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Existence of solutions for two types of generalized versions of the Cahn-Hilliard equation

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EXISTENCE OF SOLUTIONS FOR TWO TYPES OF GENERALIZED VERSIONS OF THE CAHN-HILLIARD EQUATION

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Abstract. We show existence of solutions to two types of generalized anisotropic Cahn-Hilliard problems: In the first case, we assume the mobility to be dependent on the concentration and its gradient, where the system is supplied with dynamic boundary conditions. In the second case, we deal with classical no-flux boundary conditions where the mobility depends on concentration u, gradient of concentration ∇u and the chemical potential $\Delta u - s'(u)$. The existence is shown using a newly developed generalization of gradient flows by the author and the theory of Young measures.

Keywords: Cahn-Hilliard; anisotropic behavior; gradient flow; curve of maximal slope; entropy

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1. Introduction

This work deals with the existence of solutions to a variety of Cahn-Hilliard models generalizing the applications in [17]. In what follows, we will introduce three types of equations that will be discussed in this paper, where we use some notation and Hilbert spaces as they are introduced below in Section 2.

1.1. Introductory example: Cahn-Hilliard equations on a closed manifold. The first problem in most parts was treated in [17] and we will not spend too much effort discussing it; we rather consider it as an introductory exercise for the other two problems, as it will help to improve understanding of the method. In the aforementioned paper, the author developed and applied a generalized concept of gradient flows to the following problem:

Given a bounded and open domain $\Omega \subset \mathbb{R}^n$, $n \leqslant 3$, with a smooth boundary Γ and outer normal $\boldsymbol{n}_\Gamma$, show existence of solutions to the following problem in some suitable Hilbert space:

$$
\begin{aligned}
(1.1) \qquad \partial_t u + \operatorname{div}[A(u,\nabla u)\nabla(\Delta u - s'(u))] &= 0 \text{ on } (0,T]\times\Omega,\\
[A(u,\nabla u)\nabla(\Delta u - s'(u))]\cdot \boldsymbol{n}_\Gamma = \nabla u\cdot \boldsymbol{n}_\Gamma &= 0 \text{ on } (0,T]\times\Gamma,\\
u(0) &= u_0 \text{ for } t=0,
\end{aligned}
$$

where we assume for some bounded interval $(a,b)\subset\mathbb{R}$, $0\in(a,b)$ that $u_0(x)\in(a,b)$ for all $x\in\Omega$, $s(u)=s_0(u)+s_1(u)$ with

(i) $s_0\in C^2((a,b))$ convex and $\lim_{x\to a} s_0'(x)=-\infty$, $\lim_{x\to b} s_0'(x)=\infty$,

(ii) $s_1\in C^2(\mathbb{R})$.

Furthermore, we will assume that $A\colon \mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^{n\times n}$ is Lipschitz continuous, bounded and uniformly elliptic, which means there is a constant $C>0$ such that $C^{-1}|\xi|^2\leqslant (A(c,d)\xi)\cdot\xi\leqslant C|\xi|^2$ for all $(c,d)\in\mathbb{R}\times\mathbb{R}^n$ and all $\xi\in\mathbb{R}^n$. We will use this problem in order to introduce the basic concepts of the theory. The weak formulation of the above problem reads

$$
\begin{aligned}
(1.2) \qquad \int_0^T\int_\Omega \partial_t u\psi - \int_0^T\int_\Omega (A(u,\nabla u)\nabla(\Delta u - s'(u)))\cdot\nabla\psi &= 0\\
\forall\,\psi\in L^2(0,T;H^1_{(0)}(\Omega)),&\\
\nabla u\cdot\boldsymbol{n}_\Gamma = 0 \quad \text{on } (0,T]\times\Gamma, \qquad u(0)=u_0 \quad \text{for } t=0.&
\end{aligned}
$$

Though there is a huge literature on the Cahn-Hilliard equation (we refer to [1], [3] and references therein), there seems to be only few results on concentration dependent mobility $A(u)$, among the most cited being Cahn, Elliot and Novick-Cohen [5]. Other works are by Elliot and Garcke [10], Liu, Qi and Yin [21], Liu [6], the one dimensional treatments by Dal Passo, Giacomelli and Novick-Cohen [7] and Liu [20] and the works by Novick-Cohen [31], [32]. In these works, $A(\cdot)$ is assumed to be either strictly monotone or Lipschitz continuous.

To the author's knowledge there is so far no existence result for (1.2) with the mobility depending on ∇u. A study of a viscous Cahn-Hilliard equation with the mobility depending on fractional derivatives of order smaller than 1 can be found in [26]. A recent result treating (1.1) (for A depending only on u) as a gradient flow in the Wasserstein space is due to Lisini, Matthes and Savaré [19]. Numerical studies of (1.1) can be found in [44], [45].

Rossi [37] and Grasselli, Miranville, Rossi and Schimperna [14] deal with a Cahn-Hilliard equation of the form

$$\partial_t u - \Delta\alpha(w) = 0, \quad w = s_0'(u) - \Delta u.$$

The function $\alpha\colon \mathbb{R} \to \mathbb{R}$ is then strictly monotone with $C(|r|^{2p}+1) \leqslant \alpha(r) \leqslant C^{-1}(|r|^{2p}+1)$ for some $C > 0$ and $p \geqslant 0$. Below, we will treat a Lipschitz-dependence of A on w with A being strictly positive, see Subsection 1.4. Also note that our result applies to A depending simultaneously on u, ∇u, and w.

The existence result for (1.2) can be formulated as follows:

Theorem 1.1. *For $0 < T < \infty$ and any $u_0 \in H^1_{(0)}(\Omega)$ there exists $u \in H^1(0,T;H^{-1}_{(0)}(\Omega)) \cap L^2(0,T;H^2(\Omega))$ satisfying (1.2) with $u(t,x) \in (a,b)$ for a.e. $(t,x) \in (0,T) \times \Omega$, and there is a positive constant $C \in \mathbb{R}$ such that the estimate*

$$\|\partial_t u\|^2_{L^2(0,t;H^{-1}_{(0)})} + \|\Delta u - s_0'(u)\|^2_{L^2(0,t;H^1_{(0)})} + \|u\|^2_{L^2(0,t;H^2)} \leqslant C(\mathcal{S}(u_0) - \mathcal{S}(u(t))) \tag{1.3}$$

holds for all $t \in (0,T)$, where

$$\mathcal{S}(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 + \int_\Omega s(u). \tag{1.4}$$

For Ω being a bounded domain with smooth boundary Γ, we can also ask for existence of a solution to the problem

$$\begin{gathered} \partial_t u + \operatorname{div}_\Gamma(A(u,\nabla_\Gamma u)\nabla_\Gamma(\Delta_\Gamma u - s'(u))) = 0 \text{ on } (0,T] \times \Gamma, \\ u(0) = u_0 \text{ for } t = 0, \end{gathered}$$

where $\operatorname{div}_\Gamma$, ∇_Γ, and Δ_Γ are the tangential divergence, tangential gradient, and Laplace-Beltrami operator on Γ. To this aim, let $T_x\Gamma$ be the tangential space to Γ at $x \in \Gamma$ and $T\Gamma := \bigcup_{x\in\Gamma}\{x\} \times T_x\Gamma$ the tangential bundle. We suppose that s has the properties as above and $A\colon \mathbb{R} \times T\Gamma \to \mathbb{R}^{n\times n}$ is Lipschitz continuous, bounded and uniformly elliptic, which means there is a constant $C > 0$ such that $C^{-1}|\xi|^2 \leqslant (A(u,c,d)\xi)\cdot\xi \leqslant C|\xi|^2$ for all $u \in \mathbb{R}$, $(c,d) \in T\Gamma$ and all $\xi \in T_c\Gamma$. The weak formulation for all $\psi \in L^2(0,T;H^1_{(0)}(\Gamma))$ reads

$$\begin{gathered} \int_0^T\int_\Gamma \partial_t u\psi - \int_0^T\int_\Gamma (A(u,\nabla_\Gamma u)\nabla_\Gamma(\Delta_\Gamma u - s'(u)))\cdot\nabla_\Gamma\psi = 0, \\ u(0) = u_0 \quad \text{for } t = 0. \end{gathered} \tag{1.5}$$

This problem is of particular interest for numerical simulations in vesicles formation in biological membranes, see Lowengrub, Rätz, Voigt [22], as well as Mercker and coworkers [23], [25], [24]. A former mathematical study of the Cahn-Hilliard and the Allen-Cahn equations on manifolds can be found in [36]. The aforementioned publication has its focus on singularities of the manifolds and assumes $A \equiv \text{const}$.

Theorem 1.2. *For $0 < T < \infty$ and any $u_0 \in H^1_{(0)}(\Gamma)$ there exists $u \in H^1(0,T;H^{-1}_{(0)}(\Gamma)) \cap L^2(0,T;H^2(\Gamma))$ satisfying* (1.5) *and there is a positive constant $C \in \mathbb{R}$ such that the estimate*

$$\|\partial_t u\|^2_{L^2(0,t;H^{-1}_{(0)}(\Gamma))} + \|\Delta u - s'(u)\|^2_{L^2(0,t;H^1_{(0)}(\Gamma))} + \|u\|^2_{L^2(0,t;H^2(\Gamma))} \leqslant C(\mathcal{S}(u_0) - \mathcal{S}(u(t)))$$

holds for all $t \in (0,T)$, where

$$\mathcal{S}(u) := \int_\Gamma \frac{1}{2}|\nabla_\Gamma u|^2 + \int_\Gamma s(u).$$

The earliest proof of existence for the Cahn-Hilliard equation the author is aware of is for $A(\cdot,\cdot) \equiv 1$, a smooth convex function $s_0 \colon \mathbb{R} \to \mathbb{R}$ and a small concave perturbation s_1, and was given in [11]. Former attempts to the Cahn-Hilliard equation using an energy functional $\mathcal{S}$ with s_0 like above and s_1 a small concave perturbation were in [1], [8], [18], [27]. This form of s seems to be more physical (for a choice $(a,b) = (-1,1)$) as it forces the difference of the concentrations to remain between the fixed boundaries -1 and 1.

