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A determinant formula for the relative class number of an imaginary abelian number field

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Abstract. We give a new formula for the relative class number of an imaginary abelian number field K by means of determinant with elements being integers of a cyclotomic field generated by the values of an odd Dirichlet character associated to K. We prove it by a specialization of determinant formula of Hasse.

1 Introduction

There are lots of formulas for the relative class number of an imaginary abelian number field K by means of determinant (see [5] for bibliography). In this paper we give such a new formula. We prove it by a specialization of the determinant formula for generalized group matrix which appears in [2, § 13]. The key idea is a transformation of generalized Bernoulli numbers and a transformation of their product over the odd characters to one over the even characters. In our formula, elements of the determinant are integers of a cyclotomic field generated by the values of an odd Dirichlet character associated to K, whereas elements of the determinants are rational numbers for known formulas. We may regard our formula as an imaginary version of Hasse's formula [2, § 16, (3)], which expresses the class number of a real abelian number field by means of determinant with elements being logarithms of cyclotomic units of its cyclic subfields.

2 Results

Let K be an imaginary abelian number field of degree n and with conductor f, and let K_0 be the maximal real subfield of K. Let H_0 be the subgroup of the group $(\mathbf{Z}/f\mathbf{Z})^\times$ of reduced residue classes modulo f corresponding to K_0. Let X_0 be the set of Dirichlet characters associated to K_0.

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We assume that the Dirichlet characters χ associated to K, which we call characters of K for short, are primitive and that, as usual, $\chi(x) = 0$ for an integer x not relatively prime to the conductor $f(\chi)$ of χ.

We classify the group X_0 by the following equivalence $\sim$: for characters χ, $\psi \in X_0$ let $\chi \sim \psi$ if and only if there exists an integer m such that m is relatively prime to n_χ and that $\psi = \chi^m$, where n_χ is the order of χ. We call the classes classified by this equivalence Frobenius classes. Let $\{\psi_0\}$ be a system of representatives of the Frobenius classes. For a representative ψ_0 let t_{ψ_0} be an integer such that the quotient group $(\mathbf{Z}/f\mathbf{Z})^\times/H_{\psi_0}$ is generated by a class represented by $t_{\psi_0} \bmod f$, where $H_{\psi_0} = \{x \bmod f \in (\mathbf{Z}/f\mathbf{Z})^\times \,;\, \psi_0(x) = 1\}$.

We fix an odd character χ_1^* of K. As we will see, the elements of the determinant of our formula are integers of the field generated by the values of the character χ_1^*.

For an even character χ_0 of K and for an element $a \bmod f$ of $(\mathbf{Z}/f\mathbf{Z})^\times$ let

$$u_{\chi_0}(a) = -\chi_1^*(a) \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^{f} \chi_1^*(x) R_f(ax)\,,$$

where $R_f(a)$ is the least positive residue modulo f of a. Then we define a matrix U by

$$U = (u_{\psi_0}(st_{\psi_0}^{-k}))_{(s \bmod f)H_0\,;\,\psi_0,\,0\le k\le \varphi(n_{\psi_0})-1}\,,$$

where $(s \bmod f)H_0$ runs in the rows over the quotient group $(\mathbf{Z}/f\mathbf{Z})^\times/H_0$, which is isomorphic to the Galois group G_0 of K_0; ψ_0 and k run in the columns: $\{\psi_0\}$ is a system defined above and φ is the Euler totient function. Here, $t_{\psi_0}^{-k} \bmod f$ is the inverse of $t_{\psi_0}^{k} \bmod f$, i.e., $t_{\psi_0}^{-k}$ is an integer satisfying $t_{\psi_0}^{-k} t_{\psi_0}^{k} \equiv 1 \pmod{f}$.

With the notation above we have the following

Theorem 1. *For an imaginary abelian number field K of degree n and with conductor f, we have*

$$\det U = \pm \frac{(2f)^{n/2} c\, g^*}{Q\, w} h^*$$

where h^* is the relative class number of K, Q is the Hasse unit index of K, w is the number of roots of unity in K, and g^* is defined by

$$g^* = \prod_{\chi_1} \prod_{p|f} \bigl(1 - \chi_1(p)\bigr)$$

where the products $\prod_{\chi_1}$ and $\prod_{p|f}$ are taken over the odd characters χ_1 of K and the prime numbers p dividing f, respectively, and c is a natural number expressed by

$$c = \prod_{p|n_0} p^{\frac{1}{2}\sum_{p^k|n_0}\left(q(\frac{n_0}{p^k}) - \frac{n_0}{p^k}\right)}\,,$$

where the product $\prod_{p|n_0}$ and the sum $\sum_{p^\mu|n_0}$ are taken over prime numbers p dividing $n_0 = n/2$ and the powers of p dividing n_0, respectively, and $q(m)$ is the number of solutions of $x^m = 1$ in G_0.

