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POSITIVITY OF GREEN'S MATRIX OF NONLOCAL BOUNDARY VALUE PROBLEMS

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Abstract. We propose an approach for studying positivity of Green's operators of a nonlocal boundary value problem for the system of n linear functional differential equations with the boundary conditions $n_i x_i - \sum_{j=1}^{n} m_{ij} x_j = \beta_i$, $i = 1, \ldots, n$, where n_i and m_{ij} are linear bounded "local" and "nonlocal" functionals, respectively, from the space of absolutely continuous functions. For instance, $n_i x_i = x_i(\omega)$ or $n_i x_i = x_i(0) - x_i(\omega)$ and $m_{ij} x_j = \int_0^{\omega} k(s) x_j(s)\,\mathrm{d}s + \sum_{r=1}^{n_{ij}} c_{ijr} x_j(t_{ijr})$ can be considered. It is demonstrated that the positivity of Green's operator of nonlocal problem follows from the positivity of Green's operator for auxiliary "local" problem which consists of a "close" equation and the local conditions $n_i x_i = \alpha_i$, $i = 1, \ldots, n$.

Keywords: functional differential equation; nonlocal boundary value problem; positivity of Green's operator; fundamental matrix; differential inequalities

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1. Introduction

Ordinary differential equations with integral boundary conditions arise in the theory of turbulence [22], in the theory of Markov processes [9], in heat flow problems [12], [16], [14], [24], [25], in the study of the response of a spherical cap [3], [5], [20]. In the references in [6], one can find references to works on applications of nonlocal problems in the modeling of thermostats, beams and suspension bridges.

Questions of representation of solutions and solvability of nonlocal problems for functional differential equations were considered in [4], [17], [18], [13]. Positivity of solutions for nonlocal boundary value problems for ordinary differential equations was studied in [11], [10], [27], [26], [28]. The method is to reduce nonlocal boundary value problems to the Hammerstein integral equation and then scrupulous analysis of Green's functions leads researchers to estimates (of the norm or spectral radii in linear case and the fact of a contraction in nonlinear one) of integral operators and conclusions about positivity of solutions. It looks that some of these results can be generalized also to particular cases of delay or functional differential differential equations, where Green's functions of ordinary differential equations could be used. For functional differential equations, forms of Green's functions are essentially more complicated. That is why quite a different approach was proposed for nonlocal problems with functional differential equations [1], [8], where various results on positivity/negativity of Green's functions were obtained. One of the main ideas is to obtain a connection between sign-constancy of Green's functions for different problems with functional differential equations. This approach presents a basic method for the analysis of solution's positivity (see, for example, Theorem 15.3 in [1]). The main results are obtained in the form of theorems about differential inequalities. Choosing the test functions, researchers can get coefficient tests for positivity of Green's functions. Note that all these works concern positivity of solutions to nonlocal problems only for scalar differential equations and not for systems. There are almost no results on positivity of solutions of nonlocal problems in the case of systems. Among the results we can note results on existence [17], [18], [6], [15], [21] and on positivity of solution-vectors in [15]. In this paper we try to present an approach to the study of positivity of components of solution-vectors for systems of functional differential equations.

Consider the system

$$(Mx)(t) \equiv x'(t) + (Bx)(t) = f(t), \quad t \in [0, \omega], \tag{1.1}$$

where $x = \operatorname{col}(x_1, \ldots, x_n)$, $B\colon C^n_{[0,\omega]} \to L^n_{\infty[0,\omega]}$ is a linear continuous operator, $C^n_{[0,\omega]}$ and $L^n_{\infty[0,\omega]}$ are the spaces of continuous and essentially bounded functions $y\colon [0,\omega] \to \mathbb{R}^n$, respectively, $f \in L^n_{\infty[0,\omega]}$. Let $l\colon D^n_{[0,\omega]} \to \mathbb{R}^n$ be a linear bounded functional from the space of absolutely continuous vector functions. The general representation of the functional l is

$$lx = \Psi x(0) + \int_0^\omega \Phi(s) x'(s) \,\mathrm{d}s, \tag{1.2}$$

where Ψ is an $n \times n$ constant matrix and $\Phi(s)$ is an $n \times n$ matrix with elements $\phi_{ij} \in L^n_{\infty[0,\omega]}$. Note that functionals of the forms $lx = \operatorname{col}\Big\{ \sum\limits_{j=1}^{n_i} r_{ij} x_j(t_{ij}) \Big\}_{i,j=1}^{n}$, $lx = \operatorname{col}\Big\{ \sum\limits_{j=1}^{n} \int_0^\omega k_{ij}(s) x_j(s) \,\mathrm{d}s \Big\}_{i,j=1}^{n}$, and all their linear combinations and superpositions can be considered.