1.2. Short sketch of the mathematical approach. The proofs of Theorems 1.1 and 1.2 are based on a recent result by the author [17]. The basic idea is to consider (1.2) as a gradient flow in $H^{-1}_{(0)}(\Omega)$ of the functional $\mathcal{S}$ given in (1.4) and with respect to local scalar products $g_u(\cdot,\cdot)$. The scalar products $g_u(\cdot,\cdot)$ are only defined in $u \in H^1_{(0)}(\Omega) \subset H^{-1}_{(0)}(\Omega)$. For $r_1, r_2 \in \mathcal{H}$ we define

$$g_u(r_1,r_2) = \int_\Omega \nabla p_1^u A(u,\nabla u) \nabla p_2^u,$$

where $p_i^u \in H^1_{(0)}(\Omega)$ solves

$$-\operatorname{div}(A(u,\nabla u)\nabla p_i^u) = r_i \quad \text{for } i = 1,2,$$

with boundary condition $A(u,\nabla u)\nabla p_i^u \cdot \boldsymbol{n}_\Gamma = 0$. As we will see below, with the Fréchet-subdifferential $\mathrm{d}\mathcal{S}$, the problem can be formulated as the gradient flow

$$g_u(\partial_t u, \varphi) = -\langle \mathrm{d}\mathcal{S}(u), \varphi\rangle_{H^{-1}_{(0)}} \quad \forall\, \varphi \in L^2(0,T;H^{-1}_{(0)}(\Omega)).$$

1.3. Cahn-Hilliard equation with dynamic boundary conditions and nonlinear mobility. The theory of Cahn-Hilliard equation with dynamic boundary condition is rather young. Mathematical studies and references can be found in

Miranville and Zelik [28], Gilardi, Miranville and Schimperna [13], Gal [12] and the initial work by Racke and Zheng [34]. From the modeling point of view, note that the equations derived below fall within the modeling framework developed in Heida [16], [15] or by Qian, Wang and Sheng [33].

Here, we prove existence of a solution to the problem

$$\begin{aligned}
\partial_t u &= \operatorname{div}(A(u,\nabla u)\nabla(s'(u)-\Delta u)) \quad \text{on } \Omega,\\
0 &= A(u,\nabla u)\nabla(s'(u)-\Delta u)\cdot \boldsymbol{n}_\Gamma \quad \text{on } \Gamma,\\
\partial_t u &= A_\Gamma(u)(\Delta_\Gamma u - s_\Gamma'(u) - \nabla u\cdot \boldsymbol{n}_\Gamma) \quad \text{on } \Gamma,
\end{aligned}$$

with $u(0,\cdot) = u_0(\cdot)$ for $t = 0$ on Ω and Γ, and we assume A and s to be given like in Section 1.1. A_Γ is assumed to be bounded and Lipschitz continuous with $0 < C \leqslant A_\Gamma(\cdot)$ for some positive constant C and we assume again

(i) $s_0 \in C^2((a,b))$ convex and $\lim\limits_{x\to a} s_0'(x) = -\infty$, $\lim\limits_{x\to b} s_0'(x) = \infty$,

(ii) $s_1 \in C^2(\mathbb{R})$,

(iii) $s_\Gamma = s_0 + s_2$ with $s_2 \in C^2(\mathbb{R})$.

The existence to the above problem in case $A = \mathrm{Id}$, $A_\Gamma = 1$ was treated in the above references for different forms of s and s_Γ. Note that the first and third equation of the problem are not coupled directly through boundary integrals but only through $\nabla u \cdot \boldsymbol{n}_\Gamma$. Thus, the weak formulation of the problem splits for all $\psi, \varphi \in C^1(0,T;C^\infty(\overline{\Omega}))$ into two parts:

$$\begin{aligned}
&\int_0^T\int_\Omega \partial_t u\psi - \int_0^T\int_\Omega (A(u,\nabla u)\nabla(s'(u)-\Delta u))\cdot\nabla\psi = 0, \qquad (1.6)\\
&\int_0^T\int_\Gamma \partial_t E(u)\varphi - \int_0^T\int_\Gamma A_\Gamma(E(u))(\Delta_\Gamma E(u) - s_\Gamma'(E(u)) - \nabla u\cdot\boldsymbol{n}_\Gamma)\varphi = 0
\end{aligned}$$

together with the initial condition, where we use $E(u)$ to denote the trace of u on Γ and P_0 the projection operator defined below in (2.2). Our existence result then reads as follows:

Theorem 1.3. *For $0 < T < \infty$ and any $u_0 \in H^1_{(0)}(\Omega)\cap H^2(\Omega)$ there exists $u \in H^1(0,T;H^{-1}_{(0)}(\Omega))\cap L^2(0,T;H^2(\Omega))$ with $E(u) \in H^1(0,T;L^2(\Gamma))\cap L^2(0,T;H^2(\Gamma))$, as well as $P_0(s'(u)-\Delta u) \in L^2(0,T;H^1_{(0)}(\Omega))$ satisfying (1.6), and there is a positive constant $C \in \mathbb{R}$ such that the estimate*

$$\begin{aligned}
&\|u\|^2_{H^1(0,T;H^{-1}_{(0)}(\Omega))\cap L^2(0,T;H^1(\Omega))} + \|P_0(\Delta u - s_0'(u))\|^2_{L^2(0,t;H^1_{(0)})}\\
&\quad + \|\Delta_\Gamma E(u) - \nabla u\cdot\boldsymbol{n}_\Gamma\|_{L^2(0,T;L^2(\Gamma))} + \|Eu\|^2_{H^1(0,T;L^2(\Gamma))\cap L^2(0,T;H^1(\Gamma))}\\
&\qquad \leqslant C(\mathcal{S}(u_0) - \mathcal{S}(u(t)))
\end{aligned}$$

holds for all $t \in (0,T)$, *where*

$$\mathcal{S}(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 + \int_\Omega s(u) + \int_\Gamma \frac{1}{2}|\nabla_\Gamma E(u)|^2 + \int_\Gamma s_\Gamma(Eu).$$

Note that the usual way for treating such equations is different from the gradient flow theory. In the usual approach, the Cahn-Hilliard problem with dynamic boundary conditions

$$\begin{aligned}
\partial_t u &= \operatorname{div}(\nabla(s'(u) - \Delta u)) \quad \text{on } \Omega,\\
0 &= \nabla(s'(u) - \Delta u) \cdot \boldsymbol{n}_\Gamma \quad \text{on } \Gamma,\\
\partial_t u &= (\Delta_\Gamma u - s'_\Gamma(u) - \nabla u \cdot \boldsymbol{n}_\Gamma) \quad \text{on } \Gamma,
\end{aligned}$$

is reformulated (for the moment informally) as

$$\begin{aligned}
-\Delta_N^{-1}\partial_t u &= -(s'(u) - \Delta u) + \langle \mu \rangle \quad \text{on } \Omega,\\
\langle \mu \rangle &= \langle s'(u) \rangle - \langle \Delta u \rangle,
\end{aligned}$$

where $\langle u \rangle := \fint_\Omega u$ and Δ_N^{-1} is the inverse Laplacian for Neumann boundary conditions. This formulation allows to perform integration by parts in the term Δu and thus to treat the problem in one single weak formulation of the form

$$\begin{aligned}
\int_0^T\int_\Omega -\Delta_N^{-1}\partial_t u\psi + \int_0^T\int_\Omega (s'(u)\psi + \nabla u \cdot \nabla\psi)&\\
+ \int_0^T\int_\Gamma \partial_t u\psi + \int_0^T\int_\Gamma (\nabla_\Gamma u \cdot \nabla_\Gamma \psi + s'_\Gamma(u)\psi) &= 0.
\end{aligned}$$

However, for the nonlinear dependence of the mobility on $u, \nabla u$, the operator Δ_N^{-1} would have to be replaced by a time-dependent operator, imposing lots of technical difficulties.

1.4. Cahn-Hilliard equation with mobility depending on the chemical potential. The third type of the Cahn-Hilliard equation is a generalization of the first type with an additional dependence on the "curvature" term $w := -\Delta u + s'(u)$ (see below). Thus, we write down the problem as

$$\begin{aligned}
\partial_t u - \operatorname{div}(A(u, \nabla u, w)\nabla w) = 0 \text{ on } (0,T] \times \Omega,\\
w + \Delta u - s'(u) = 0 \text{ on } (0,T] \times \Omega,\\
(A(u, \nabla u, w)\nabla w) \cdot \boldsymbol{n}_\Gamma = \nabla u \cdot \boldsymbol{n}_\Gamma = 0 \text{ on } (0,T] \times \Gamma,\\
u(0) = u_0 \text{ for } t = 0,
\end{aligned}$$

where $s(u) = s_0(u) + s_1(u)$ with $s_0(u) = |u|^p$ for some $p \geqslant 2$, and $s_1 \in C_b^{3,1}(\mathbb{R})$ is a three times continuously differentiable mapping with bounded derivatives up to order 2.