We remark here that the elements $u_{\chi_0}(a)$ and the matrix U depend on the character χ_1^*, as we see in the examples below, and that, in addition, U depends on the choice of integers t_{ψ_0}. In fact, we have different U's for different t_{ψ_0}'s in the case of $K = \mathbf{Q}(\zeta_7)$, the 7th cyclotomic field. Moreover, we note that the matrix U never coincides with any matrix in known formulas, because U always contains a constant column corresponding to the principal character $\psi_0 = 1$.

As seen by definition, the number g^* may be zero and then remains a problem of how to construct such a formula in Theorem 1 in case of $g^* = 0$.

For the cyclotomic fields of prime power conductor we have the following corollaries.

Corollary 1. *For the cyclotomic field $K = \mathbf{Q}(\zeta_{p^\rho})$ of conductor p^ρ $(\rho \geq 1)$, p an odd prime, we have*

$$\begin{aligned}\det U &= \det\bigl(u_{\psi_0}(g^i t_{\psi_0}^{-k})\bigr)_{0\leq i\leq \frac{p^{\rho-1}(p-1)}{2}-1\,;\,\psi_0, 0\leq k\leq \varphi(n_{\psi_0})-1}\\ &= \pm(2p^\rho)^{\frac{p^{\rho-1}(p-1)}{2}-1}h^*,\end{aligned}$$

where g is a primitive root modulo p^ρ.

For the field $K = \mathbf{Q}(\zeta_{p^\rho})$ we can take $t_{\psi_0} = g$ for every $\psi_0 \neq 1$ and $t_{\psi_0} = 1$ for $\psi_0 = 1$.

Corollary 2. *For the cyclotomic field $K = \mathbf{Q}(\zeta_{2^\rho})$ of conductor 2^ρ $(\rho \geq 2)$ we have*

$$\begin{aligned}\det U &= \bigl(u_{\psi_0}(5^i t_{\psi_0}^{-k})\bigr)_{0\leq i\leq 2^{\rho-2}-1\,;\,\psi_0, 0\leq k\leq \varphi(n_{\psi_0})-1}\\ &= \pm\, 2^{(\rho+1)2^{\rho-2}-\rho}\, h^*.\end{aligned}$$

For the field $K = \mathbf{Q}(\zeta_{2^\rho})$ we can take $t_{\psi_0} = 5$ for every $\psi_0 \neq 1$ and $t_{\psi_0} = 1$ for $\psi_0 = 1$.

Here we give examples. We adopt the basic characters which Hasse used in [2]. For an odd prime p let χ_p be an odd character modulo p of order $p-1$ and ψ_{p^ρ} $(\rho \geq 2)$ an even character modulo p^ρ of order $p^{\rho-1}$; in addition $\psi_{p^\rho}^p = \psi_{p^{\rho-1}}$. For the prime 2 let χ_4 be the odd character modulo 4 and ψ_{2^ρ} $(\rho \geq 3)$ an even character modulo 2^ρ of order $2^{\rho-2}$; in addition $\psi_{2^\rho}^2 = \psi_{2^{\rho-1}}$. The subscript of a basic character denotes the conductor.

For the following calculation of the values of $u_{\chi_0}(a)$, we use the identity

$$\sum_{\substack{x=1\\(x,f)=1\\\chi_0(x)=1}}^{f} \chi_1^*(x)R_f(ax) = \sum_{\substack{x=1\\(x,f)=1\\\chi_0(x)=1}}^{[f/2]} \chi_1^*(x)\bigl(2R_f(ax)-f\bigr)\,.$$

Example 1. Let $K = \mathbf{Q}(\zeta_5)$, i.e., $p = 5$, $\rho = 1$. Take $g = 2$ and $\chi_1^* = \chi_5$. Then $\{\psi_0\} = \{1, \chi_5^2\}$ and

$$\begin{aligned}u_1(a) &= -\chi_5(a)\bigl(2R_5(a) - 5 + i\bigl(2R_5(2a)-5\bigr)\bigr)\,,\\ u_{\chi_5^2}(a) &= -\chi_5(a)\bigl(2R_5(a)-5\bigr)\,.\end{aligned}$$

Consequently

$$U = \begin{pmatrix} u_1(1) & u_{\chi_5^2}(1) \\ u_1(2) & u_{\chi_5^2}(2) \end{pmatrix} = \begin{pmatrix} 3+i & 3 \\ 3+i & i \end{pmatrix}$$

and hence $\det U = -2\cdot 5$. Otherwise, by Corollary 1 and [2, Tafel II], $\det U = \pm(2\cdot 5)^{\frac{5-1}{2}-1}\cdot 1 = \pm 2\cdot 5$.