If the homogeneous boundary value problem $(Mx)(t) = 0$, $t \in [0,\omega]$, $i = 1,\dots,n$, $lx = 0$, has only the trivial solution, then the boundary value problem

$$(Mx)(t) = f(t), \quad t \in [0,\omega], \; lx = \beta, \tag{1.3}$$

has for each $f \in L^n_{\infty[0,\omega]}$ and $\beta \in \mathbb{R}^n$ a unique solution, which has the representation [4]

$$x(t) = \int_0^\omega G(t,s)f(s)\,\mathrm{d}s + X(t)\beta, \quad t \in [0,\omega], \tag{1.4}$$

where $X(t)$ is the fundamental matrix of the system $(Mx)(t) = 0$, $t \in [0,\omega]$, such that $lX = I$ (I is a unit matrix), $G(t,s)$ is called Green's matrix of problem (1.3). The operator $G\colon L^n_{\infty[0,\omega]} \to C^n_{[0,\omega]}$ defined by the equality

$$(Gf)(t) = \int_0^\omega G(t,s)f(s)\,\mathrm{d}s, \quad t \in [0,\omega], \tag{1.5}$$

is called Green's operator of problem (1.3). The main purpose of this paper is to study the following property formulated first by Tchaplygin [23]: when does it follow from the conditions

$$(Mx)(t) \geqslant (My)(t), \quad t \in [0,\omega], \; lx \geqslant ly, \tag{1.6}$$

that

$$x(t) \geqslant y(t), \quad t \in [0,\omega]? \tag{1.7}$$

We understand these inequalities as inequalities for the corresponding components. The property (1.6) $\Longrightarrow$ (1.7) is the basis of the approximation method [23] known then as a monotone technique.

System (1.1) can be written also in the form

$$(M_i x)(t) \equiv x_i'(t) + \sum_{j=1}^{n} (B_{ij}x_j)(t) = f_i(t), \quad t \in [0,\omega], \; i = 1,\dots,n, \tag{1.8}$$

where $x = \mathrm{col}(x_1,\dots,x_n)$, $B_{ij}\colon C^1_{[0,\omega]} \to L^1_{\infty[0,\omega]}$ are linear continuous operators, $C^1_{[0,\omega]}$ and $L^1_{\infty[0,\omega]}$ are the spaces of continuous and essentially bounded functions $y\colon [0,\omega] \to \mathbb{R}^1$, respectively, $f_i \in L^1_{\infty[0,\omega]}$. The operators B_{ij} can be, for example, of the forms $(B_{ij}x)(t) = \sum\limits_{k=1}^{m} p_{ijk}(t)x(h_k(t))$, $t \in [0,\omega]$, $x(\xi) = 0$ for $\xi \notin [0,\omega]$,

$(B_{ij}x)(t) = \int_0^\omega K_{ij}(t,s)x(s)\,\mathrm{d}s,\ t \in [0,\omega]$, and also their linear combinations or superpositions.

The paper is built as follows. In Introduction we describe previous results on nonlocal problems and formulate the purpose of the paper. In Section 2 we propose our method for studying nonlocal problems. This method presents a sort of a right regularization developing the idea of Azbelev's W-transform. In Section 3 we study nonlocal problem on the semiaxis basing our approach on the Cauchy problem for systems of delay differential equations. In Section 4 auxiliary results for scalar functional differential equations are formulated. The main results on positivity of Green's operator are obtained in Section 5. Open problems and ways for possible development in studying nonlocal problems are formulated in Section 6.

2. Desciprtion of right regularization scheme for studying positivity of Green's operators

Consider the auxiliary boundary value problem

$$(M_0x)(t) \equiv x'(t) + (B_0x)(t) = z(t),\ l_0x = \alpha,\ t \in [0,\omega], \tag{2.1}$$

where $x = \mathrm{col}(x_1,\dots,x_n)$, $z \in L^n_{\infty[0,\omega]}$, $B_0\colon C^n_{[0,\omega]} \to L^n_{\infty[0,\omega]}$ is a linear continuous operator and $l_0\colon D^n_{[0,\omega]} \to \mathbb{R}^n$ is a linear bounded functional. When does positivity of Green's operator of problem (2.1) imply positivity of Green's operator of the given problem (1.3)? Our results claim that in the case of smallness of $\|B - B_0\|$, $\|l - l_0\|$ and under additional simple conditions that the operator $B - B_0$ and the functional $l - l_0$ are, for example, negative, this conclusion is true.

It should be stressed that we can choose the auxiliary problem (2.1) such that the homogeneous boundary value problem

$$(M_0x)(t) = 0, \quad t \in [0,\omega],\ l_0x = 0, \tag{2.2}$$

has only the trivial solution. In this case the boundary value problem

$$(M_0x)(t) = z(t), \quad t \in [0,\omega],\ l_0x = \alpha, \tag{2.3}$$

has for each $z \in L^n_{\infty[0,\omega]}$ and $\alpha \in \mathbb{R}^n$ a unique solution, which has the representation [4]

$$x(t) = \int_0^\omega G_0(t,s)z(s)\,\mathrm{d}s + X_0(t)\alpha, \quad t \in [0,\omega], \tag{2.4}$$

where $X_0(t)$ is the fundamental matrix of the equation

$$(M_0x)(t) = 0, \quad t \in [0,\omega], \tag{2.5}$$

such that $l_0X_0 = I$.

