José Manuel Mazón; Julio Daniel Rossi; Julián Toledo
On optimal matching measures for matching problems related to the Euclidean distance

Mathematica Bohemica, Vol. 139 (2014), No. 4, 553–566

ON OPTIMAL MATCHING MEASURES FOR MATCHING PROBLEMS RELATED TO THE EUCLIDEAN DISTANCE

José Manuel Mazón, València, Julio Daniel Rossi, Alicante,
Julián Toledo, València

(Received September 10, 2013)

Abstract. We deal with an optimal matching problem, that is, we want to transport two measures to a given place (the target set) where they will match, minimizing the total transport cost that in our case is given by the sum of two different multiples of the Euclidean distance that each measure is transported. We show that such a problem has a solution with an optimal matching measure supported in the target set. This result can be proved by an approximation procedure using a p-Laplacian system. We prove that any optimal matching measure for this problem is supported on the boundary of the target set when the two multiples that affect the Euclidean distances involved in the cost are different. Moreover, we present simple examples showing uniqueness or non-uniqueness of the optimal measure.

Keywords: mass transport; Monge-Kantorovich problem; p-Laplacian equation

MSC 2010: 49J20, 49J45, 45G10

1. Introduction

We are interested in an optimal matching problem (see [5], [8], [14], and [15]) that consists in transporting two commodities (say nuts and screws, we assume that we have the same total number of nuts and screws) to prescribed locations, the target set (say factories where we assemble the nuts and the screws) in such a way that they match there (each factory receives the same number of nuts and of screws) and the total cost of the operation, measured in terms of multiples of the Euclidean distance that the commodities are transported, is minimized. That is, for one unit of mass of

The first and third authors have been partially supported by the Spanish Ministerio de Economía y Competitividad and FEDER, project MTM2012-31103. The second author is partially supported by the Spanish Ministerio de Economía y Competitividad under grants MTM2010-18128 and MTM2011-27998.

nuts that is transported from x to z and one unit of mass of screws from y to z, we pay the cost

$$A|x-z|+B|y-z|,$$

where z is a point where we match the unit of nuts with the unit of screws and belongs to the target set, A, B are positive constants and we denote by $|\cdot|$ the Euclidean distance. With the occurrence of the multiplicative constants we are taking into account that the cost of transporting nuts and screws can be different (for example due to different weights).

This problem, which we describe in mathematical precise terms in Section 2, was first treated in [12] where the authors prove the following results (which we also describe more precisely in Section 2):

- ▷ This optimal matching problem has a solution, that is, an optimal matching measure supported on the target set and a pair of optimal transport maps that send each of the commodities to the optimal matching measure. This solution can be obtained by two different methods. One can use directly the classical Monge-Kantorovich's mass transport theory or one can approximate a pair of Kantorovich potentials by solutions to a system of PDEs of p-Laplacian type taking the limit as $p \to \infty$.
- ▷ One can always obtain a solution of the optimal matching problem with a matching measure supported on the boundary of the target set.

When one considers the sum of two Euclidean distances as cost (taking both multiplicative constants equal to one) one can find simple examples, see [12] and Section 4 in this paper, that show that there are configurations for which there are optimal measures supported in the interior of the target set. We reproduce these examples in Section 4. Our main goal here is to show that this is not possible when we have two *different* multiplicative constants.

Theorem 1.1. *Let $A \neq B$, then any optimal matching measure is supported on the boundary of the target set.*

For the sum of two Euclidean distances there are examples of non-uniqueness for the optimal matching measure and also examples of uniqueness, see [12]. Here we also provide examples that show that, even for two different multiplicative constants, we may have non-uniqueness of the optimal matching measure, but we also include examples of uniqueness.

Let us end this introduction with some remarks concerning our bibliography. Optimal matching problems for uniformly convex cost were analysed in [1], [3]–[5], [8] and have implications in the economic theory (hedonic markets and equilibria), see [5]–[9] and references therein. However, when one considers the Euclidean distance as cost new difficulties appear since we deal with a non-uniformly convex cost. The method to solve the problem by taking limit as $p \to \infty$ in a system of PDE's of p-Laplacian type relies on a procedure of solving mass transport problems introduced by Evans and Gangbo in [10] which proves to be quite fruitful, see [2], [11], [13]. We have to remark that the limit as $p \to \infty$ in the system requires some care since the system is nontrivially coupled and therefore the estimates for one component are related to the others, and we believe that it is interesting in its own right, see [12].