Furthermore, we will assume that $A\colon\ \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times n}$ is Lipschitz continuous, bounded and uniformly elliptic, which means there is a constant $C > 0$ such that $C^{-1}|\xi|^2 \leqslant (A(a,b,c)\xi)\cdot\xi \leqslant C|\xi|^2$ for all $(a,b,c) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$ and all $\xi \in \mathbb{R}^n$. The weak formulation of the above problem reads

$$(1.7)\quad \begin{aligned} &\int_0^T\int_\Omega \partial_t u\psi + \int_0^T\int_\Omega (A(u,\nabla u,w)\nabla w)\cdot\nabla\psi = 0 \quad \forall\,\psi \in L^2(0,T;H^1_{(0)}(\Omega)),\\ &w = -\Delta u + s'(u), \quad \nabla u\cdot\boldsymbol{n}_\Gamma = 0 \quad \text{on } (0,T]\times\Gamma, \quad u(0) = u_0 \quad \text{for } t = 0. \end{aligned}$$

for which the following existence theorem holds:

Theorem 1.4. *For $0 < T < \infty$ and any $u_0 \in H^1_{(0)}(\Omega)$ there exists $u \in H^1(0,T;H^{-1}_{(0)}(\Omega)) \cap L^2(0,T;H^2(\Omega))$, $w \in L^2(0,T;H^1_{(0)}(\Omega))$ satisfying (1.7) and there is a positive constant $C \in \mathbb{R}$ such that the estimate*

$$\|\partial_t u\|^2_{L^2(0,t;H^{-1}_{(0)})} + \|\Delta u - P_0(s_0'(u))\|^2_{L^2(0,t;H^1_{(0)})} + \|u\|^2_{L^2(0,t;H^2)} \leqslant C(\mathcal{S}(u_0) - \mathcal{S}(u(t)))$$

holds for all $t \in (0,T)$, where

$$\mathcal{S}(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 + \int_\Omega s(u).$$

The last result is of particular interest for the sharp interface limit. This limit is obtained by replacing $\mathcal{S}$ by

$$\mathcal{S}^\varepsilon(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{\varepsilon^2}\int_\Omega s(u)$$

and solving a sequence of problems

$$\begin{aligned} \partial_t u^\varepsilon - \operatorname{div}(A(u^\varepsilon,\nabla u^\varepsilon,w^\varepsilon)\nabla w^\varepsilon) = 0 \quad \text{on } (0,T]\times\Omega,\\ w^\varepsilon + \Delta u^\varepsilon - \frac{1}{\varepsilon^2}s'(u^\varepsilon) = 0 \quad \text{on } (0,T]\times\Omega,\\ (A(u^\varepsilon,\nabla u^\varepsilon,w^\varepsilon)\nabla w^\varepsilon)\cdot\boldsymbol{n}_\Gamma = \nabla u^\varepsilon\cdot\boldsymbol{n}_\Gamma = 0 \quad \text{on } (0,T]\times\Gamma,\\ u^\varepsilon(0) = u_0^\varepsilon \quad \text{for } t = 0. \end{aligned}$$

For the corresponding sequence of solutions u^ε, we expect

$$u^\varepsilon \to u,$$

where $u \in BV(\Omega)$ with $u(\cdot) \in \{-1, 1\}$ almost surely, ∇u being equal to a varifold γ with curvature κ, satisfying $\partial_t \gamma = \kappa$ in a weak sense. We refer to the work by Röger and Schätzle [35], Mugnai and Röger [29], [30] or the survey by Serfaty [39]. Note that with regard to the limit equations, the dependence of A on u and ∇u leads to anisotropic behavior of the limit problem, where the surface velocity may depend on the normal direction $\boldsymbol{n}$. The quantity w^ε should converge to the curvature κ and thus the dependence of A on w may affect the limit equations as the velocity may then depend nonlinearly on κ. A rigorous study of these reflections is, unfortunately, beyond the scope of this article. Two interesting modeling papers on the subject are by Taylor and Cahn [41] and by Torabi, Lowengrub, Voigt and Wise [43].

1.5. Outline of the paper. In Section 2 we will introduce some standard Hilbert spaces which will be frequently used in this paper and collect some basic facts on them. We will furthermore introduce basic notation for the work with boundary derivatives. In Section 3 we will introduce some functional analytical tools, in particular the theory of Young measures, whereas in Section 4, we will introduce the theory of gradient flows in the way it is presented in [17].

Since we introduced the three types of problems by complexity of their analysis, we will then go on first treating the problems from Subsection 1.1, making the reader familiar with the method and notation in Section 5. The second step will be a generalization to dynamic boundary conditions in Section 6, making it necessary to look for a suitable Hilbert space in order to apply the gradient flow theory. Finally, we will include the dependence of mobility on curvature and prove Theorem 1.4 in Section 7.

2. Notation and preliminaries

For any Hilbert space $\mathcal{H}$, we denote by $L^p(0, T; \mathcal{H})$ the Bochner space of L^p-functions over $(0, T]$ having values in $\mathcal{H}$ and by $H^1(0, T; \mathcal{H})$ the space of functions $u \in L^2(0, T; \mathcal{H})$ having $\partial_t u \in L^2(0, T; \mathcal{H})$. Furthermore, by $C([0, T], \mathcal{H})$ we denote the continuous functions from $[0, T]$ to $\mathcal{H}$, by $C^k([0, T], \mathcal{H})$ the k-times continuously differentiable functions and by $AC([0, T]; \mathcal{H})$ the set of absolutely continuous functions over $[0, T]$.

2.1. Sobolev spaces on Ω. In order to study the examples below, we will frequently make use of the following Banach and Hilbert spaces: We consider an open, bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$ and outer normal vector $\boldsymbol{n}_\Gamma$. Let $W^k_p(\Omega)$ denote the usual L^p-Sobolev space and $W^k_{p,0}(\Omega)$ the closure of

$C_0^\infty(\Omega)$ in $W_p^k(\Omega)$. We will also make use of the notation

$$H^k(\Omega) := W_2^k(\Omega) \quad \text{and} \quad H_0^k(\Omega) := W_{2,0}^k(\Omega). \tag{2.1}$$

We use the definition of the fractional Sobolev spaces $W_p^s(\Omega)$ and $W_p^s(\Gamma)$ as given in Adams [2] and set $W_2^s(\Omega) = (W_{2,0}^{-s}(\Omega))^{-1}$ for $s < 0$.

Let $H^{-1}(\Omega)$ denote the dual of $H_0^1(\Omega)$. Furthermore, we introduce

$$H_{(0)}^1(\Omega) := \left\{ \varphi \in H^1(\Omega) \colon \int_\Omega \varphi = 0 \right\}$$

with the scalar product

$$\langle \varphi, \psi \rangle_{H_{(0)}^1} := \int_\Omega \nabla\varphi \cdot \nabla\psi \quad \forall\, \varphi, \psi \in H_{(0)}^1(\Omega)$$

and its dual space $H_{(0)}^{-1}(\Omega)$ with the scalar product

$$\langle \varphi, \psi \rangle_{H_{(0)}^{-1}} := \langle \nabla\Delta_N^{-1}\varphi, \nabla\Delta_N^{-1}\psi \rangle_{L^2} \quad \forall\, \varphi, \psi \in H_{(0)}^{-1}(\Omega),$$

where Δ_N is the Laplace operator with Neumann boundary conditions. More generally, we define

$$L_{(m)}^2(\Omega) := \left\{ f \in L^2(\Omega) \colon \int_\Omega f = m \right\}, \quad C_{(0)}^k(\overline{\Omega}) := L_{(0)}^2(\Omega) \cap C^k(\overline{\Omega}) \quad \forall\, k \in \mathbb{N} \cup \{\infty\}$$

and denote by

$$P_0 \colon L^2(\Omega) \to L_{(0)}^2(\Omega), \quad f \mapsto f - \int_\Omega f \tag{2.2}$$

the orthogonal projection onto $L_{(0)}^2(\Omega)$. For simplicity, we may sometimes omit the (Ω) if the context is clear (e.g. H^1 instead of $H^1(\Omega)$). Then, $-\Delta_N \colon H_{(0)}^1(\Omega) \to H_{(0)}^{-1}(\Omega)$ is the Riesz isomorphism.

Lemma 2.1 ([17]). *Let $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ have the property that there is $0 < C \leqslant 1$ such that $C|\xi|^2 \leqslant \xi A(x) \xi \leqslant C^{-1}|\xi|^2$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$. For $\varphi \in H_{(0),n}^{-1}(\Omega)$ let $p_\varphi \in H_{(0)}^1(\Omega)$ solve*

$$-\operatorname{div}(A\nabla p_\varphi) = \varphi \quad \text{on } \Omega, \quad (A\nabla p_\varphi)\cdot \boldsymbol{n}_\Gamma = 0 \quad \text{on } \Gamma.$$

Then there is $0 < G \leqslant 1$ depending only on C such that for all $\varphi \in H_{(0)}^{-1}(\Omega)$ we have

$$G\|\varphi\|_{H_{(0)}^{-1}}^2 \leqslant \int_\Omega \nabla p_\varphi \cdot (A\nabla p_\varphi) \leqslant G^{-1}\|\varphi\|_{H_{(0)}^{-1}}^2,$$

2.2. Sobolev spaces on Γ. Since Γ is C^∞, we may introduce the tangential gradient ∇_Γ in the following way: On Γ, let $\boldsymbol{n}_\Gamma$ be the normal vector field and for each arbitrary C^∞-vector field $\boldsymbol{a}\colon \overline{\Omega} \to \mathbb{R}^3$, let us define the normal part a_n and the tangential part $\boldsymbol{a}_\tau$ on Γ via

$$a_n := \boldsymbol{a} \cdot \boldsymbol{n}_\Gamma, \quad \boldsymbol{a}_\tau := \boldsymbol{a} - a_n \boldsymbol{n}_\Gamma.$$

We define the normal derivative

$$\partial_n a := \nabla a \cdot \boldsymbol{n}_\Gamma$$

and the tangential gradient ∇_Γ for any scalar a through

$$\nabla_\Gamma a := (\nabla a)_\tau = \nabla a - \boldsymbol{n}_\Gamma \partial_n a.$$