Introduce the operator $\Delta B = B - B_0$ and functional $\Delta l = l - l_0$. After substituting (2.4) into (1.3) we get the system

$$z(t) + \{\Delta B[G_0z + X_0\alpha]\}(t) = f(t), \quad t \in [0,\omega], \tag{2.6}$$

$$\alpha + \Delta l[G_0z + X_0\alpha] = \beta, \tag{2.7}$$

for finding $z(t) \in L^n_{\infty[0,\omega]}$ and $\alpha \in \mathbb{R}^n$. This system can be considered as a system

$$y(t) = (Ay)(t) + g(t), \tag{2.8}$$

where $y = \operatorname{col}\{x,\alpha\}$ with the corresponding operator $A\colon L^{2n}_{\infty[0,\omega]} \to L^{2n}_{\infty[0,\omega]}$ It is clear from the form of equation (2.7) that for every $g = \operatorname{col}\{f,\beta\}$ we get a constant n-vector α as a part of the solution-vector $y = \operatorname{col}\{x,\alpha\}$.

Let us introduce the n-vector $E = \operatorname{col}\{1,\ldots,1\}$, and denote by $\|X_0\|$—the norm of the fundamental matrix of equation (2.5), $\|\Delta B\|$—the norm of the operator ΔB: $C^n_{[0,\omega]} \to L^n_{\infty[0,\omega]}$ and $\|\Delta l\|$—the norm of the functional $\Delta l\colon C^n_{[0,\omega]} \to \mathbb{R}^n$,

$$\|G_0\| = \max_{t\in[0,\omega]} \int_0^\omega |G_0(t,s)|E(s)\,\mathrm{d}s, \quad t \in [0,\omega]. \tag{2.9}$$

Noted above for system (2.8) allows us to obtain the following assertion for unique solvability of the system (2.6), (2.7).

Theorem 2.1. *Let the boundary value problem* (2.1) *have only the trivial solution and let the following inequalities be fulfilled:*

$$\|\Delta B\|\{\|G_0\| + \|X_0\|\} < 1, \tag{2.10}$$

$$\|\Delta l\|\{\|G_0\| + \|X_0\|\} < 1. \tag{2.11}$$

Then the boundary value problem (1.3) *has for each $f \in L^n_{\infty[0,\omega]}$ and $\alpha \in \mathbb{R}^n$ a unique solution.*

Let us use positivity/negativity of Green's operator G_0 and the fundamental matrix X_0 of auxiliary problem (2.1) together with positivity/negativity of the operator ΔB and functional Δl in order to obtain positivity of Green's operator G and the fundamental matrix X of the given boundary value problem (1.3).

Theorem 2.2. *Let the auxiliary boundary value problem* (2.1) *have only the trivial solution, and let its Green's operator G_0 be positive and the fundamental matrix X_0, satisfying the condition $l_0X_0 = I$, be nonnegative. Assume also that*

$$(-\Delta B) \text{ is a positive operator and } (-\Delta l) \text{ is a positive functional,} \tag{2.12}$$

and there exist n-vectors $u \in L^n_{\infty[0,\omega]}$ and $\gamma \in \mathbb{R}^n$ with positive components and a positive ε such that

$$u(t) + (\Delta B G_0 u)(t) + (\Delta B)X_0(t)\gamma \geqslant \varepsilon, \quad t \in [0,\omega], \tag{2.13}$$

$$\gamma + \Delta l G_0 u + \Delta l X_0 \gamma > 0. \tag{2.14}$$

Then the boundary value problem (1.3) *is uniquely solvable, its Green's operator G is positive and for nonnegative f and β the solution of* (1.3) *is nonnegative.*

To prove Theorem 2.2 we use the results of paragraphs 5.6 and 5.7 of the book by Krasnosel'skii and his coauthors [19] about estimates of the spectral radii of positive operators for the operator $A\colon L^{2n}_{\infty[0,\omega]} \to L^{2n}_{\infty[0,\omega]}$. From inequalities (2.13), (2.14) we get the estimate $\varrho(A) < 1$ for its spectral radius $\varrho(A)$. Then we have $y = (I-A)^{-1}g = g + Ag + A^2g + \ldots$ and for every nonnegative f and β we get a nonnegative y.

R e m a r k 2.1. Theorem 2.2 can be interpreted as a theorem about integral inequality. In the frame of the traditional approach, called also the perturbation scheme, we add the integral inequality also for functionals defining the boundary conditions. This allows us to study not only problems with "close" to each other operators B and B_0, but also ones with "close" to each other boundary conditions defined by the functionals l and l_0. The smalness of ΔB and Δl is defined, for example, by (2.13) and (2.14).

R e m a r k 2.2. Difficulties in the study of many boundary value problems are also connected with their non-separated boundary conditions. It is a complicated problem to construct Green's operators of the "model" problem (2.1) even for auxiliary equations in the case of non-separated boundary conditions. Our idea is to choose the "model" problem with separated conditions, i.e., the matrices Ψ and Φ (see the description of functional (1.2)) for l_0 are diagonal ones or in other words $l_0x = \operatorname{col}\{l_{01}x_1, l_{02}x_2, \ldots, l_{0n}x_n\}$. The functional l can also include non-diagonal elements, but they should be "small enough".