2. A description of the optimal matching problem and its p-Laplacian approximation

To write the optimal matching problem in mathematical terms, we fix two nonnegative compactly supported functions $f^+, f^- \in L^\infty$, with supports X_+, X_-, respectively, satisfying the mass balance condition $M_0 := \int_{X_+} f^+ = \int_{X_-} f^-$. We also consider a compact set D (the target set). Then we take a large bounded domain Ω such that it contains all the relevant sets, the supports of f^+ and f^-, X_+, X_- and the target set D. For simplicity we will assume that Ω is a convex $C^{1,1}$ bounded open set. We also assume that

$$X_+ \cap X_- = \emptyset, \quad (X_+ \cup X_-) \cap D = \emptyset \quad \text{and} \quad (X_+ \cup X_-) \cup D \subset\subset \Omega.$$

Whenever T is a map from a measure space (X, μ) to an arbitrary space Y, we denote by $T\#\mu$ the pushforward measure of μ by T. Explicitly, $(T\#\mu)[B] = \mu[T^{-1}(B)]$ (the measurable sets for $T\#\mu$ are exactly the ones such that $T^{-1}(B)$ is μ-measurable). When we write $T\#f = g$, where f and g are nonnegative functions, this means that the measure having density f is pushed-forward to the measure having density g.

For Borel functions $T_\pm \colon \Omega \to \Omega$ such that $T_+\#f^+ = T_-\#f^-$, we consider the functional

$$\mathcal{F}_{A,B}(T_+, T_-) := \int_\Omega A|x - T_+(x)| f^+(x)\,\mathrm{d}x + \int_\Omega B|y - T_-(y)| f^-(y)\,\mathrm{d}y,$$

where, as in the introduction, $|\cdot|$ denotes the Euclidean norm and A, B are positive constants.

The optimal matching problem can be stated as the minimization problem

$$\min_{(T_+,T_-)\in \mathcal{A}_D(f^+,f^-)} \mathcal{F}_{A,B}(T_+, T_-), \tag{2.1}$$

where $\mathcal{A}_D(f^+, f^-) := \{(T_+, T_-) \colon\ T_\pm(X_\pm) \subset D,\ T_+\#f^+ = T_-\#f^-\}$.

If $(T_+^*, T_-^*) \in \mathcal{A}_D(f^+, f^-)$ is a minimizer of the optimal matching problem (2.1), we shall call the measure $\mu^* := T_+^* \# f^+ = T_-^* \# f^-$ a *matching measure* to the problem. Note that there is no reason why a matching measure should be absolutely continuous with respect to the Lebesgue measure. In fact we shall see examples of matching measures that are singular (see Example 4.1).

Let us denote by $\mathcal{M}(D, M_0) := \{\mu \in \mathcal{M}^+(\Omega)\colon \operatorname{supp}(\mu) \subset D,\ \mu(\Omega) = M_0\}$ the set of all possible matching measures. Given $\mu \in \mathcal{M}(D, M_0)$, we denote $\mathcal{A}(f^\pm, \mu) := \{T_\pm\colon T_\pm \# f^\pm = \mu\}$ and we have that

$$
\begin{aligned}
\text{(2.2)}\qquad \inf_{(T_+,T_-)\in\mathcal{A}_D(f^+,f^-)} \mathcal{F}_{A,B}(T_+, T_-) &= \inf_{\mu\in\mathcal{M}(D,M_0)} \inf_{(T_+,T_-)\in\mathcal{A}(f^+,f^-,\mu)} \mathcal{F}_{A,B}(T_+, T_-) \\
&= \inf_{\mu\in\mathcal{M}(D,M_0)} \{AW_1(f^+, \mu) + BW_1(f^-, \mu)\}
\end{aligned}
$$

where $\mathcal{A}(f^+, f^-, \mu) := \{(T_+, T_-)\colon T_+ \in \mathcal{A}(f^+, \mu),\ T_- \in \mathcal{A}(f^-, \mu)\}$, and where $W_1(\cdot, \cdot)$ denotes the 1-Wasserstein distance (its definition is given in [16] (see also [17]) as the value of the optimal mass transport problem with the Euclidean distance as the cost between its two arguments). Indeed, observe that given $(T_+, T_-) \in \mathcal{A}_D(f^+, f^-)$, if we define $\mu := T_+ \# f^+$, we have that $\mu \in \mathcal{M}(D, M_0)$ and $(T_+, T_-) \in \mathcal{A}(f^+, f^-, \mu)$.

Note that on the right-hand side of (2.2) we are considering all possible measures supported in D with total mass M_0 and then we minimize the total transport cost. This is probably the most natural way of looking at the optimal matching problem and it is equivalent to our previous formulation. We have the following existence theorem, for the proof we refer to [12].

Theorem 2.1. *The optimal matching problem* (2.1) *has a solution, that is, there exist Borel functions* $(T_+^*, T_-^*) \in \mathcal{A}_D(f^+, f^-)$ *such that*

$$
\mathcal{F}_{A,B}(T_+^*, T_-^*) = \inf_{(T_+,T_-)\in\mathcal{A}_D(f^+,f^-)} \mathcal{F}_{A,B}(T_+, T_-).
$$

Moreover, we can obtain a solution $(\widetilde{T}_+, \widetilde{T}_-)$ *of the optimal matching problem* (2.1) *with a matching measure supported on the boundary of* D.