For a smooth manifold, this is equivalent to the Levi-Civita connection on Γ. Thus, we may understand any vector field $\boldsymbol{f}_\tau$ tangential to Γ as an element of the $T\Gamma$, and define the divergence

$$\operatorname{div}_\Gamma \boldsymbol{f}_\tau := \operatorname{trace} \nabla_\Gamma \boldsymbol{f}_\tau,$$

where we find for any sufficiently regular $\boldsymbol{f}$:

$$\operatorname{div} \boldsymbol{f} = \operatorname{div}_\Gamma \boldsymbol{f}_\tau + \partial_n(\boldsymbol{f}_n).$$

The mean curvature of Γ is defined by

$$\kappa_\Gamma := \operatorname{trace}(\nabla_\Gamma \boldsymbol{n}_\Gamma)$$

and we have the following important result, which can be found for example in [4]:

Lemma 2.2 ([4], Lemma 3.4)**.** *Let Γ be a closed surface. For any $f \in C^1(\Gamma)$ we have*

$$\int_\Gamma \nabla_\Gamma f = \int_\Gamma f \kappa_\Gamma \boldsymbol{n}_\Gamma.$$

Furthermore, for any tangentially differentiable field $\boldsymbol{q}$ it follows that

$$\int_\Gamma \operatorname{div}_\Gamma \boldsymbol{q} = \int_\Gamma \kappa_\Gamma \boldsymbol{q} \cdot \boldsymbol{n}_\Gamma.$$

The Laplace-Beltrami operator Δ_Γ on Γ is defined as $\Delta_\Gamma f := \operatorname{div}_\Gamma \nabla_\Gamma f$. For a nice introduction to surface gradients and the Laplace-Beltrami operator not based on the Levi-Civita connection, we refer to Buscaglia and Ausas [4].

R e m a r k 2.3. Lemma 2.2 implies for the closed surface Γ that

$$-\int_\Gamma g\Delta_\Gamma f = \int_\Gamma \nabla_\Gamma g\cdot\nabla_\Gamma f \quad \forall\, f,g\in C^2(\overline{\Omega}).$$

Via localization, projection and interpolation, we can introduce $W_2^s(\Gamma)$ for $s\in\mathbb{R}$ [2]. Note that

$$\|u\|^2_{W_2^1(\Gamma)} = \int_\Gamma |\nabla_\Gamma u|^2 + \int_\Gamma u^2.$$

For $u\in C^2(\overline{\Omega})$, we set $E_\Gamma(u) := u|_\Gamma$, the trace of u on Γ, and $\partial_n u := \nabla u\cdot \boldsymbol{n}_\Gamma$, with $E_\Gamma(u)$, $\partial_n u$ both being functions on Γ. Like in Ω, consider the space

$$\begin{aligned} H^1_{(0)}(\Gamma) &:= \Big\{u\in W_2^1(\Gamma)\colon \int_\Gamma u = 0\Big\},\\ \|u\|^2_{H^1_{(0)}(\Gamma)} &:= \int_\Gamma |\nabla_\Gamma u|^2 \end{aligned} \tag{2.3}$$

and introduce $H^{-1}_{(0)}(\Gamma)$ in an obvious way. We summarize the main embedding results of interest from [2] in a short lemma:

Lemma 2.4. *The operators* $E_\Gamma\colon W_2^k(\Omega)\to W_2^{k-1/2}(\Gamma)$, $k\geqslant 1$, *and* $\partial_n\colon W_2^k(\Omega)\to W_2^{k-3/2}(\Gamma)$, $k\geqslant 2$, *are continuous. Furthermore,* $W_2^{k_1}(\Omega)\hookrightarrow W_2^{k_2}(\Omega)$, $W_2^{k_1}(\Gamma)\hookrightarrow W_2^{k_2}(\Gamma)$ *are continuous and compact for all* $k_1>k_2$ *and* $k_1,k_2\in\mathbb{R}$.

R e m a r k 2.5. Note that there is $0<C<1$ such that

$$C\|u\|_{W_2^1(\Omega)} \leqslant \|\nabla u\|_{L^2(\Omega)} + \|E_\Gamma(u)\|_{L^2(\Gamma)} \leqslant C^{-1}\|u\|_{W_2^1(\Omega)},$$

i.e. the last chain of inequalities shows an equivalence of norms on $W_2^1(\Omega)$.

Furthermore, for simplicity of notation, we write

$$u\equiv E_\Gamma(u)\in L^2(\Gamma) \quad \forall\, u\in W_2^1(\Omega) \tag{2.4}$$

and thus we do not distinguish between $W_2^1(\Omega)$-functions and their traces, whenever this will not cause confusion. Finally, we have the following result, which can be found for example in the book by Temam [42]:

Lemma 2.6 ([42], Theorem 1.2). *Let*

$$E(\Omega) := \{u \in L^2(\Omega)^n \colon \operatorname{div} u \in L^2(\Omega)\}.$$

Then the operator

$$\partial_n \colon E(\Omega) \to H^{-1/2}(\Omega), \quad u \mapsto u \cdot \boldsymbol{n}_\Gamma$$

is continuous.

3. Functional analytical tools and young measures

3.1. Tools from functional analysis. We state two fundamental results from functional analysis which are known in various versions, among which we will use the following:

Theorem 3.1 (Egorov's theorem for $L^2(0,T;\mathcal{H})$). *Let $\mathcal{H}$ be a Hilbert space and $(v_n)_{n\in\mathbb{N}} \subset L^2(0,T;\mathcal{H})$ a sequence such that $v_n \to v \in L^2(0,T;\mathcal{H})$ strongly and pointwise for a.e. $t \in (0,T)$. Then for any $\varepsilon > 0$ there is $K_\varepsilon \subset (0,T)$ compact with $\mathcal{L}((0,T) \setminus K_\varepsilon) < \varepsilon$ such that $v_n \to v$ uniformly on K_ε.*

Theorem 3.2 (Lusin). *For a Banach space $\mathcal{B}$, let $f \in L^p(0,T;\mathcal{B})$ for some $1 \leqslant p < \infty$. Then for each $\varepsilon > 0$ there is a compact set $K^\varepsilon \subset (0,T)$ such that $\mathcal{L}((0,T) \setminus K^\varepsilon) < \varepsilon$ and $f \in C(K^\varepsilon;\mathcal{B})$.*

3.2. Young measures. For a separable metric space E, we denote by $\mathcal{B}(E)$ the Borel σ-algebra, where $\mathcal{L}(0,T)$ is the Lebesgue σ-algebra on $(0,T)$ and $\mathcal{L}(0,T)\otimes\mathcal{B}(E)$ is the product σ-algebra. Let $\mathcal{M}(0,T;E)$ denote the set of measurable functions over $(0,T)$ with values in E. An $\mathcal{L}(0,T) \otimes \mathcal{B}(E)$-measurable function $h\colon (0,T) \times E \to (-\infty,\infty]$ is a normal integrand if $v \mapsto h(t,v)$ is lower semicontinuous for all $t \in (0,T)$.

For a Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ denote the Borel σ-algebra with respect to $\|\cdot\|_{\mathcal{H}}$. We say that an $\mathcal{L}\otimes\mathcal{B}(\mathcal{H})$-measurable functional $h\colon (0,T)\times\mathcal{H} \to (-\infty,\infty]$ is a weakly normal integrand if

$$v \mapsto h_t(v) := h(t,v) \quad \text{is sequentially weakly l.s.c. for a.e. } t \in (0,T).$$

Definition 3.3 (Time dependent parametrized measures). A parametrized measure in E is a family $\boldsymbol{\nu} := \{\nu_t\}_{t\in(0,T)}$ of Borel probability measures on E such that

$$t \in (0,T) \mapsto \nu_t(B) \quad \text{is } \mathcal{L}\text{-measurable for all } B \in \mathcal{B}(E).$$

We denote by $\mathcal{Y}(0,T;E)$ the set of all parametrized measures.

For computations below, the most important result on parametrized measures is a generalization of Fubini's theorem [9]: For every parameterized measure $\boldsymbol{\nu} = \{\nu_t\}_{t\in(0,T)}$ there exists a unique measure ν on $\mathcal{L}(0,T)\otimes\mathcal{B}(E)$ defined by

$$\nu(I\times A) = \int_I \nu_t(A)\,\mathrm{d}t \quad \forall\, I\in\mathcal{L}(0,T),\ A\in\mathcal{B}(E).$$

Moreover, for every $\mathcal{L}(0,T)\otimes\mathcal{B}(E)$-measurable function $h\colon (0,T)\times E\to[0,\infty]$, the function

$$t\mapsto \int_E h(t,\xi)\,\mathrm{d}\nu_t(\xi)$$

is $\mathcal{L}(0,T)$-measurable and the Fubini integral representation holds:

$$\int_{(0,T)\times E} h(t,\xi)\,\mathrm{d}\nu(t,\xi) = \int_0^T \left(\int_E h(t,\xi)\,\mathrm{d}\nu_t(\xi)\right)\mathrm{d}t. \tag{3.1}$$

If ν is concentrated on the graph of a measurable function $u\colon (0,T)\to E$, then $\nu_t = \delta_{u(t)}$ for a.e. $t\in(0,T)$, where $\delta_{u(t)}$ denotes the Dirac measure carried by $\{u(t)\}$. In this case, by (3.1):

$$\int_{(0,T)\times E} h(t,\xi)\,\mathrm{d}\nu(t,\xi) = \int_0^T h(t,u(t))\,\mathrm{d}t.$$

For calculations below, we will study the following situation: given two Hilbert spaces $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, we will consider a mapping $g_\bullet(\cdot,\cdot)\colon\ \widetilde{\mathcal{H}}\times\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ being continuous in $\widetilde{\mathcal{H}}$ and bilinear continuous in $\mathcal{H}$ with

$$C^{-1}\|\xi\|_{\mathcal{H}}^2 \leqslant g_u(\xi,\xi) \leqslant C\|\xi\|_{\mathcal{H}}^2 \quad \forall\, u\in\widetilde{\mathcal{H}},\ \xi\in\mathcal{H}$$

for some constant C and

$$g_{u_m}(v_m,\varphi)\to g_u(v,\varphi) \quad \forall\,\varphi\in\mathcal{H}, \tag{3.2}$$

whenever $u_m\to u$ strongly in $\widetilde{\mathcal{H}}$ and $v_m\rightharpoonup v$ weakly in $\mathcal{H}$. Starting from Section 4 below, we will assume $\widetilde{\mathcal{H}}\hookrightarrow\mathcal{H}$ continuously, which is actually not needed for the results in this section.