R e m a r k 2.3. The conditions of Theorem 2.2 imply the property (1.6) $\Longrightarrow$ (1.7). This allows us to estimate the solution-vector $x(t)$ of boundary value problem (1.3).

R e m a r k 2.4. It is clear that in the case of negative Green's operators G_0 and nonpositive fundamental matrices X_0 we have to require that ΔB and Δl be positive operators and functionals and to assume that $u(t)+(\Delta B G_0 u)(t)+(\Delta B)X_0(t)\gamma \leqslant -\varepsilon$, $t \in [0,\omega]$, $\gamma + \Delta l G_0 u + \Delta l X_0 \gamma < 0$.

In the previous results we need estimates of Green's matrix $G_0(t,s)$ and the fundamental matrix $X_0(t)$. This fact leads to the corresponding difficulties in the study of positivity of Green's matrices. In the following assertion we come up with the idea to use differential inequalities. The assertion is an analog of theorems about differential inequalities. We have to know only that G_0 is positive and the fundamental matrix X_0 satisfying the condition $l_0 X_0 = I$ is nonnegative, and do not assume their estimates.

Theorem 2.3. *Let the auxiliary boundary value problem* (2.1) *have only the trivial solution, let its Green's operator* G_0 *be positive and the fundamental matrix* X_0, *satisfying the condition* $l_0 X_0 = I$, *be nonnegative, condition* (2.12) *be fulfilled and let there exist* n*-vector* $v \in D^n_{\infty[0,\omega]}$ *with positive components and positive* ε *such that*

$$u(t) \equiv v'(t) + (Bv)(t) \geqslant \varepsilon, \quad t \in [0,\omega], \tag{2.15}$$

$$\gamma \equiv lv > 0. \tag{2.16}$$

Then the boundary value problem (1.3) *is uniquely solvable, its Green's operator* G *is positive and for nonnegative* f *and* β *the solution of* (1.3) *is nonnegative.*

P r o o f. Proof follows from the fact that the function $u(t)$ defined by (2.15) and the constant γ defined by (2.16) satisfy inequalities (2.13), (2.14). Reference to Theorem 2.2 completes the proof. $\square$

Analogously we can obtain the following assertion.

Theorem 2.4. *Let the auxiliary boundary value problem* (2.1) *have only the trivial solution, let its Green's operator* $(-G_0)$ *be positive and the fundamental matrix* X_0, *satisfying the condition* $l_0 X_0 = I$, *be nonpositive, the condition*

(2.17) ΔB *is a positive operator and* Δl *is a positive functional*

be fulfilled and let there exist an n*-vector* $v \in D^n_{\infty[0,\omega]}$ *with positive components and a positive* ε *such that*

$$u(t) \equiv v'(t) + (Bv)(t) \leqslant -\varepsilon, \quad t \in [0,\omega], \tag{2.18}$$

$$\gamma \equiv lv < 0. \tag{2.19}$$

Then the boundary value problem (1.3) *is uniquely solvable, its Green's operator G is negative and for nonnegative f and β the solution of* (1.3) *is nonpositive.*

3. Positivity of Green's operator in the case of nonlocal Cauchy problem with non-separated boundary conditions

Consider the delay system

$$(M_i x)(t) \equiv x_i'(t) + \sum_{j=1}^{n} p_{ij}(t) x_j(t - \tau_{ij}(t)) = f_i(t),\ t \in [0,\infty), \quad i = 1,\dots,n, \tag{3.1}$$

where

$$x_i(\xi) = 0,\ \xi < 0, \quad i = 1,\dots,n, \tag{3.2}$$

with the boundary conditions

$$x_i(0) - \sum_{j=1}^{n} m_{ij} x_j = \beta_i, \quad i = 1,\dots,n \tag{3.3}$$

where $m_{ij}\colon C^1_{[0,\infty)} \to \mathbb{R}^1$ $(i,j = 1,\dots,n)$ are linear bounded functionals. Consider the auxiliary problem (3.1), (3.2), (3.4), where

$$x_i(0) = \alpha_i, \quad i = 1,\dots,n. \tag{3.4}$$

The general solution of problem (3.1), (3.2), (3.4) can be represented in the form

$$x(t) = \int_0^t C(t,s) f(s)\,\mathrm{d}s + X(t)\alpha, \tag{3.5}$$

where $x = \operatorname{col}\{x_1,\dots,x_n\}$, $\alpha = \operatorname{col}\{\alpha_1,\dots,\alpha_n\}$, $C(t,s)$ is the $n \times n$ Cauchy matrix, $X(t)$ is an $n \times n$ fundamental matrix such that $X(0) = I$.