The limit as $p \to \infty$ in a p-Laplacian system. In this section we show that we can follow the ideas of Evans-Gangbo [10], to get the matching measure and Kantorovich potentials for the transports involved at the same time; we refer to [12] for details. Let us begin with the following statement:

$$
W_{f^\pm}^D := \inf_{(T_+,T_-)\in\mathcal{A}_D(f^+,f^-)} \mathcal{F}_{A,B}(T_+, T_-) = \sup_{\substack{v,w\in W^{1,\infty}(\Omega) \\ |\nabla v|_\infty \leqslant A,\, |\nabla w|_\infty \leqslant B \\ v\leqslant w \text{ in } D}} \int_\Omega v f^+ - w f^-.
$$

This result is the starting point of our variational approach to the problem via a p-Laplacian system in this section. Take $p > N$ in this section and recall that, for simplicity, we assumed that Ω is a convex $C^{1,1}$ bounded open set. Let us consider the variational problem

$$\min_{\substack{(v,w)\in W^{1,p}(\Omega)\times W^{1,p}(\Omega)\\ v\leqslant w \text{ in } D}} \frac{1}{p}\int_\Omega \frac{1}{A^p}|Dv|^p + \frac{1}{p}\int_\Omega \frac{1}{B^p}|Dw|^p - \int_\Omega vf^+ + \int_\Omega wf^-. \tag{2.3}$$

Standard tools from variational analysis show that there exists a minimizer (v_p, w_p) of (2.3). In addition, any two minimizers differ by a constant, that is, if (v_p, w_p) and $(\widetilde{v}_p, \widetilde{w}_p)$ are minimizers then there exists a constant c with $v_p = \widetilde{v}_p + c$ and $w_p = \widetilde{w}_p + c$.

We can pass to the limit as $p \to \infty$ in the sequence of minimizer functions. In fact, up to a subsequence,

$$\lim_{p\to\infty} (v_p, w_p) = (v_\infty, w_\infty) \quad \text{uniformly},$$

where (v_∞, w_∞) is a solution of the variational problem

$$\max_{\substack{v,w\in W^{1,\infty}(\Omega)\\ |\nabla v|_\infty\leqslant A,\, |\nabla w|_\infty\leqslant B\\ v\leqslant w \text{ in D}}} \int_\Omega vf^+ - wf^-. \tag{2.4}$$

Note that the constraint $|\nabla v|_\infty \leqslant A$, $|\nabla w|_\infty \leqslant B$ is equivalent to

$$|v(x) - v(y)| \leqslant A|x-y|, \quad |w(x) - w(y)| \leqslant B|x-y|.$$

The limit (v_∞, w_∞) gives a pair of Kantorovich potentials for our optimal matching problem. But in fact this limit procedure gives much more since it allows us to identify the optimal matching measure (see [12]).

Concerning the PDE that is solved in this limit procedure we have: Let (v_p, w_p) be minimizer functions of problem (2.3). Then there exists a positive Radon measure h_p of mass M_0 such that

$$\begin{cases} -\mathrm{div}\Big(\dfrac{1}{A^p}|\nabla v_p|^{p-2}\nabla v_p\Big) = f^+ - h_p & \text{in } \Omega,\\ \dfrac{1}{A^p}|\nabla v_p|^{p-2}\nabla v_p\cdot\eta = 0 & \text{on } \partial\Omega,\end{cases}$$

$$\begin{cases} -\mathrm{div}\Big(\dfrac{1}{B^p}|\nabla w_p|^{p-2}\nabla w_p\Big) = h_p - f^- & \text{in } \Omega,\\ \dfrac{1}{B^p}|\nabla w_p|^{p-2}\nabla w_p\cdot\eta = 0 & \text{on } \partial\Omega.\end{cases}$$

This positive measure h_p is supported on $\{x \in D\colon\ v_p(x) = w_p(x)\}$. Here η denotes the normal unit exterior vector to the boundary of Ω.

Moreover, we have that up to a subsequence,

$$h_p \rightharpoonup h_\infty \quad \text{as } p \to \infty, \text{ weakly}^* \text{ as measures},$$

with h_∞ a positive Radon measure of mass M_0 supported on $\{x \in D\colon\ v_\infty(x) = w_\infty(x)\}$. In addition, (v_∞, w_∞) satisfies

$$\begin{aligned} &Av_\infty \text{ is a Kantorovich potential for the transport of } f^+ \text{ to } h_\infty,\\ &Bw_\infty \text{ is a Kantorovich potential for the transport of } h_\infty \text{ to } f^-, \end{aligned}$$

with respect to the Euclidean distance. A Kantorovich potential for the mass transport of f^+ to h_∞ is a 1-Lipschitz function w such that $\int_\Omega wf^+ - \int_\Omega w dh_\infty = W_1(f^+, h_\infty)$.