Corollary 3.4 ([17]). *As a consequence of* (3.2), *we find for $u_n\to u$ strongly in $\widetilde{\mathcal{H}}$ and $\varphi_n\rightharpoonup\varphi$ weakly in $\mathcal{H}$:*

$$g_u(\varphi,\varphi)\leqslant \liminf_{n\to\infty} g_{u_n}(\varphi_n,\varphi_n).$$

The following statement is a generalization of [38], Theorem 3.2 and a direct consequence of Corollary 3.4 and Theorem 4.3 of [40].

Theorem 3.5 ([17]). *Let $\{v_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^p(0,T;\mathcal{H})$ for some $p>1$, and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $L^p(0,T;\widetilde{\mathcal{H}})$ with $u_n \to u \in L^p(0,T;\widetilde{\mathcal{H}})$ pointwise a.e. in $(0,T)$. Then there exists a subsequence $k\mapsto v_{n_k}$ and a parametrized measure $\boldsymbol{\nu}=\{\nu_t\}_{t\in(0,T)}\in\mathcal{Y}(0,T;\mathcal{H})$ such that for a.e. $t\in(0,T)$*

$$\limsup_{k\to\infty}\|v_{n_k}(t)\|_{\mathcal{H}}<\infty,\quad \nu_t \text{ is concentrated on } L(t):=\bigcap_{q=1}^{\infty}\overline{\{v_{n_k}(t)\colon\ k\geqslant q\}}^{w}$$

of weak limit points of $\{v_n\}_{n\in\mathbb{N}}$, and

$$\liminf_{k\to\infty}\int_0^T h(t,v_{n_k}(t))\,\mathrm{d}t\geqslant\int_0^T\left(\int_{\mathcal{H}}h(t,\xi)\,\mathrm{d}\nu_t(\xi)\right)\mathrm{d}t$$

for every weakly normal integrand h such that $h^-(\cdot,v_{n_k}(\cdot))$ is uniformly integrable and

$$\liminf_{k\to\infty}\int_0^T g_{u_m}(v_m(t),v_m(t))\,\mathrm{d}t\geqslant\int_0^T\left(\int_{\mathcal{H}}g_u(\xi,\xi)\,\mathrm{d}\nu_t(\xi)\right)\mathrm{d}t. \tag{3.3}$$

In particular,

$$\int_0^T\left(\int_{\mathcal{H}}\|\xi\|_{\mathcal{H}}^p\,\mathrm{d}\nu_t(\xi)\right)\leqslant\liminf_{k\to\infty}\int_0^T\|v_{n_k}\|_{\mathcal{H}}^p\,\mathrm{d}t,$$

and, setting

$$v(t):=\int_{\mathcal{H}}\xi\,\mathrm{d}\nu_t(\xi),\quad \text{we have}\quad v_{n_k}\rightharpoonup v \text{ in } L^p(0,T;\mathcal{H}).$$

Finally, if $\nu_t=\delta_{v(t)}$ for a.e. $t\in(0,T)$, then

$$\langle v_{n_k},w\rangle_{\mathcal{H}}\to\langle v,w\rangle_{\mathcal{H}}\quad\text{in } L^1(0,T)\quad\forall\,w\in L^q(0,T;\mathcal{H}),\quad\frac{1}{p}+\frac{1}{q}=1,$$

and up to extraction of a further subsequence independent of t (still denoted by v_{n_k})

$$v_{n_k}(t)\rightharpoonup v(t)\quad\text{for a.e. } t\in(0,T).$$

The theory developed in [17] deals with equations of the form

$$\partial_t u \in -\nabla_{l,u}\mathcal{S}(u) + f(t) \tag{4.1}$$

with $\mathcal{S}$ being a (possibly nonconvex) lower semicontinuous entropy functional on a Hilbert space $\mathcal{H}$, $\nabla_{l,u}\mathcal{S}$ being the *limiting subgradient* with respect to a *densely defined metric structure* $g_\bullet$ and $f \in L^2(0,T;\mathcal{H})$.

More precisely, consider Hilbert spaces $\mathcal{H}_0 \hookrightarrow \widetilde{\mathcal{H}} \hookrightarrow \mathcal{H}$ with the set $B(\mathcal{H})$ of positive definite continuous bilinear forms. We then use the following terms and notation:

Definition 4.1. We call any tuple $(\mathcal{H}_0, \widetilde{\mathcal{H}}, \mathcal{H}, g)$ of Hilbert spaces $\mathcal{H}_0$, $\widetilde{\mathcal{H}}$, $\mathcal{H}$ and a mapping $g_\bullet \colon \widetilde{\mathcal{H}} \to B(\mathcal{H})$ satisfying (1) and (2) an *entropy space*:

(1) $\mathcal{H}_0 \hookrightarrow \widetilde{\mathcal{H}} \hookrightarrow \mathcal{H}$, where the embeddings are dense, and the embedding $\mathcal{H}_0 \hookrightarrow \widetilde{\mathcal{H}}$ is compact. We denote by $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{\widetilde{\mathcal{H}}}$, $\|\cdot\|_{\mathcal{H}_0}$ the respective norms and by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ the scalar product on $\mathcal{H}$.

(2) g is a densely defined metric in the following sense: There is a positive constant $1 \leqslant G^* < \infty$ such that

$$\frac{1}{\sqrt{G^*}}|\langle x,y\rangle_{\mathcal{H}}| \leqslant |g_u(x,y)| \leqslant \sqrt{G^*}|\langle x,y\rangle_{\mathcal{H}}| \quad \forall\, u \in \widetilde{\mathcal{H}},\ \forall\, x,y \in \mathcal{H}, \tag{4.2}$$

for all $u \in \widetilde{\mathcal{H}}$ and $g_\bullet$ is strong-weak-continuous in the following sense: if $u_n \to u$ strongly in $\widetilde{\mathcal{H}}$ and $\varphi_n \rightharpoonup \varphi$ weakly in $\mathcal{H}$ as $n \to \infty$, then

$$g_{u_n}(\varphi_n, \psi) \to g_u(\varphi,\psi) \quad \text{as } n \to \infty\ \forall\, \psi \in \mathcal{H}. \tag{4.3}$$

This means that with every point $u \in \widetilde{\mathcal{H}}$ we associate a local scalar product and local norm

$$\langle x,y\rangle_{g(u)} := g_u(x,y), \quad \|x\|_{g(u)} := \sqrt{g_u(x,x)} \quad \forall\, x,y \in \mathcal{H}.$$

We denote by $\tilde{g}_u$ the unique automorphism on $\mathcal{H}$ such that

$$g_u(v,\varphi) = \langle \tilde{g}_u(v), \varphi\rangle_{\mathcal{H}} \quad \forall\, \varphi \in \mathcal{H}. \tag{4.4}$$

We will assume that $\mathcal{S}$ is a proper functional $\mathcal{S}\colon \mathcal{H} \to (-\infty,\infty]$. Then we define the set-valued subdifferential $\mathrm{d}\mathcal{S}(u)$ at $u \in D(\mathcal{S}) \cap \widetilde{\mathcal{H}}$ through

$$\delta \in \mathrm{d}\mathcal{S}(u) \Leftrightarrow \langle \delta, v\rangle_{\mathcal{H}} \leqslant \liminf_{h \searrow 0} \frac{\mathcal{S}(u+hv) - \mathcal{S}(u)}{h} \quad \forall\, v \in \mathcal{H}. \tag{4.5}$$

For a convex and lower semicontinuous $\mathcal{S}$, the last definition is equivalent to the usual definition of the Fréchet-subdifferential used by Rossi and Savaré [38]

$$d^f\mathcal{S}(u) = \{\xi \in \mathcal{H}\colon\ \mathcal{S}(w) - \mathcal{S}(u) - \langle \xi, w-u\rangle_{\mathcal{H}} \geqslant o(|w-u|) \text{ as } w \to u\},$$

where the above Landau notation should be understood as

$$\liminf_{w\to u} \frac{\mathcal{S}(w) - \mathcal{S}(u) - \langle \xi, w-u\rangle_{\mathcal{H}}}{|w-u|} \geqslant 0. \tag{4.6}$$

For non-convex functionals, it is evident that $d^f\mathcal{S}(u) \subset \mathrm{d}\mathcal{S}(u)$ but, in infinite dimension, (4.5) need not imply (4.6).

The subgradient $\nabla_u\mathcal{S}(u)$ of $\mathcal{S}$ in $u \in \widetilde{\mathcal{H}} \cap D(\mathrm{d}\mathcal{S})$ is defined by

$$\delta \in \nabla_u\mathcal{S}(u) \Leftrightarrow \exists \tilde{\delta} \in \mathrm{d}\mathcal{S}(u)\colon\ g_u(\delta, v) := \langle \tilde{\delta}, v\rangle_{\mathcal{H}} \quad \forall\, v \in \mathcal{H}, \tag{4.7}$$

where the index u refers to the local metric. If no confusion occurs, we write $\nabla\mathcal{S}(u) = \nabla_u\mathcal{S}(u)$.