Theorem 3.1. *Assume that $p_{ij} \leqslant 0$ for $i \neq j$, $\tau_{ij} \geqslant 0$, $i,j = 1,\dots,n$, functionals $m_{ij}\colon C^1_{[0,\infty)} \to \mathbb{R}^1$ are positive ones,*

$$\int_{t-\tau_{ii}(t)}^{t} p_{ii}(s)\,\mathrm{d}s \leqslant \frac{1}{\mathrm{e}}, \quad t \geqslant 0,\ i = 1,\dots,n, \tag{3.6}$$

there exists a constant vector $v = \operatorname{col}\{v_1, \dots, v_n\}$ *with all positive components and a positive* ε *such that*

$$\sum_{j=1}^{n} p_{ij}(t)v_j \geqslant \varepsilon, \quad t \geqslant 0,\ i = 1, \dots, n, \tag{3.7}$$

and

$$v_i - \sum_{j=1}^{n} m_{ij}v_j > 0, \quad i = 1, \dots, n. \tag{3.8}$$

Then the boundary value problem (3.1), (3.2) *and* (3.3) *is uniquely solvable for every* $f \in L^n_{[0,\infty)}$, $\alpha \in \mathbb{R}^n$, *its Green's operator* G *is positive, and for every nonnegative* f *and* β, *the solution is nonnegative.*

Lemma 3.1 ([1], [2]). *Assume that* $p_{ij} \leqslant 0$ *for* $i \neq j$, $\tau_{ij} \geqslant 0$, $i, j = 1, \dots, n$, *and inequality* (3.6) *is fulfilled. Then the Cauchy matrix satisfies the inequality* $C(t,s) \geqslant 0$ *for* $t \geqslant s \geqslant 0$, *the fundamental matrix such that* $X(0) = I$ *satisfies the inequality* $X(t) \geqslant 0$ *for* $t \geqslant 0$. *If, in addition, condition* (3.7) *is fulfilled, then the matrices* $C(t,s)$ *and* $X(t)$ *satisfy exponential estimates, i.e., there exist positive numbers* N *and* a *such that*

$$C_{i,j}(t,s) \leqslant N\mathrm{e}^{-a(t-s)}, \quad X_{ij}(t) \leqslant N\mathrm{e}^{-at}, \quad i,j = 1, \dots, n,\ 0 \leqslant s \leqslant t < \infty. \tag{3.9}$$

Proof of Theorem 3.1 follows from Lemma 3.1 and Theorem 2.3.

Remark 3.1. Let us describe the types of functionals which can be studied by the method proposed above. The functionals

$$m_{ij}x = \sum_{k=1}^{n_{ij}} r_{ijk}x_k(t_{ijk}), \quad m_{ij}x = \sum_{k=1}^{n_{ij}} \int_0^\infty R_{ijk}(s)x_j(h_{ijk}(s))\,\mathrm{d}s,$$

where r_{ijk} and n_{ij} are real numbers and $R_{ijk}(t)$ functions are summable on the semiaxis, and all their linear combinations and superpositions are allowed. It is clear that for sufficiently small $|r_{ijk}|$ and $\int_0^\infty |R_{ijk}(s)|\,\mathrm{d}s$ the inequalities (3.8) will be fulfilled.

Remark 3.2. Let us explain how the vector v in Theorem 3.1 can be found. Define the matrix $Q = \{q_{ij}\}$, where $q_{ij} = -\operatorname*{ess\,sup}_{t\geqslant 0} |p_{ij}(t)|$, if $i \neq j$, and $q_{ii} = \operatorname*{ess\,inf}_{t\geqslant 0} p_{ii}(t)$, $i, j = 1, \dots, n$. The vector v can be found as $v = Q^{-1}E$.

4. Positivity of Green's operators in the scalar case

Let us consider the scalar equation

$$(Mx)(t) \equiv x'(t) + (Bx)(t) = f(t), \quad t \in [0,\omega], \tag{4.1}$$

coupled with the boundary condition

$$lx = \alpha, \tag{4.2}$$

where $B\colon C^1_{[0,\omega]} \to L^1_{\infty[0,\omega]}$ is a linear continuous operator, $l\colon D^1_{[0,\omega]} \to \mathbb{R}^1$ is a linear bounded functional defined on the space of scalar absolutely continuous functions. Using the general form of the functional $l\colon D^1_{[0,\omega]} \to \mathbb{R}^1$ we can write (4.2) in the form

$$lx \equiv \psi x(0) + \int_0^\omega \varphi(s)x'(s)\,\mathrm{d}s = \alpha. \tag{4.3}$$

We consider also the one-point problem (4.1), (4.4), where

$$x(\omega) = \alpha, \tag{4.4}$$

which is an important particular case of boundary condition (4.3) ($\psi = 1$, $\varphi(t) \equiv 1$), and the periodic problem (4.1), (4.5), where

$$x(0) - x(\omega) = \alpha. \tag{4.5}$$

Define the operator $N\colon C_{[0,\omega]} \to C_{[0,\omega]}$ by the formula

$$(Nx)(t) = \int_t^\omega (Bx)(s)\,\mathrm{d}s. \tag{4.6}$$

Definition 4.1. Let us say that the problem (4.1), (4.3) satisfies the condition Θ if

$$\frac{\int_s^\omega \phi(\xi)C'_\xi(\xi,s)\,\mathrm{d}\xi + \phi(s)}{\theta + \int_0^\omega \phi(s)C'_s(s,0)\,\mathrm{d}s} < 0. \tag{4.7}$$

For equation (4.1), we propose the following assertion about nine equivalences.