We conclude that the measure h_∞ is an optimal matching measure for the optimal matching problem (2.1).

3. Localizing the support of optimal matching measures in the nonsymmetric case $A \neq B$

Let us show that, in any space dimension, and for any configuration of the data f^+, f^- and D, any possible optimal measure is supported on ∂D when $A \neq B$. This has to be contrasted with the case in which $A = B$ where we can have optimal measures supported in the interior of D (see Example 4.1 in the next section).

P r o o f of Theorem 1.1. We argue by contradiction. Hence, assume that there exists an optimal measure μ_0 with a nontrivial part of it supported in D°, the interior of D. Let (T_+^*, T_-^*) be the solution of the optimal matching problem (2.1) obtained in Theorem 2.1. Then, since $T_\pm^* \# f^\pm = \mu_0$, there exists z_0 a point in D° in the support of μ_0, $x_0 \in X_+$ a Lebesgue point of T_+^* and $y_0 \in X_-$ a Lebesgue point of T_-^*, such that $T_+^*(x_0) = z_0$ and $T_-^*(y_0) = z_0$.

Now, computing the derivative of

$$G(z) = A|x_0 - z| + B|y_0 - z|$$

with respect to z at z_0 we get

$$DG(z_0) = -A\frac{x_0 - z_0}{|x_0 - z_0|} - B\frac{y_0 - z_0}{|y_0 - z_0|}.$$

Since $A \neq B$ and $(x_0 - z_0)/|x_0 - z_0|$ and $(y_0 - z_0)/|y_0 - z_0|$ are unitary vectors, we conclude that

$$DG(z_0) \neq 0. \tag{3.1}$$

Therefore, the main idea of the proof is that we can move some mass of μ near z_0 so that the cost diminishes. Note that since we have $z_0 \in D^\circ$, such change of moving mass to a nearby point is possible since we remain in D. So, let us fix $\delta > 0$ such that $B_\delta(z_0) \subset D$.

By (3.1) and a continuity argument, there exists a positive number η and a point $z_1 \in B_\delta(z_0)$ such that

$$A|x_0 - z_1| + B|y_0 - z_1| < A|x_0 - z_0| + B|y_0 - z_0| - \eta.$$

Now, using again a continuity argument we can find a small $r_0 > 0$ such that

$$A|x - z_1| + B|y - z_1| < A|x - z_0| + B|y - z_0| - \frac{\eta}{2}$$

for every $x \in B_{r_0}(x_0)$ and every $y \in B_{r_0}(y_0)$. Therefore, we can choose $r \leqslant \min\{r_0, \delta\}$ satisfying

$$r < \frac{\eta}{2(A+B)} \tag{3.2}$$

and such that

$$A|x - z_1| + B|y - z_1| < A|x - z_0| + B|y - z_0| - \frac{\eta}{2} \tag{3.3}$$

for every $x \in B_r(x_0)$ and every $y \in B_r(y_0)$.

Since x_0 and y_0 are Lebesgue points of T_+^* and T_-^*, respectively, we have

$$\lim_{\delta \to 0} \frac{1}{|B_\delta(x_0)|} \int_{B_\delta(x_0)} |T_+^*(x) - z_0| \,\mathrm{d}x = 0, \quad \lim_{\delta \to 0} \frac{1}{|B_\delta(y_0)|} \int_{B_\delta(y_0)} |T_-^*(x) - z_0| \,\mathrm{d}x = 0.$$

Therefore, inside the two balls $B_r(x_0)$ and $B_r(y_0)$ we can find two sets E_1 and E_2, respectively, of positive measure, such that

$$T_+^*(E_1) \subset B_r(z_0) \quad \text{and} \quad T_-^*(E_2) \subset B_r(z_0). \tag{3.4}$$

Also we can assume that

$$\int_{E_1} f^+(x) \,\mathrm{d}x = \int_{E_2} f^-(y) \,\mathrm{d}y = k > 0. \tag{3.5}$$

Note that, thanks to the mass balance condition (3.5), we have an optimal transport map $x = S(y)$ that sends $f^-\chi_{E_2}$ to $f^+\chi_{E_1}$. In particular, S satisfies

$$\int_{E_1} H(x) f^+(x)\,\mathrm{d}x = \int_{E_2} H(S(y)) f^-(y)\,\mathrm{d}y$$

for every continuous function H. Hence,

$$\int_{E_1} A|x - z_i| f^+(x)\,\mathrm{d}x = \int_{E_2} A|S(y) - z_i| f^-(y)\,\mathrm{d}y, \quad i = 0, 1.$$