In what follows, we denote the *local slope* by

$$|\partial\mathcal{S}|(u) := \limsup_{w\to u,\ w\in D(\mathcal{S})} \frac{|\mathcal{S}(u) - \mathcal{S}(w)|}{\|u-w\|_{g(u)}}, \tag{4.8}$$

implying

$$\begin{aligned}
\sup_{\delta\in\nabla_u\mathcal{S}(u)} \|\delta\|_{g(u)} &\leqslant \sup_{\delta\in\nabla_u\mathcal{S}(u)} \frac{\langle \tilde{g}_u(\delta), \delta\rangle_{\mathcal{H}}}{\|\delta\|_{g(u)}} \\
&\leqslant \sup_{\delta\in\nabla_u\mathcal{S}(u)} \liminf_{h\searrow 0} \frac{\mathcal{S}(u + h\tilde{g}_u(\delta)) - \mathcal{S}(u)}{h\|\tilde{g}_u(\delta)\|_{\mathcal{H}}} \\
&\leqslant \limsup_{w\to u,\ w\in D(\mathcal{S})} \frac{|\mathcal{S}(u) - \mathcal{S}(w)|}{\|u-w\|_{g(u)}} = |\partial\mathcal{S}|(u) \quad \forall\, u \in D(\mathrm{d}\mathcal{S})
\end{aligned} \tag{4.9}$$

and in case $\mathrm{d}\mathcal{S}$ is single valued, $|\partial\mathcal{S}|(u) = \|\nabla\mathcal{S}(u)\|_{g(u)}$.

Finally, for every subset $A \subset \mathcal{H}$ we define the affine hull $\operatorname{aff} A$ and its minimal section A° through

$$\begin{gathered}
\operatorname{aff} A := \Big\{\sum_i t_i a_i\colon\ a_i \in A,\ t_i \in \mathbb{R},\ \sum_i t_i = 1\Big\},\\
|A^\circ| := \inf_{\xi\in A} \|\xi\|_{\mathcal{H}}, \quad A^\circ := \{\xi \in A\colon\ \|\xi\|_{\mathcal{H}} = |A^\circ|\}.
\end{gathered}$$

Definition 4.2 ([38], [17])**.** We say that for any $u \in \mathcal{H}$, $\xi \in \mathcal{H}$ is an element of the limiting subdifferential $\mathrm{d}_l\mathcal{S}(u)$ of $\mathcal{S}$ in u if there are $u_n \in \mathcal{H}$ with $u_n \to u$

strongly and $\xi_n \in \mathrm{d}\mathcal{S}(u_n)$ such that $\xi_n \rightharpoonup \xi$ weakly in $\mathcal{H}$. The limiting subgradient is defined through

$$\nabla_{l,u}\mathcal{S}(u) = \tilde{g}_u^{-1}(\mathrm{d}_l\mathcal{S}(u)).$$

Thus, equation (4.1) has to be understood in the sense of

$$g_u(\partial_t u, \varphi) \in \langle \mathrm{d}_l S(u), \varphi\rangle_{\mathcal{H}} + g_u(f, \varphi) \quad \forall\, \varphi \in L^2(0,T;\mathcal{H}). \tag{4.10}$$

In case that the graph of $(\mathcal{S},\, \mathrm{d}\mathcal{S})$ is strongly-weakly closed in $\mathcal{H}\times\mathcal{H}\times\mathbb{R}$, i.e.

$$\left.\begin{matrix} \xi_n \in \mathrm{d}\mathcal{S}(v_n), \quad r_n = \mathcal{S}(v_n) \\ v_n \to v, \quad \xi_n \rightharpoonup \xi, \quad r_n \to r \end{matrix}\right\} \Rightarrow \xi \in \mathrm{d}\mathcal{S}(v),\ r = \mathcal{S}(v), \tag{4.11}$$

we find $\mathrm{d}_l\mathcal{S} = \mathrm{d}\mathcal{S}$. As explained by Rossi and Savaré [38], this condition yields closedness and convexity of $\mathrm{d}\mathcal{S}$, the continuity condition

$$v_n \to v,\ \sup_n(|\partial\mathcal{S}(v_n)|, \mathcal{S}(v_n)) < \infty \Rightarrow \mathcal{S}(v_n) \to \mathcal{S}(v) \quad \text{as } n \nearrow \infty \tag{4.12}$$

and the the following chain rule: If $v \in H^1(0,T;\mathcal{H})$, $\xi \in L^2(0,T;\mathcal{H})$ with $\xi(t) \in \mathrm{d}_l\mathcal{S}(v(t))$ for a.e. $t \in (0,T)$, and $\mathcal{S}\circ v$ is a.e. equal to a function s of bounded variation, then

$$\frac{\mathrm{d}}{\mathrm{d}t}s(t) = \langle \xi, v'(t)\rangle_{\mathcal{H}}. \tag{4.13}$$

Lemma 4.3 (See [38]). *If $\mathcal{S}$ is convex, condition* (4.11) *is fulfilled. In particular,* (4.13) *holds.*

For the rest of the paper, we assume that $\mathcal{S}$ is an entropy functional in the following sense:

Definition 4.4. Let $(\mathcal{H}_0, \widetilde{\mathcal{H}}, \mathcal{H}, g)$ be an entropy space with $G^* > 1$. We say that $\mathcal{S}\colon \mathcal{H} \to (-\infty, \infty]$ is an entropy functional on $(\mathcal{H}_0, \widetilde{\mathcal{H}}, \mathcal{H}, g)$ if it satisfies:

(1) $D(\mathcal{S}) \subset \widetilde{\mathcal{H}}$ and $\mathcal{S}\colon \mathcal{H} \to \mathbb{R}$ is proper, lower semicontinuous, i.e. the domain $D(\mathcal{S})$ of $\mathcal{S}$ is nonempty.

(2) $\mathcal{S} + \|\cdot\|_{\mathcal{H}}$ has compact sublevels, i.e. there exists $\tau_* > 0$ such that the sets

$$\Big\{ v \in \mathcal{H}\colon\ \mathcal{S}(v) + \frac{1}{2\tau}\min\Big\{1, \frac{1}{\sqrt{G^*}}\Big\}\|v\|_{\mathcal{H}}^2 < C \Big\}$$

are compact for any $\tau < \tau_*$ and any $C > 0$ and there is a constant $S_0 > 0$ such that

$$\mathcal{S}(v) + \frac{1}{2\tau_*}\min\Big\{1, \frac{1}{\sqrt{G^*}}\Big\}\|v\|_{\mathcal{H}}^2 \geqslant -S_0. \tag{4.14}$$

(3) $\mathcal{S}$ satisfies the estimate

$$\|u\|_{\mathcal{H}_0} \leqslant C(\mathcal{S}(u) + |\partial S|^2(u) + 1).$$

We close this section stating the first of the three existence theorems from [17] which we will use below:

Theorem 4.5. *Let $\mathcal{H}_0$, $\widetilde{\mathcal{H}}$, $\mathcal{H}$, g and $\mathcal{S}$ satisfy Definitions* 4.1 *and* 4.4 *with* $\mathrm{d}_l\mathcal{S}(u)$ *being convex and closed for all $u \in \mathcal{H}$, and*

$$\mathcal{S}(u) = \mathcal{S}_{\mathcal{H}}(u) + \mathcal{S}_{\widetilde{\mathcal{H}}}(u) \tag{4.15}$$

with functionals $\mathcal{S}_{\mathcal{H}}\colon \mathcal{H} \to \mathbb{R}$ being proper, lower semicontinuous, and $\mathcal{S}_{\widetilde{\mathcal{H}}}\colon D(\mathcal{S}) \subset \widetilde{\mathcal{H}} \to \mathbb{R}$ being continuous with respect to $\widetilde{\mathcal{H}}$. Furthermore, let $f \in L^2(0,T;\mathcal{H})$. Then, for each $u_0 \in \mathcal{H}_0$ and every $0 < T \in \mathbb{R}$, there exists a solution $u \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{H}_0)$ to (4.10)*, satisfying the Lyapunov inequality*

$$\frac{1}{2}\int_0^t \|\partial_t u\|^2_{g(u)} + \frac{1}{2}\int_0^t |(f - \nabla_l \mathcal{S}(u))^\circ|^2 + \mathcal{S}(u(t)) \leqslant \mathcal{S}(u(0)) + \int_0^t \langle f, u\rangle_{\mathcal{H}} \quad \text{for a.e. } t \in (0,T). \tag{4.16}$$

If $\mathcal{S}$ additionally fulfils the continuity assumption (4.12) *then there is a negligible set $\mathcal{N} \subset (0,T)$ such that*

$$\frac{1}{2}\int_s^t |\partial_t u|^2 + \frac{1}{2}\int_s^t |(f - \nabla_l \mathcal{S}(u))^\circ|^2 + \mathcal{S}(u(t)) \leqslant \mathcal{S}(u(s)) + \int_s^t \langle f, u\rangle_{\mathcal{H}} \quad \forall\, t \in (s,T),\ \forall\, s \in (0,T)\setminus\mathcal{N}.$$

5. Proofs of Theorems 1.1 and 1.2

We introduce the spaces

$$\mathcal{H} := H^{-1}_{(0)}(\Omega), \quad \widetilde{\mathcal{H}} := H^1_{(0)}(\Omega), \quad \mathcal{H}_0 := H^2(\Omega),$$

such that we find $\mathcal{H}_0 \hookrightarrow \widetilde{\mathcal{H}} \hookrightarrow L^2(\Omega) \hookrightarrow \mathcal{H}$ with all embeddings being dense and compact.