Theorem 4.1. *Let $B\colon C_{[0,\omega]} \to L_{[0,\omega]}$ be a positive Volterra nonzero operator. Then the following assertions are equivalent:*

1) *there exists a positive absolutely continuous function v such that*

$$Mv(t) \leqslant 0, \quad v(\omega) - \int_t^\omega Mv(s)\,\mathrm{d}s > 0 \quad \text{for } t \in [0,\omega), \tag{4.8}$$

2) *the spectral radius of the operator $N\colon C_{[0,\omega]} \to C_{[0,\omega]}$ is less than one,*
3) *the problem (4.1), (4.4) is uniquely solvable, and its Green's function $G(t,s)$ is negative for $0 \leqslant t < s \leqslant \omega$ and nonpositive for $0 \leqslant s \leqslant t \leqslant \omega$,*
4) *a nontrivial solution of the homogeneous equation $(Mx)(t) = 0$, $t \in [0,\omega]$ has no zeros on $[0,\omega]$,*
5) *the Cauchy function $C(t,s)$ of equation (4.1) is positive for $0 \leqslant s \leqslant t \leqslant \omega$,*
6) *there exists a positive continuous function v such that $v(t) > (Nv)(t)$, $t \in [0,\omega)$,*
7) *the periodic problem (4.1), (4.5) is uniquely solvable, and its Green's function $P(t,s)$ is positive for $0 \leqslant s \leqslant t \leqslant \omega$,*
8) *there exists a positive essentially bounded function u such that*

$$B\mathrm{e}^{\int_s^t u(\xi)\,\mathrm{d}\xi}(t) \leqslant u(t), \quad t \in [0,\omega]. \tag{4.9}$$

If in addition the condition Θ is fulfilled, then the following assertion is included in the list of the equivalences:

9) *the problem (4.1), (4.3) is uniquely solvable and its Green's function $P(t,s)$ is positive for $t,s \in [0,\omega]$.*

Theorem 4.1 was proved in [1], [8].

R e m a r k 4.1. For a wide class of boundary value problems, for example for many generalized periodic problems, the condition Θ is fulfilled. Here let us discuss only problems with the general form of boundary condition. Let us assume that $\theta > 0$ and $\phi(s) < -\varepsilon < 0$, then it follows from Theorem 4.1 that on the nonoscillation interval $C(t,s) > 0$ and consequently in the case of a positive operator B the derivative satisfies the inequality $C'_t(t,s) \leqslant 0$ for $0 \leqslant s \leqslant t \leqslant \omega$. It is obvious that the denominator in (4.7) is positive. The numerator will be negative if the interval $[0,\omega]$ is small enough.

5. Positivity of Green's operators in the case of nonlocal problem with non-separated boundary conditions

In Theorem 3.1, the Cauchy problem was used as a model problem (2.3). The results of this section are based on positivity/negativity of Green's matrices of several other problems for system (1.8). Consider the diagonal equations

$$(m_i x_i)(t) \equiv x_i'(t) + (B_{ii}x_i)(t) = f_i(t), \ t \in [0,\omega], \tag{5.1}$$

coupled with the boundary conditions

$$l_i x_i = 0, \tag{5.2}$$

for $i = 1,\dots,n$, where $B_{ii}\colon C^1_{[0,\omega]} \to L^1_{\infty[0,\omega]}$ is a linear continuous operator, $l\colon D^1_{[0,\omega]} \to \mathbb{R}^1$ is a linear bounded functional defined on the space of scalar absolutely continuous functions.

Denote by $g_i(t,s)$ Green's function of problem (5.1), (5.2). Define the operator $K\colon C^n_{[0,\omega]} \to C^n_{[0,\omega]}$ by the formula

$$(Kx)(t) = \operatorname{col}\Bigl\{-\int_0^\omega g_i(t,s) \sum_{j=1, j\neq i}^{n} (B_{ij}x_j)(s)\,\mathrm{d}s\Bigr\}_{i=1}^{n}, \quad t \in [0,\omega]. \tag{5.3}$$

Let us start with the following assertion proved in [1].

Theorem 5.1. *Let the following conditions be fulfilled:*

1) *n scalar boundary value problems* (5.1), (5.2) *are uniquely solvable, their Green's functions $g_i(t,s)$, $i = 1,\dots,n$ preserve their signs;*
2) *the nondiagonal operators B_{ij}, $i,j = 1,\dots,n$, $j \neq i$, are positive or negative such that the operator $K\colon C^n_{[0,\omega]} \to C^n_{[0,\omega]}$ determined by the formula* (5.3) *is positive.*

Then the assertions a), b) *are equivalent and each of them implies* c).

a) *There exists a vector function $v \in C^n_{[0,\omega]}$ with positive absolutely continuous components $v_i\colon [0,\omega] \to [0,\infty)$ such that the solution w of the problem*

$$(m_i w_i)(t) \equiv w_i'(t) + (B_{ii}w_i)(t) = (M_i v)(t), \quad t \in [0,\omega], \ l_i w_i = l_i v_i, \tag{5.4}$$

is positive for $t \in [0,\omega]$, $i = 1,\dots,n$.