Using this together with (3.3) we obtain that

$$\begin{aligned}
\int_{E_1} & A|x - z_1| f^+(x)\,\mathrm{d}x + \int_{E_2} B|y - z_1| f^-(y)\,\mathrm{d}y \\
&= \int_{E_2} (A|S(y) - z_1| + B|y - z_1|) f^-(y)\,\mathrm{d}y \\
&\leqslant \int_{E_2} (A|S(y) - z_0| + B|y - z_0|) f^-(y)\,\mathrm{d}y - k\frac{\eta}{2} \\
&= \int_{E_1} A|x - z_0| f^+(x)\,\mathrm{d}x + \int_{E_2} B|y - z_0| f^-(y)\,\mathrm{d}y - k\frac{\eta}{2}.
\end{aligned} \tag{3.6}$$

Now let us define

$$\widetilde{T}_+(x) = \begin{cases} T_+^*(x), & x \in X_+ \setminus E_1, \\ z_1, & x \in E_1, \end{cases} \quad \text{and} \quad \widetilde{T}_-(y) = \begin{cases} T_-^*(y), & y \in X_- \setminus E_2, \\ z_1, & y \in E_2. \end{cases}$$

This pair corresponds to the transport of f^+ and f^- to the measure $(M_0 - k)\mu + k\delta_{z_1}$ that is supported in D.

Using (3.4), (3.6) and (3.2), for such choice of the transport maps we have

$$\begin{aligned}
\int_\Omega & A|x - \widetilde{T}_+(x)| f^+(x)\,\mathrm{d}x + \int_\Omega B|y - \widetilde{T}_-(y)| f^-(y)\,\mathrm{d}y \\
&= \int_{X_+ \setminus E_1} A|x - T_+^*(x)| f^+(x)\,\mathrm{d}x + \int_{X_- \setminus E_2} B|y - T_-^*(y)| f^-(y)\,\mathrm{d}y \\
&\quad + \int_{E_1} A|x - z_1| f^+(x)\,\mathrm{d}x + \int_{E_2} B|y - z_1| f^-(y)\,\mathrm{d}y \\
&\leqslant \int_{X_+ \setminus E_1} A|x - T_+^*(x)| f^+(x)\,\mathrm{d}x + \int_{X_- \setminus E_2} B|y - T_-^*(y)| f^-(y)\,\mathrm{d}y \\
&\quad + \int_{E_1} A|x - z_0| f^+(x)\,\mathrm{d}x + \int_{E_2} B|y - z_0| f^-(y)\,\mathrm{d}y - k\frac{\eta}{2}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{X_+\setminus E_1} A|x - T_+^*(x)| f^+(x)\,\mathrm{d}x + \int_{X_-\setminus E_2} B|y - T_-^*(y)| f^-(y)\,\mathrm{d}y \\
&\quad + \int_{E_1} A|x - T_+^*(x) + T_+^*(x) - z_0| f^+(x)\,\mathrm{d}x \\
&\quad + \int_{E_2} B|y - T_-^*(y) + T_-^*(y) - z_0| f^-(y)\,\mathrm{d}y - k\frac{\eta}{2} \\
&\leqslant \int_{X_+} A|x - T_+^*(x)| f^+(x)\,\mathrm{d}x + \int_{X_-} B|y - T_-^*(y)| f^-(y)\,\mathrm{d}y \\
&\quad + \int_{E_1} A|T_+^*(x) - z_0| f^+(x)\,\mathrm{d}x + \int_{E_2} B|T_-^*(y) - z_0| f^-(y)\,\mathrm{d}y - k\frac{\eta}{2} \\
&\leqslant \int_{X_+} A|x - T_+^*(x)| f^+(x)\,\mathrm{d}x + \int_{X_-} B|y - T_-^*(y)| f^-(y)\,\mathrm{d}y \\
&\quad + \int_{E_1} Arf^+(x)\,\mathrm{d}x + \int_{E_2} Brf^-(y)\,\mathrm{d}y - k\frac{\eta}{2} \\
&= \int_{X_+} A|x - T_+^*(x)| f^+(x)\,\mathrm{d}x + \int_{X_-} B|y - T_-^*(y)| f^-(y)\,\mathrm{d}y \\
&\quad + Ark + Brk - k\frac{\eta}{2} \\
&< \int_{X_+} A|x - T_+^*(x)| f^+(x)\,\mathrm{d}x + \int_{X_-} B|y - T_-^*(y)| f^-(y)\,\mathrm{d}y,
\end{aligned}
$$

which is a contradiction with the fact that μ is an optimal matching measure. □

Remark 3.1. A possible strategy to show that there is an optimal matching measure supported on ∂D in the case $A = B$ can be to take a sequence $A_n \neq B$ such that $A_n \to A = B$ and consider the limit of the corresponding optimal matching measures μ_n (which are supported on $\partial\Omega$). This limit would give an optimal measure for $A = B$ supported on ∂D. In [12] the fact that there is always (regardless of whether $A = B$ or not) an optimal measure supported on ∂D was proved directly using techniques from optimal mass transport theory.