Taking $g = 2$ and $\chi_1^* = \chi_5^3$, we have

$$U = \begin{pmatrix} 3-i & 3 \\ 3-i & -i \end{pmatrix}$$

and hence $\det U = -2\cdot 5$.

Example 2. Let $K = \mathbf{Q}(\zeta_{2^3})$, i.e., $p = 2, \rho = 3$. Take $\chi_1^* = \chi_4$. Then $\{\psi_0\} = \{1, \psi_{2^3}\}$ and

$$\begin{aligned} u_1(a) &= -2\chi_4(a)\big(R_{2^3}(a) - R_{2^3}(3a)\big)\,, \\ u_{\psi_{2^3}}(a) &= -2\chi_4(a)\big(R_{2^3}(a) - 4\big)\,. \end{aligned}$$

Consequently

$$U = \begin{pmatrix} u_1(1) & u_{\psi_{2^3}}(1) \\ u_1(5) & u_{\psi_{2^3}}(5) \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 4 & -2 \end{pmatrix}$$

and hence $\det U = -2^5$. Otherwise, by Corollary 2 and [2, Tafel II], $\det U = \pm 2^{(3+1)2^{3-2}-3}\cdot 1 = \pm 2^5$.

Taking $\chi_1^* = \chi_4\psi_8$, we have

$$U = \begin{pmatrix} 8 & 6 \\ 8 & 2 \end{pmatrix}$$

and hence $\det U = -2^5$.

Example 3. Let $K = \mathbf{Q}(\sqrt{-3}, \sqrt{5})$. Take $\chi_1^* = \chi_3$. Then $\{\psi_0\} = \{1, \chi_5^2\}$ and

$$\begin{aligned} u_1(a) &= -2\chi_3(a)\big(R_{15}(a) - R_{15}(2a) + R_{15}(4a) + R_{15}(7a) - 15\big)\,, \\ u_{\chi_5^2}(a) &= -2\chi_3(a)\big(R_{15}(a) + R_{15}(4a) - 15\big)\,. \end{aligned}$$

Consequently

$$U = \begin{pmatrix} u_1(1) & u_{\chi_5^2}(1) \\ u_1(2) & u_{\chi_5^2}(2) \end{pmatrix} = \begin{pmatrix} 10 & 20 \\ 10 & -10 \end{pmatrix}$$

and hence $\det U = -2^2\cdot 3\cdot 5^2$. Otherwise, since $c = 1$, $g^* = 2$, $w = 2\cdot 3$ and $Q = 1$, which is obtained by [2, Tafel II], we have by Theorem 1

$$\det U = \pm\frac{(2f)^{n/2} c\, g^*}{Qw} h^* = \pm\frac{(2\cdot 15)^2\cdot 1\cdot 2}{1\cdot 2\cdot 3}\cdot 1 = \pm 2^2\cdot 3\cdot 5^2.$$

Taking $\chi_1^* = \chi_3\chi_5^2$, we have

$$U = \begin{pmatrix} 30 & 20 \\ 30 & 10 \end{pmatrix}$$

and hence $\det U = -2^2\cdot 3\cdot 5^2$.

3 The determinant of a generalized group matrix

In the second chapter of the book [2] Hasse gave two transformations of the class number formula for a real abelian number field; the first transformation is an application of summations $\sum_x \chi(x)A_f(x)$ to the group matrix, $A_f(x)$ an ordinary distribution (cf. [2, p. 18] or [4, Lemma 12.15]), and the second transformation is one for summations $\sum_s \chi(s)u_\chi(s)$ and for the matrix $U_{\mathfrak{G}}$ (see Lemma 1).

By the first transformation, replacing the distribution $A_f(x)$ in [2, p. 18] with

$$A_f(x) = -\left(\frac{R_f(x)}{f} - \frac{1}{2}\right),$$

we can obtain the formula of Girstmair [1] with Maillet determinant for the relative class number of an imaginary abelian number field with conductor f.