b) *The spectral radius of the operator $K\colon C^n_{[0,\omega]} \to C^n_{[0,\omega]}$ is less than one.*

c) *The boundary value problem* (1.8), (5.2) *is uniquely solvable for every right hand side* $f = \operatorname{col}(f_1, \dots, f_n)$ *such that* $f_i \in L^n_{\infty[0,\omega]}$, $i = 1, \dots, n$, *and elements of its Green's matrix preserve sign and satisfy the inequalities*

$$g_i(t,s)G_{ij}(t,s) \geqslant 0, \quad t,s \in [0,\omega], \tag{5.5}$$

while

$$|G_{ii}(t,s)| \geqslant |g_i(t,s)|, \quad t,s \in [0,\omega], \tag{5.6}$$

for $i, j = 1, \dots, n$.

Consider the equation

$$(M_i x)(t) \equiv x_i'(t) + \sum_{j=1}^{n} p_{ij}(t) x_j(t - \tau_{ij}(t)) = f_i(t), \quad t \in [0,\omega],\ i = 1, \dots, n, \tag{5.7}$$

where

$$x_i(\xi) = 0, \quad \xi \notin [0,\omega], \tag{5.8}$$

with boundary conditions

$$-x_i(\omega) + \sum_{j=1}^{n} m_{ij} x_j = \beta_i, \quad i = 1, \dots, n. \tag{5.9}$$

As a model problem we can take the one-point problem (5.7), (5.8), (5.10), where

$$-x_i(\omega) = \alpha_i, \quad i = 1, \dots, n. \tag{5.10}$$

Denote $P(t) = \varepsilon + \max\limits_{1\leqslant i\leqslant n} \sum\limits_{j=1}^{n} p_{ij}(t)$, where ε is a positive number, $\tau(t) = \max\limits_{1\leqslant i,j\leqslant n} \tau_{ij}(t)$.

Theorem 5.2. *Assume that* $p_{ij} \geqslant 0$, $\tau_{ij} \geqslant 0$ *for* $i, j = 1, \dots, n$, *the functionals* $m_{ij}\colon C^1_{[0,\infty)} \to \mathbb{R}^1$ *are positive ones,*

$$\int_{t-\tau(t)}^{t} P(s)\,\mathrm{d}s \leqslant \frac{1}{\mathrm{e}}, \quad t \geqslant 0,\ i = 1, \dots, n, \tag{5.11}$$

and

$$\exp\Big\{-\mathrm{e}\int_0^{\omega} P(s)\,\mathrm{d}s\Big\} > \sum_{j=1}^{n} m_{ij}\Big\{-\mathrm{e}\int_0^{t} P(s)\,\mathrm{d}s\Big\}, \quad i = 1, \dots, n. \tag{5.12}$$

Then the boundary value problem (5.7), (5.8) *and* (5.9) *is uniquely solvable for every* $f \in L^n_{[0,\infty)}$, $\alpha \in \mathbb{R}^n$, *its Green's operator* G *is negative, and for every nonpositive* f *and* β, *the solution is nonnegative.*

Proof. Let us consider the diagonal equation

$$(m_i x)(t) \equiv x_i'(t) + p_{ii}(t)x_i(t - \tau_{ii}(t)) = f_i(t), \quad t \in [0, \omega], \tag{5.13}$$

coupled with condition (5.10). The vector-function

$$v_i(t) = \exp\left\{-\mathrm{e}\int_0^t P(s)\,\mathrm{d}s\right\}, \quad i = 1, \ldots, n, \tag{5.14}$$

satisfies the assertion 1) of Theorem 4.1. This, according to Theorem 4.1, implies that Green's functions $g_i(t,s)$ of all diagonal problems (5.13), (5.10) are nonpositive for $(t,s) \in (0,\omega) \times (0,\omega)$, and $g_i(t,s) < 0$ for $0 < t < s < \omega$, $i = 1, \ldots, n$. Vector-function (5.14) satisfies also the assertion a) of Theorem 5.1. According to Theorem 5.1, Green's operator G_0 of problem (5.7), (5.8), (5.10) is negative. It is clear that X_0 is nonpositive. Inequality (5.12) implies condition (2.19). Now, Theorem 2.4 completes the proof of Theorem 5.2. □

Remark 5.1. It is clear that the inequality

$$\exp\left\{-\mathrm{e}\int_0^\omega P(s)\,\mathrm{d}s\right\} > \sum_{j=1}^n m_{ij} 1, \quad i = 1, \ldots, n, \tag{5.15}$$

can be set instead of (5.12) in Theorem 5.2.