To end this section, we observe that, as was done for the case $A = B$ in [12] (we refer to that reference for the proof) we can characterize when the optimal matching measure is a delta in the general case.

Theorem 3.1. *Assume that there is a point $z_0 \in D$ such that for any pair of points $x \in X_+$ and $y \in X_-$ we have*

$$\min_{z\in D}\{A|x - z| + B|y - z|\} = A|x - z_0| + B|y - z_0|. \tag{3.7}$$

Then the measure $M_0\delta_{z_0}$ is an optimal matching measure.

Conversely, if $M_0\delta_{z_0}$ is an optimal matching measure, then for any pair of points $x \in X_+$ and $y \in X_-$ we have (3.7).

It is easy to see that, for D convex, condition (3.7) is equivalent to

$$\Big\langle A\frac{x-z_0}{|x-z_0|}+B\frac{y-x_0}{|y-z_0|}, z-z_0\Big\rangle \leqslant 0 \quad \text{for all } x\in X_+,\ y\in X_- \text{ and } z\in D$$

(note that z_0 must belong to ∂D when $A\neq B$).

4. Examples

Let us now show the uniqueness and non-uniqueness of the optimal matching measure.

Example 4.1. Consider the optimal matching problem for the data: $\Omega =]-4,4[$, $f^+ = b\chi_{]-3,-2[} + (1-b)\chi_{]2,3[}$, $f^- = \chi_{]-2,-1[}$ and $D=[0,1]$, where $0\leqslant b\leqslant 1$ is fixed.

▷ *A symmetric cost.* First, let us describe in detail what happens when $A=B$ (we can assume that $A=B=1$). In this case any matching measure in D is of the form $\mu = b\delta_0+\nu$ for any positive Radon measure ν of mass $1-b$, supported on D.

Indeed, it is easy to see that for

$$T_+^*(x) = \begin{cases} 0 & \text{if } -3<x<-2,\\ t_+^*(x) & \text{in the other case,}\end{cases}$$

where t_+^* is any optimal transport map transporting $(1-b)\chi_{]2,3[}$ to ν, and

$$T_-^*(x) = \begin{cases} 0 & \text{if } -2<x<-2+b,\\ t_-^*(x) & \text{in the other case,}\end{cases}$$

where t_-^* is any optimal transport map transporting $\chi_{]-2+b,-1[}$ to ν, we have

$$\mathcal{F}_{1,1}(T_+^*,T_-^*) = 4.$$

Also, for

$$v^*(x) := \begin{cases} -x & \text{if } x\leqslant 0,\\ x & \text{if } x\geqslant 0,\end{cases}$$

and

$$w^*(x) = x,$$

we have

$$\int_\Omega v^*(x)f^+(x)\,\mathrm{d}x - w^*(x)f^-(x)\,\mathrm{d}x = 4.$$

Then our assertion follows from

$$\begin{aligned}\int_\Omega v^*(x)f^+(x)\,\mathrm{d}x - w^*(x)f^-(x)\,\mathrm{d}x &\leqslant \sup_{\substack{v,w\in W^{1,\infty}(\Omega)\\ |v'|_\infty,\,|w'|_\infty\leqslant 1\\ v\leqslant w \text{ in } D}} \int_\Omega vf^+ - wf^-\\ &= \inf_{(T_+,T_-)\in\mathcal{A}_D(f^+,f^-)} \mathcal{F}_{1,1}(T_+,T_-) \leqslant \mathcal{F}_{1,1}(T_+^*,T_-^*).\end{aligned}$$

We distinguish three cases:

1. If $b = 1$, δ_0 is the unique matching measure.
2. If $0 < b < 1$, there are infinitely many matching measures but all of them with singular part.
3. If $b = 0$, we have that any positive Radon measure of mass 1 supported on D is a matching measure. Moreover, only in this case, the cost of the matching problem is the same as the cost of the classical transport problem of f^+ to f^-.

So we cannot expect uniqueness of the matching measure in general, but it may hold for some special configurations of the masses and the target set. Uniqueness of the matching measure holds in one dimension if and only if the target set D is located to the left or to the right from the supports of f^+ and f^-, while if there is some mass of f^+ to the left of D and some mass of f^- to the right (or vice versa) then there are infinitely many optimal matching measures.