For the proof of our formula we need the following lemmas. Let $\mathfrak{G}$ be an abelian group of order n and $\mathfrak{X}$ the group of characters of $\mathfrak{G}$. For $\chi \in \mathfrak{X}$ let

$$\mathfrak{H}_\chi = \{x \in \mathfrak{G}; \chi(x) = 1\}.$$

For $s \in \mathfrak{G}$ and $\chi \in \mathfrak{X}$ let $u_\chi(s)$ be a complex-valued function satisfying the following conditions:

(i) $u_\chi(s) = u_{\chi^\nu}(s)$ for $s \in \mathfrak{G}$ and $\nu \in \mathbf{Z}$ relatively prime to the order n_χ of χ.

(ii) $u_\chi(s) = u_\chi(s')$ for $s,\ s' \in \mathfrak{G}$ with $\chi(s) = \chi(s')$.

We classify the group $\mathfrak{X}$ by the Frobenius equivalence defined as in § 2. Let $\{\psi\}$ be a system of representatives of the Frobenius classes of $\mathfrak{X}$. For a character ψ let t_ψ be a representative of a generator $t_\psi\mathfrak{H}_\psi$ of the cyclic group $\mathfrak{G}/\mathfrak{H}_\psi$. Then we define a matrix $U_{\mathfrak{G}}$ by

$$U_{\mathfrak{G}} = \left(u_\psi(st_\psi^{-k})\right)_{s\in\mathfrak{G};\psi,0\le k\le\varphi(n_\psi)-1},$$

where s runs in the rows, and ψ and k run in the columns.

Lemma 1. *[2, §14] For the matrix $U_{\mathfrak{G}}$ we have*

$$\det U_{\mathfrak{G}} = \pm\, c_{\mathfrak{G}} \prod_{\chi\in\mathfrak{X}} \sum_{s\, mod\, \mathfrak{H}_\chi} \chi(s)u_\chi(s),$$

where $c_{\mathfrak{G}}$ is a positive number defined by

$$c_{\mathfrak{G}} = \pm \frac{1}{\det(\chi(s))_{s\in\mathfrak{G},\chi\in\mathfrak{X}}} \prod_\psi \left(\left(\frac{n}{n_\psi}\right)^{\varphi(n_\psi)} \det\left(\psi(t_\psi)^{ik}\right)_{\substack{1\le i\le n_\psi \\ (i,n_\psi)=1 \\ 0\le k\le\varphi(n_\psi)-1}} \right)$$

and $s \bmod \mathfrak{H}_\chi$ in the sum $\sum_{s \bmod \mathfrak{H}_\chi}$ runs over the quotient group $\mathfrak{G}/\mathfrak{H}_\chi$.

Lemma 2. *[2, §14 and §15] For an abelian group $\mathfrak{G}$ of order n the number $c_{\mathfrak{G}}$ is a natural number and holds*

$$c_{\mathfrak{G}} = \prod_{p|n} p^{\frac{1}{2}\sum_{p^k|n}\left(q(\frac{n}{p^k}) - \frac{n}{p^k}\right)},$$

where the product and summation are taken over the prime numbers p dividing n and over the powers of p dividing n, and $q(m)$ is the number of solutions of $x^m = 1$ in $\mathfrak{G}$. Therefore $c_{\mathfrak{G}} = 1$ if and only if $\mathfrak{G}$ is cyclic.

4 Proof of Theorem 1

Proof of Theorem 1. We start with the arithmetic class number formula for h^*,

$$h^* = Qw \prod_{\chi_1} \left(-\frac{1}{2} B_{1,\chi_1} \right).$$

For any odd character χ_1 of K we have

$$B_{1,\chi_1} = \frac{1}{f(\chi_1)} \sum_{a=1}^{f(\chi_1)} \chi_1(a)a = \frac{1}{f} \sum_{a=1}^{f} \chi_1(a)a$$

and like as [4, Lemma 8.7] we have

$$\sum_{\substack{a=1 \\ (a,f)=1}}^{f} \chi_1(a)a = \prod_{p|f} \big(1 - \chi_1(p)\big) \cdot \sum_{a=1}^{f} \chi_1(a)a\,.$$

In fact, if $p \mid f$, we have $\chi(p) \sum_{a=1}^{f} \chi(a)a = \sum_{b=1}^{f/p} \chi(pb)(pb)$ and hence

$$\begin{aligned} \prod_{p|f}(1 - \chi_1(p)) \cdot \sum_{a=1}^{f} \chi_1(a)a &= \sum_{a=1}^{f} \chi(a)a + \sum_{\substack{d|f \\ d>1}} \Big(\sum_{\substack{d'|d \\ d'>1}} \mu(d') \Big) \chi(d)d \\ &= \sum_{a=1}^{f} \chi(a)a - \sum_{\substack{d|f \\ d>1}} \chi(d)d = \sum_{\substack{a=1 \\ (a,f)=1}}^{f} \chi_1(a)a\,, \end{aligned}$$

where $\mu(\cdot)$ is the Möbius function.