Consider now the nonlocal problem (5.7), (5.8), (5.16), where

$$x_i(0) - x_i(\omega) - \sum_{j=1}^n m_{ij} x_j = \beta_i, \; i = 1, \ldots, n. \tag{5.16}$$

As a model problem we can take the periodic problem (5.7), (5.8), (5.17), where

$$x_i(0) - x_i(\omega) = \alpha_i, \; i = 1, \ldots, n. \tag{5.17}$$

Denote

$$\chi(t - \tau_{ij}(t)) = \begin{cases} 1, & t - \tau_{ij}(t) \in [0, \omega], \\ 0, & t - \tau_{ij}(t) \notin [0, \omega]. \end{cases} \tag{5.18}$$

Theorem 5.3. *Assume that* $p_{ij} \leqslant 0$ *for* $i \neq j$, $p_{ii} \geqslant 0$, $i, j = 1, \dots, n$, *functionals* $m_{ij}\colon C^1_{[0,\infty)} \to \mathbb{R}^1$ *are positive, at least one of the two conditions either*

$$\int_{t-\tau_{ii}(t)}^{t} p_{ii}(s)\,\mathrm{d}s \leqslant \frac{1}{\mathrm{e}}, \quad \tau_{ii}(t) \geqslant 0,\ t \geqslant 0,\ i = 1, \dots, n, \tag{5.19}$$

or

$$\int_0^{\omega} p_{ii}(s)\,\mathrm{d}s < 1, \quad i = 1, \dots, n, \tag{5.20}$$

is fulfilled, there exist positive ε *and continuous* $0 < P_*(t) < \min\limits_{1 \leqslant i \leqslant n} \sum\limits_{j=1}^{n} p_{ij}(t)$ *such that the inequalities*

$$\sum_{j=1}^{n} p_{ij}(t)\chi(t - \tau_{ij}(t)) \geqslant \varepsilon > 0, \quad t \in [0,\omega],\ i = 1, \dots, n, \tag{5.21}$$

$$\sum_{j=1}^{n} p_{ij}(t)\chi(t - \tau_{ij}(t)) \exp\left\{ \int_{t-\tau_{ij}(t)}^{t} P_*(\xi)\,\mathrm{d}\xi \right\} - P_*(t) \geqslant \varepsilon > 0, \quad t \in [0,\omega],\ i = 1, \dots, n, \tag{5.22}$$

and

$$1 - \exp\left\{ -\int_0^{\omega} P_*(\xi)\,\mathrm{d}\xi \right\} - \sum_{j=1}^{n} m_{ij} \exp\left\{ -\int_0^{t} P_*(\xi)\,\mathrm{d}\xi \right\} > 0, \quad i = 1, \dots, n, \tag{5.23}$$

are satisfied.

Then the boundary value problem (5.7), (5.8) *and* (5.16) *is uniquely solvable for every* $f \in L^n_{[0,\infty)}$, $\alpha \in \mathbb{R}^n$, *its Green's operator* G *is positive, and for every nonnegative* f *and* β, *the solution is nonnegative.*

R e m a r k 5.2. It is clear that the inequality

$$1 - \exp\left\{ -\int_0^{\omega} P_*(\xi)\,\mathrm{d}\xi \right\} - \sum_{j=1}^{n} m_{ij} 1 > 0, \quad i = 1, \dots, n \tag{5.24}$$

can be set instead of (5.23) in the formulation of Theorem 5.3.

P r o o f. If we put $v(t) = \exp\{-\int_0^t p_{ii}(\xi)\,\mathrm{d}\xi\}$ into the assertion 1) of Theorem 4.1, we get that condition (5.19) implies, according to Theorem 4.1, positivity of Green's function $g_i(t,s)$ of the problem consisting of the diagonal equation (5.13) and the periodic boundary condition (5.17). Positivity of Green's function $g_i(t,s)$ of problem

(5.13), (5.17) follows also from condition (5.20) [13]. It follows from inequalities (5.21) that the n-vector $v(t) = \mathrm{col}\{v_1(t), \dots, v_n(t)\} = \mathrm{col}\{1, \dots, 1\}$ satisfies the assertion a) of Theorem 5.1. Positivity of Green's operator of system (5.7), (5.8), (5.17) follows now from Theorem 5.1. The vector-function $v(t) = \{v_1(t), \dots, v_n(t)\}$, where $v_i(t) = \exp\bigl\{-\int_0^t P_*(\xi)\,\mathrm{d}\xi\bigr\}$, $i = 1, \dots, n$, satisfies Theorem 2.3 which completes the proof. □

6. Discussion and open problems

Results about positivity/negativity of Green's operators open the way for studying nonlinear functional differential systems. Researchers could use the known scheme of quasi-linearization developed in the books [4], [18] and special nonlinear approaches [18] for the analysis of systems of nonlinear equations. Another direction to develop this topic is connected with the analysis of nonlinear nonlocal boundary conditions. Results on problems with nonlinear conditions are presented in the survey paper [7]. It is clear that an analog of Theorem 2.1 can be obtained in the case of systems of nonlinear functional differential equations coupled with nonlinear conditions, where, for example, contraction of the corresponding operators instead of the estimates of the spectral radius of the operator A should be obtained.

Assertion 9) of Theorem 4.1 allows researchers to study nonlocal problems directly, constructing Green's functions of scalar equations. This idea can work, for example, in the case of "diagonal" nonlocal conditions. It would be interesting to use this possibility and to get results about positivity of Green's operators without the assumption about positivity of Green's operator for "close" local problems.

Nonlocal boundary conditions can be interpreted as a sort of feedback control. It would be interesting to find examples of this control in applications.

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