Moreover, in one dimension there is necessarily a singular part in an optimal measure if the masses f^+ and f^- have some part of both of them to the left or to the right of D, while if f^+ is completely on the right and f^- completely on the left of D then there are optimal matching measures without singular part (note that these measures without singular part are not supported on ∂D).

Now, let us come back to the symmetric situation given in the case $b = 0$. In this case we can also compute optimal pairs (v_p, w_p) (we leave this to the reader, details can be found in [12]). For this sequence (v_p, w_p) we obtain that the limit of the measures h_p is the matching measure $h_\infty = \delta_0/2 + \delta_1/2$. Remark that this measure is supported on ∂D. Note that (v_p, w_p) is unique up to a constant, that is, any other minimizer is of the form $(v_p + c, w_p + c)$, c constant. Therefore, this example shows that not every possible optimal matching measure can be obtained using this approximation procedure.

▷ *A non symmetric cost.* Now consider $A \neq B$, for example, we take $A = 1$ and $B > 1$. Then the unique matching measure (for any b) is $\mu = \delta_0$.

Indeed, in this case, the optimal transport maps are easy to find. We have

$$T_+^*(x) = 0 \quad \text{and} \quad T_-^*(x) = 0.$$

To compute the total cost with these two maps we have to compute

$$\int_\Omega |x - T_+^*(x)| f^+(x)\,\mathrm{d}x = b\int_{-3}^{-2} -x\,\mathrm{d}x + (1-b)\int_2^3 x\,\mathrm{d}x = \frac{5}{2}$$

and

$$B\int_\Omega |y - T_-^*(y)| f^-(y)\,\mathrm{d}y = B\int_{-2}^{-1} -y\,\mathrm{d}y = B\frac{3}{2}.$$

Hence, for these $T_\pm^*$ we have

$$\mathcal{F}_{1,B}(T_+^*, T_-^*) = \frac{5}{2} + B\frac{3}{2}.$$

Now, let us compute a pair of potentials. In this case, we need to impose $|v'|_\infty \leqslant 1$ and $|w'|_\infty \leqslant B$. Let us take

$$v^*(x) := \begin{cases} -x & \text{if } x \leqslant 0, \\ x & \text{if } x \geqslant 0, \end{cases}$$

and

$$w^*(x) = Bx.$$

With these two potentials we get

$$\int_\Omega v^*(x) f^+(x)\,\mathrm{d}x - w^*(x) f^-(x)\,\mathrm{d}x = \frac{5}{2} + B\frac{3}{2}.$$

Then our assertion follows again from the duality argument:

$$\begin{aligned}\int_\Omega v^*(x) f^+(x)\,\mathrm{d}x - w^*(x) f^-(x)\,\mathrm{d}x &\leqslant \sup_{\substack{v,w\in W^{1,\infty}(\Omega)\\ |v'|_\infty\leqslant 1,\, |w'|_\infty\leqslant B\\ v\leqslant w \text{ in } D}} \int_\Omega vf^+ - wf^- \\ &= \inf_{(T_+,T_-)\in\mathcal{A}_D(f^+,f^-)} \mathcal{F}_{1,B}(T_+,T_-) \leqslant \mathcal{F}_{1,B}(T_+^*, T_-^*).\end{aligned}$$

Hence in this case we have uniqueness of the matching measure regardless of the value of b. In fact, if $b \neq 0$ it is clear that the part of f^+ that is on the left of 0 must be taken there and the rest of it also goes to zero since, as $B > A$, it is more expensive to take mass to the right from zero. When $b = 0$ the same argument applies.

In the general case, uniqueness of the optimal matching measure holds in one dimension for $A \neq B$ for any possible configuration of f^+ and f^- when the target set is an interval, $[a, b]$. To see this fact just argue as in the proof of Theorem 1.1 in the previous section using the fact that the function

$$G(z) = A|x - z| + B|y - z|$$

has a unique minimum in $[a, b]$ that is given by b for $x, y \geqslant b$, by a for $x, y \leqslant a$, by b if $x \leqslant a$, $y \geqslant b$ and by a if $y \leqslant a$, $x \geqslant b$ (to describe the location of the minimum we assumed that $A < B$).

Therefore, to obtain a non-uniqueness example with $A \neq B$ we have to go up to dimension two.