Therefore, putting

$$S(\chi_1) = \sum_{\substack{a=1 \\ (a,f)=1}}^{f} \chi_1(a)a\,,$$

we have by the arithmetic class number formula for h^*

$$\frac{(-2f)^{n/2} g^* h^*}{Q\,w} = \prod_{\chi_1} S(\chi_1)$$

and hence our task is to show that the product of the right-hand side is $\pm c^{-1} \det U$.

Recall that χ_1^* is a fixed odd character of K. For an even character χ_0 of K let

$$H_{\chi_0} = \{ x \bmod f \in (\mathbf{Z}/f\mathbf{Z})^\times : \chi_0(x) = 1 \}\,.$$

Choose a system of representatives $s \bmod f$ of $(\mathbf{Z}/f\mathbf{Z})^\times / H_{\chi_0}$. Then, for an odd character $\chi_1 = \chi_0 \chi_1^*$ of K we have

$$S(\chi_1) = S(\chi_0 \chi_1^*) = \sum_{s \bmod H_{\chi_0}} \chi_0(s) u_{\chi_0}(s)\,,$$

where

$$u_{\chi_0}(s) = \chi_1^*(s) \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^{f} \chi_1^*(x) R_f(sx)\,.$$

Therefore we have

$$\prod_{\chi_1} S(\chi_1) = \prod_{\chi_0} \sum_{s \bmod H_{\chi_0}} \chi_0(s) u_{\chi_0}(s)\,,$$

where the product $\prod_{\chi_0}$ is taken over the even characters χ_0 of K.

Here we use Lemmas 1 and 2 by letting $\mathfrak{G}$ be the group $(\mathbf{Z}/f\mathbf{Z})^\times/H_0$ and by replacing n by $n/2$, χ by χ_0, $U_{\mathfrak{G}}$ by U, $c_{\mathfrak{G}}$ by c, and $u_\psi(s)$ by $u_{\psi_0}(s)$.

To use Lemma 1, we need to check the $u_{\chi_0}(s)$ for meeting the conditions (i) and (ii) in § 3. First let ν be an integer relatively prime to the order of χ_0. Then $\chi_0^\nu(x) = 1$ if and only if $\chi_0(x) = 1$. Hence

$$\begin{aligned} u_{\chi_0^\nu}(s) &= \chi_1^*(s) \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0^\nu(x)=1}}^{f} \chi_1^*(x) R_f(sx) \\ &= u_{\chi_0}(s). \end{aligned}$$

Secondly let s, s' be integers relatively prime to f satisfying $\chi_0(s) = \chi_0(s')$. Hence

$$\begin{aligned} u_{\chi_0}(s') &= \chi_1^*(s') \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^{f} \chi_1^*(x) R_f(s'x) \\ &= \chi_1^*(s') \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^{f} \chi_1^*\big(s(s')^{-1}x\big) R_f\big(s' \cdot s(s')^{-1}x\big) \\ &= \chi_1^*(s') \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^{f} \chi_1^*(s)\chi_1^*(s')^{-1}\chi_1^*(x) R_f(sx) \\ &= \chi_1^*(s) \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^{f} \chi_1^*(x) R_f(sx) \\ &= u_{\chi_0}(s)\,. \end{aligned}$$

Here $(s')^{-1} \bmod f$ is the inverse of $s' \bmod f$. Therefore we have checked the conditions.

Consequently, by Lemma 1 we obtain

$$\frac{(-2f)^{n/2}g^*h^*}{Q\,w} = \prod_{\chi_1} S(\chi_1) = \frac{1}{\pm c}\det U\,,$$

that is,

$$\det U = \pm\frac{(2f)^{n/2}c\,g^*}{Q\,w}h^*$$

and by Lemma 2 we immediately obtain the expression of c. This completes the proof.

Corollaries 1 and 2 are directly obtained by Theorem 1, because for the cyclotomic fields K of prime power conductors we have $g^* = 1$ by definition, $c = 1$ by Lemma 2 and $Q = 1$ by [2, Satz 27].

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