Definition 5.1. Let $\mathcal{S}\colon \mathcal{H} \to (-\infty, \infty]$ be given through (1.4) with $\mathcal{S}(u) := \infty$ for all $u \notin \widetilde{\mathcal{H}}$. Then we consider the restriction $\widetilde{\mathcal{S}} := \mathcal{S}\big|_{L^2}$ of $\mathcal{S}$ to $L^2(\Omega)$ and define the set valued L^2-subdifferentials $(\delta\mathcal{S}/\delta u)(u) \subset L^2(\Omega)$ and $(\delta^0\mathcal{S}/\delta u)(u) \subset L^2_{(0)}(\Omega)$ at $u \in D(\widetilde{\mathcal{S}})$ through:

$$u \in D(\widetilde{\mathcal{S}})\colon\ \delta \in \frac{\delta\mathcal{S}}{\delta u}(u) \Leftrightarrow \langle \delta, v\rangle_{L^2} \leqslant \lim_{h\searrow 0} \frac{\widetilde{\mathcal{S}}(u+hv) - \widetilde{\mathcal{S}}(u)}{h} \quad \forall\, v \in L^2(\Omega),$$

$$u \in D(\widetilde{\mathcal{S}}) \cap L^2_{(0)}(\Omega)\colon\ \delta \in \frac{\delta^0\mathcal{S}}{\delta u}(u) \Leftrightarrow \langle \delta, v\rangle_{L^2} \leqslant \lim_{h\searrow 0} \frac{\widetilde{\mathcal{S}}(u+hv) - \widetilde{\mathcal{S}}(u)}{h} \quad \forall\, v \in L^2_{(0)}(\Omega).$$

We only prove Theorem 1.1 starting with two lemmas by Abels and Wilke. Theorem 1.2 is proved likewise.

Lemma 5.2 ([1], Lemma 4.1, Corollary 4.4)**.** *Assume $s_1 \equiv 0$, then $\mathcal{S}\colon L^2_{(0)}(\Omega) \to \mathbb{R}$ and $\mathcal{S}\colon \mathcal{H} \to \mathbb{R}$ are proper, lower semicontinuous and convex.*

Abels and Wilke [1] identified the L^2- and $\mathcal{H}$- subdifferential of $\mathcal{S}$ in the Fréchet-sense:

Lemma 5.3 ([1])**.** *Assume $s_1 \equiv 0$ and set $s_0' = \infty$ for $x \notin (a,b)$. Then, for the L^2-subdifferential of $\mathcal{S}$ defined through* (1.4)*, we have*

$$D\Big(\frac{\delta^0\mathcal{S}}{\delta u}\Big) = \big\{c \in H^2(\Omega) \cap L^2_{(0)}(\Omega)\colon\ s'(c) \in L^2(\Omega), \\ s''(c)|\nabla c|^2 \in L^1(\Omega),\ \partial_n c\big|_{\partial\Omega} = 0\big\} \tag{5.1}$$

and

$$\frac{\delta^0\mathcal{S}}{\delta u}(\tilde{u}) = -\Delta\tilde{u} + P_0 s'(\tilde{u}). \tag{5.2}$$

Moreover,

$$\|\tilde{u}\|^2_{H^2} + \|s'(\tilde{u})\|^2_{L^2} + \int_\Omega s''(\tilde{u})|\nabla\tilde{u}|^2 \leqslant C\Big(\Big\|\frac{\delta^0\mathcal{S}}{\delta u}(\tilde{u})\Big\|^2_{L^2} + \|\tilde{u}\|^2_{L^2} + 1\Big) \tag{5.3}$$

for some constant C independent of $\tilde{u}$.

For the $\mathcal{H}$-subdifferential we have

$$D(\mathrm{d}\mathcal{S}) = \Big\{c \in D\Big(\frac{\delta^0\mathcal{S}}{\delta u}\Big)\colon\ \frac{\delta^0\mathcal{S}}{\delta u}(c) \in H^1_{(0)}(\Omega)\Big\}, \tag{5.4}$$

$$\mathrm{d}\mathcal{S}(\tilde{u}) = \Delta_N(-\Delta\tilde{u} + P_0 s'(\tilde{u})), \tag{5.5}$$

and in particular,

$$\|\tilde{u}\|^2_{H^2(\Omega)} \leqslant C(\|\mathrm{d}\mathcal{S}(\tilde{u})\|^2_{\mathcal{H}} + \|\tilde{u}\|^2_{L^2(\Omega)} + 1). \tag{5.6}$$

Note that the term $+1$ in (5.3) and (5.6) was not present in the original statements. As $\mathcal{S}$ in the setting of Lemma 5.3 is convex, the graph of $(\mathrm{d}\mathcal{S}, \mathcal{S})$ is strongly-weakly closed in the sense of (4.11). In particular, this implies the chain-rule condition (4.13) and convexity of $\mathrm{d}\mathcal{S}(u)$ for all $u \in D(\mathrm{d}\mathcal{S})$.

In case $s_1 \not\equiv 0$, $\mathcal{S}\colon \mathcal{H} \to \mathbb{R}$ remains lower semicontinuous and equations (5.1)–(5.5) still hold with modified constants. Finally, the following lemma holds:

Lemma 5.4. $\mathrm{d}\mathcal{S}$ *is single valued and strong-weak closed. In particular,* (4.13) *holds.*

Proof. It is easy to verify that $\mathrm{d}\mathcal{S}(u)$ is single valued for all $u \in D(\mathrm{d}\mathcal{S})$. For $u_n \to u$ strongly in $\mathcal{H}$ and $\xi_n = \mathrm{d}\mathcal{S}(u_n)$ such that $\xi_n \rightharpoonup \xi$ weakly in $\mathcal{H}$, note that due to the boundedness of the sequences u_n and ξ_n we find the boundedness of $\|u_n\|_{\mathcal{H}_0}$ and thus $u_n \rightharpoonup u$ weakly in $H^2(\Omega)$, $u_n \to u$ strongly in $H^1_{(0)}(\Omega)$ and $u_n \to u$ a.s. in Ω up to a subsequence. Furthermore, for $w_n := -\Delta u_n + P_0 s'(u_n)$ we find $w_n \rightharpoonup \omega$ weakly in $H^1_{(0)}(\Omega)$ for some $\omega \in H^1_{(0)}(\Omega)$.

Now, let

$$\widetilde{\mathcal{S}}(u) := \mathcal{S}(u) - \int_\Omega s_1(u).$$

Then $\widetilde{\mathcal{S}}(\cdot)$ is convex and therefore, the graph of $\mathrm{d}\widetilde{\mathcal{S}}$ is strongly weakly closed by Lemma 4.3. For a further subsequence and for $\zeta_n := \mathrm{d}\widetilde{\mathcal{S}}(u_n)$ we get weak convergence of $\zeta_n \rightharpoonup \zeta = \mathrm{d}\widetilde{\mathcal{S}}(u) = -\Delta u + P_0 s_0'(u)$ in $\mathcal{H}$ and $P_0(s_1'(u_n)) \to P_0(s_1'(u))$ strongly in L^2. Thus,

$$\xi_n = \zeta_n + s_1'(u_n) \rightharpoonup \zeta + P_0(s_1'(u)) = -\Delta u + P_0(s'(u))$$

weakly in $\widetilde{\mathcal{H}}$. This implies (4.11) and thus (4.13). $\square$

For $u \in \widetilde{\mathcal{H}}$, we define for $r_1, r_2 \in \mathcal{H}$:

$$\begin{aligned} g_u(r_1, r_2) &= \int_\Omega \nabla p_1^u A(u, \nabla u) \nabla p_2^u \\ &= \int_\Omega r_1 p_2^u = \langle r_1, p_2 \rangle_{H^{-1}_{(0)}, H^1_{(0)}} = \int_\Omega r_2 p_1^u = \langle r_2, p_1 \rangle_{H^{-1}_{(0)}, H^1_{(0)}}, \end{aligned} \tag{5.7}$$

where $p_i^u \in H^1_{(0)}(\Omega)$ solves

$$-\operatorname{div}(A(u, \nabla u) \nabla p_i^u) = r_i \quad \text{for } i = 1, 2, \tag{5.8}$$

with boundary condition $A(u,\nabla u)\nabla p_i^u \cdot \boldsymbol{n}_\Gamma = 0$. It is immediate to check that g is a densely defined metric in the sense of Definition 4.1.

The above considerations together with (4.9) yield that $\mathcal{S}$ fulfils all requirements of Definition 4.4. As a consequence of Theorem 4.5 we get the existence of a solution $u \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{H}_0)$ to (4.10) and it remains to reconstruct an expression of the form (4.1):

For any $u \in D(\mathcal{S})$, $r \in L^2(\Omega)$ with p from (5.8), $\gamma \in AC(0,T;L^2(\Omega))$ with $\gamma(0)=u$, $\gamma'(0)=r$ we use Lemma 5.4 and write

$$g_{\tilde{u}}(\nabla_{\tilde{u}}\mathcal{S}, r) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(\gamma(t))\Big|_0 = \int_\Omega \frac{\delta^0\mathcal{S}}{\delta u}(\tilde{u})r = \int -\operatorname{div}\Big(A(u,\nabla u)\nabla\frac{\delta^0\mathcal{S}}{\delta u}(\tilde{u})\Big)p$$

to obtain the specific form of (4.10). Equation (4.1) in the present setting reads (note that $g_u(\partial_t u, r_2) = \langle \partial_t u, p_2\rangle_{H^{-1}_{(0)},H^1_{(0)}}$)

$$\begin{aligned} &\partial_t u \in \operatorname{div}\left(A(u,\nabla u)\nabla\frac{\delta^0\mathcal{S}}{\delta u}(\tilde{u})\right) \\ \text{or}\quad &g_u(\partial_t u, \varphi) \in -\langle \mathrm{d}_l\mathcal{S}(u), \varphi\rangle_{\mathcal{H}} \quad \forall\, \varphi \in L^2(0,T;\mathcal{H}). \end{aligned} \tag{5.9}$$

Estimate (4.16) together with the above calculations yields (1.3). Theorem 1.2 can be proved similarly having in mind that the proof of Lemma 5.3 presented by Abels and Wilke [1] is the same for a closed surface Γ with $H^1_{(0)}(\Gamma)$ defined through (2.3).