Example 4.2. Let us take in $\mathbb{R}^2$ two measures

$$f^+ = \delta_{(0,0)} \quad \text{and} \quad f^- = \delta_{(1,0)},$$

and consider $A > B = 1$. Now we will choose a target set D for which there are infinitely many optimal matching measures. Let $1 < k < A$ and consider the curve

$$\Gamma = \{z\colon\ A|z| + |z - (1,0)| = k\}.$$

On the x-axis the unique point on this curve is given by $\big((k-1)/(A-1), 0\big)$. Now we consider the equation $G(z) = A|z| + |z - (1,0)| = k$ and compute the derivative with respect to z_1 at the point $((k-1)/(A-1), 0)$ obtaining $G_{z_1}(((k-1)/(A-1), 0)) = A - 1 \neq 0$. Therefore, by the Implicit Function Theorem we have that the curve Γ passes trough $((k-1)/(A-1), 0)$ and near this point is a smooth arc that we call γ. With this in mind we choose D to be any smooth small domain such that $D \subset \{z\colon\ A|z| + |z - (1,0)| \geqslant k\}$ and the boundary of D contains a piece of the smooth arc γ of Γ near $((k-1)/(A-1), 0)$. Then it is easy to check that for any point $\tilde{z}$ on $\partial D \cap \gamma$ the measure $\delta_{\tilde{z}}$ is an optimal matching measure for our problem.

References

[1] *M. Agueh, G. Carlier*: Barycenters in the Wasserstein space. SIAM J. Math. Anal. *43* (2011), 904–924.

[2] *L. Ambrosio*: Lecture notes on optimal transport problems. Mathematical Aspects of Evolving Interfaces. Lectures given at the C.I.M.-C.I.M.E. joint Euro-summer school, Madeira, Funchal, Portugal (P. Colli et al., eds.). Lecture Notes in Math. 1812, Springer, Berlin, 2003, pp. 1–52.

[3] *G. Carlier*: Duality and existence for a class of mass transportation problems and economic applications. Advances in Mathematical Economics 5 (S. Kusuoka et al., eds.). Springer, Tokyo, 2003, pp. 1–21.
[4] *G. Carlier, I. Ekeland*: Matching for teams. Econ. Theory *42* (2010), 397–418.
[5] *P.-A. Chiappori, R. J. McCann, L. P. Nesheim*: Hedonic price equilibria, stable matching, and optimal transport: Equivalence, topology, and uniqueness. Econ. Theory *42* (2010), 317–354.
[6] *I. Ekeland*: Existence, uniqueness and efficiency of equilibrium in hedonic markets with multidimensional types. Econ. Theory *42* (2010), 275–315.
[7] *I. Ekeland*: Notes on optimal transportation. Econ. Theory *42* (2010), 437–459.
[8] *I. Ekeland*: An optimal matching problem. ESAIM, Control Optim. Calc. Var. *11* (2005), 57–71.
[9] *I. Ekeland, J. J. Heckman, L. Nesheim*: Identification and Estimates of Hedonic Models. Journal of Political Economy *112* (2004), S60–S109.
[10] *L. C. Evans, W. Gangbo*: Differential equations methods for the Monge-Kantorovich mass transfer problem. Mem. Am. Math. Soc. *137* (1999), 66.
[11] *N. Igbida, J. M. Mazón, J. D. Rossi, J. Toledo*: A Monge-Kantorovich mass transport problem for a discrete distance. J. Funct. Anal. *260* (2011), 3494–3534.
[12] *J. M. Mazón, J. D. Rossi, J. Toledo*: An optimal matching problem for the Euclidean distance. SIAM J. Math. Anal. *46* (2014), 233–255.
[13] *J. M. Mazón, J. D. Rossi, J. Toledo*: An optimal transportation problem with a cost given by the Euclidean distance plus import/export taxes on the boundary. Rev. Mat. Iberoam. *30* (2014), 277–308.
[14] *B. Pass*: Multi-marginal optimal transport and multi-agent matching problems: uniqueness and structure of solutions. Discrete Contin. Dyn. Syst. *34* (2014), 1623–1639.
[15] *B. Pass*: Regularity properties of optimal transportation problems arising in hedonic pricing models. ESAIM, Control Optim. Calc. Var. *19* (2013), 668–678.
[16] *C. Villani*: Optimal Transport. Old and New. Grundlehren der Mathematischen Wissenschaften 137, Springer, Berlin, 2009.
[17] *C. Villani*: Topics in Optimal Transportation. Graduate Studies in Mathematics 58, American Mathematical Society, Providence, 2003.

Authors' addresses: *José Manuel Mazón*, Departament d'Anàlisi Matemàtica, Facultat d'Economia, Universitat de València, Avda. Doctor Moliner 50, 46100 Burjassot, València, Spain, e-mail: `mazon@uv.es`; *Julio Daniel Rossi*, Departamento de Análisis Matemático, Universidad de Alicante, Ap. Correos 99, 03080 Alicante, Spain, on leave from Depto. de Matemática, FCEyN UBA, Ciudad Universitaria, Pab 1 (1428), Buenos Aires, Argentina, e-mail: `julio.rossi@ua.es`; *Julián Toledo*, Departament d'Anàlisi Matemàtica, Facultat d'Economia, Universitat de València, Avda. Doctor Moliner 50, 46100 Burjassot, València, Spain, e-mail: `toledojj@uv.es`.