *for $c \in (0, c^*)$, the maximum is $f_{\max} = M_2(c) > M_0(c)$,*
▷ *for $c \in (c^*, 1)$, the maximum is $f_{\max} = M_0(c) > M_2(c)$.*

Now, we reformulate the previous results in terms of oscillation of equation $(\mathrm{E_D})$.

Theorem 2.3. *Let $c^* \in (0,1)$ be such as in Theorem 2.2.*

▷ *If $c \in (0, c^*)$ and $k^2 > M_2(c)$, then equation $(\mathrm{E_D})$ is oscillatory.*
▷ *If $c \in (c^*, 1)$ and $k^2 > M_0(c)$, then equation $(\mathrm{E_D})$ is oscillatory.*
▷ *If $k^2 \leqslant M_2(c)$, then the class $\mathcal{N}_2$ is nonempty for equation $(\mathrm{E_D})$.*
▷ *If $k^2 \leqslant M_0(c)$, then the class $\mathcal{N}_0$ is nonempty for equation $(\mathrm{E_D})$.*

Proof. The first two assertions are obvious. We shall prove the third. If $k^2 \leqslant M_2(c) \leqslant f_{\max}$, then there exists a root λ_* of the characteristic equation such that $\lambda_* \in (1,2)$. Then $y_*(t) = t^{\lambda_*}$ is the corresponding solution of equation $(\mathrm{E_D})$ and it is easy to verify that $y_* \in \mathcal{N}_2$. The last assertion can be verified similarly. □

It is desirable to evaluate the value of c^*. However, actual computation leads to a cubic equation whose roots are formalized as the third root of a complex number. Instead of this, using Matlab, we find out that $c^* = 0.32049$ with the corresponding $f_{\max} = M_0(c^*) = M_2(c^*) = 2.4735$.

We illustrate all our results in the following examples.

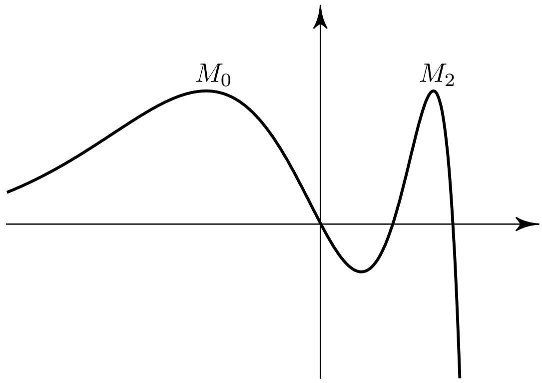

Figure 3. Graph of $f(\lambda)$ with $c^* = 0.32049$.

Example 2.1. Consider the third order Euler delay equation

$$y'''(t) + \frac{k^2}{t^3} y(0.4t) = 0, \quad t \geqslant t_0 > 0. \tag{2.2}$$

Since $0.4 > c^* = 0.32049$, then (using e.g. Matlab) we evaluate the corresponding values $M_0 = 3.9849$ and $M_2 = 1.7056$. Theorem 2.3 implies that

- ▷ equation (2.2) is oscillatory if and only if $k^2 > 3.9849$;
- ▷ if $k^2 \leqslant 3.9849$, then the class $\mathcal{N}_0$ is nonempty for equation (2.2);
- ▷ if $k^2 \leqslant 1.7056$, then the class $\mathcal{N}_2$ is nonempty for equation (2.2).

Note that for $k^2 = 3.84$ the solution $y(t) = t^{-2}$ belongs to the class $\mathcal{N}_0$ of equation (2.2).

Example 2.2. Consider the third order Euler delay equation

$$y'''(t) + \frac{k^2}{t^3} y(0.2t) = 0, \quad t \geqslant t_0 > 0. \tag{2.3}$$

Since $0.2 < c^* = 0.32049$, then we find out the corresponding values $M_2 = 5.5239$ and $M_0 = 1.2246$ and it follows from Theorem 2.3 that

- ▷ equation (2.3) is oscillatory if and only if $k^2 > 5.5239$;
- ▷ if $k^2 \leqslant 5.5239$, then the class $\mathcal{N}_2$ is nonempty for equation (2.3);
- ▷ if $k^2 \leqslant 1.2246$, then the class $\mathcal{N}_0$ is nonempty for equation (2.3).

Note that for $k^2 = 4.1926$ we have the solution $y(t) = t^{1.5}$ which belongs to the class $\mathcal{N}_2$ of equation (2.3), while for $k^2 = 1.2$ we have the solution $y_1(t) = t^{-1}$ which belongs to the class $\mathcal{N}_0$. Moreover, there are additional three nonoscillatory solutions. One solution $y_2(t) \approx t^{-1.4009839}$ from the class $\mathcal{N}_0$ and two solutions $y_3(t) \approx t^{1.1846025}$, $y_4(t) \approx t^{1.973965}$ that belong to the class $\mathcal{N}_2$.

3. Summary

In this paper we have presented a new necessary and sufficient condition for oscillation of the third order Euler differential equation with delay. Since the Euler differential equation is often used in comparison results as a reference equation, the results obtained in this paper are useful and important for future investigation of asymptotic properties of differential equations.

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