6. Proof of Theorem 1.3

6.1. The entropy space. We introduce the space $\widetilde{V}$ through

$$\widetilde{V} := H^1_{(0)}(\Omega) \times L^2(\Gamma), \quad \|u = (u_\omega, u_\gamma)\|^2_{\widetilde{V}} := \|u_\omega\|^2_{H^1(\Omega)} + \|u_\gamma\|^2_{L^2(\Gamma)},$$

where in $L^2(\Gamma)$, we take the Hausdorff measure on Γ. Note that

$$V := \{u = (u_\omega, u_\gamma) \in \widetilde{V} \colon\ E_\Gamma(u_\omega) = u_\gamma\}$$

is a closed subspace of $\widetilde{V}$, being isomorphic with $H^1_{(0)}(\Omega)$ and with the equivalent norm (cf. Remark 2.4)

$$\|u = (u_\omega, u_\gamma)\|^2_V := \|\nabla u_\omega\|^2_{L^2(\Omega)} + \|u_\gamma\|^2_{L^2(\Gamma)}.$$

We furthermore introduce

$$\|(u_\omega, u_\gamma)\|^2_{H^1_\Gamma(\Omega)} := \int_\Omega |\nabla u_\omega|^2 + \int_\Gamma |\nabla_\Gamma u_\gamma|^2,$$
$$H^1_\Gamma := \overline{\{(u_\omega, u_\gamma) \in V \colon\ u_\omega \in H^2(\Omega)\}}^{\|\cdot\|_{H^1_\Gamma(\Omega)}}$$

and the dual space $H_\Gamma^* := (H_\Gamma^1)^{-1}$. For any function $v \in H^{-1}_{(0)}(\Omega)$ having the property that there is $\tilde v \in L^2_{(0)}(\Omega)$ with

$$\int_\Omega v\psi = \langle v, \psi\rangle_{H^{-1}_{(0)}, H^1_{(0)}} = \int_\Omega \tilde v \psi \quad \forall\, \psi \in H^1_{(0)}(\Omega)$$

we formally write $\tilde v = P_0(v)$. We finally introduce the space $H^1_\Delta(\Omega)$ through

$$\begin{aligned}\|(u_\omega, u_\gamma)\|^2_{H^1_\Delta} &:= \int_\Omega (P_0(\Delta u_\omega))^2 + \int_\Gamma (\partial_n u_\omega - \Delta_\Gamma u_\gamma)^2 + \|u\|^2_{H^1_\Gamma(\Omega)},\\ H^1_\Delta &:= \overline{\{(u_\omega, u_\gamma) \in H^1_\Gamma\colon\ u_\omega \in H^3(\Omega)\}}^{\|\cdot\|_{H^1_\Delta}}.\end{aligned}$$

Remark 6.1. Since for $u \in V$, $u \in H^1_\Gamma$ or $u \in H^1_\Delta$ we have $u_\gamma = E_\Gamma(u_\omega)$ like in (2.4), we will sometimes abuse notation and not distinguish between u_γ and $E_\Gamma(u_\omega)$, i.e. we will often write $u \simeq u_\gamma \simeq u_\omega$ whenever the meaning is clear from the context.

In what follows, we will say that $u \in H^2(\Omega)$ weakly solves the system

$$\begin{aligned}-\Delta u_\omega &= f \quad \text{in } \Omega,\\ -\Delta_\Gamma u_\gamma + \partial_n u &= g \quad \text{on } \Gamma\end{aligned}$$

if and only if it is a solution to the problem

$$\int_\Omega \nabla u\cdot\nabla\varphi + \int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma\varphi = \int_\Omega f\varphi + \int_\Gamma g\varphi \quad \forall\,\varphi \in H^2(\Omega). \tag{6.1}$$

In particular, we infer in case $g = 0$ for $\varphi \equiv 1$ that $\int_\Omega f = 0$.

Lemma 6.2.

$$\begin{aligned}H^1_\Delta &= \{u \in H^1_\Gamma\colon\ \Delta u \in L^2(\Omega),\ (\partial_n u_\omega - \Delta_\Gamma u_\gamma) \in L^2(\Gamma)\}\\ &= \{(u_\omega, u_\gamma) \in H^1_\Gamma\colon\ u_\omega \in H^2(\Omega),\ u_\gamma \in H^2(\Gamma)\}.\end{aligned}$$

Proof. We show for $(u_\omega, u_\gamma) \in H^1_\Gamma$ with $u_\omega \in H^3(\Omega)$ that there is $C > 0$ independent of u such that

$$\|\Delta u\|_{L^2} \leqslant C(\|P_0(\Delta u)\|_{L^2} + \|\partial_n u_\omega - \Delta_\Gamma u_\gamma\|_{L^2(\Gamma)} + \|u\|^2_{H^1_\Gamma(\Omega)}). \tag{6.2}$$

The major point of the following argumentation is that for P_0 given in (2.2), both Δu and $P_0(\Delta u)$ are functions in $L^2(\Omega)$ and the difference $P_0(\Delta u) - \Delta u = -\int_\Omega \Delta u$ is a constant function. We then indirectly show that this constant has to be 0.

Assume (6.2) is does not hold. Then there is a sequence of functions $(u_m)_{m\in\mathbb{N}} \subset H^1_\Gamma$, $u_m \in H^3(\Omega)$ for all m, such that

$$\|\Delta u_m\|_{L^2} = 1 \geqslant m(\|P_0(\Delta u_m)\|_{L^2} + \|\partial_n u_{m,\omega} - \Delta_\Gamma u_{m,\gamma}\|_{L^2(\Gamma)} + \|u_m\|^2_{H^1_\Gamma(\Omega)}).$$

We set $\tilde{f}_m := -P_0(\Delta u_m)$, $f_m := -\Delta u_m$, $g_m := \partial_n u_m - \Delta_\Gamma u_m$ and find $\tilde{f}_m \to 0$ strongly in $L^2(\Omega)$, $g_m \to 0$ strongly in $L^2(\Gamma)$, $u_m \to 0$ strongly in H^1_Γ and $f_m \rightharpoonup f$ weakly in $L^2(\Omega)$. With $h_m := \fint_\Omega f_m$ and (2.2) we find $f_m = \tilde{f}_m + h_m$. We equally consider h_m as constant functions and write formally

$$|h_m| = \fint_\Omega |h_m| = \frac{1}{|\Omega|^{1/2}}\|h_m\|_{L^2} \leqslant \frac{1}{|\Omega|^{1/2}}(\|\Delta u_m\|_{L^2} + \|P_0(\Delta u_m)\|_{L^2}).$$

Thus, there exists $h \in \mathbb{R}$ such that $h_m \to h$ in $\mathbb{R}$ for a subsequence. Since $|h_m|\,|\Omega|^{1/2} = \|h_m\|_{L^2} \geqslant \|\Delta u_m\| - \|P_0(\Delta u_m)\| \to 1$, we get $f_m = (\tilde{f}_m + h_m) \to \tilde{h} \neq 0$ strongly in $L^2(\Omega)$, where $\tilde{h}$ is the constant function with $\tilde{h} = h$ a.e. Note that due to regularity of u and the definitions above, for any m we have

$$\int_\Omega \nabla u_m \cdot \nabla\varphi + \int_\Gamma \nabla_\Gamma u_m \cdot \nabla_\Gamma \varphi = \int_\Omega f_m\varphi + \int_\Gamma g_m\varphi \quad \forall\,\varphi \in H^2(\Omega),$$

and thus, in the limit, u is a solution to

$$\int_\Omega f\varphi = \int_\Omega \Delta u\varphi = 0 \quad \forall\,\varphi \in H^2(\Omega).$$

By our conclusions drawn from (6.1), this implies $\Delta u = P_0(\Delta u)$, a contradiction with $\Delta u - P_0(\Delta u) = h \neq 0$.

Now, considering $u \in H^1_\Delta$ and any sequence $(u_m)_{m\in\mathbb{N}} \subset H^3(\Omega)$ such that $u_m \to u$ in $\|\cdot\|_{H^1_\Delta}$, we find $\Delta u \in L^2$. Since $\partial_n u_{m,\omega} - \Delta_\Gamma u_{m,\gamma} \to \tilde{f}$ for some $\tilde{f} \in L^2(\Gamma)$, we find for all sufficiently regular $\psi \in C^3_{(0)}(\overline{\Omega})$

$$\begin{aligned}
\int_\Gamma f\psi - \int_\Omega \psi\Delta u &= \lim_{m\to\infty}\left(\int_\Gamma (\partial_n u_{m,\omega} - \Delta_\Gamma u_{m,\gamma})\psi - \int_\Omega \psi\Delta u_m\right)\\
&= \lim_{m\to\infty}\left(\int_\Gamma (\nabla_\Gamma u_{m,\gamma})\cdot\nabla_\Gamma\psi + \int_\Omega \nabla\psi\cdot\nabla u_m\right)\\
&= \left(\int_\Gamma (\nabla_\Gamma u_\gamma)\cdot\nabla_\Gamma\psi + \int_\Omega \nabla\psi\cdot\nabla u\right)\\
&= \left(\int_\Gamma (\partial_n u_\omega - \Delta_\Gamma u_\gamma)\psi - \int_\Omega \psi\Delta u\right).
\end{aligned}$$

This is the first equality of the statement.

Let $u \in H^1_\Delta$ and set $f := -\Delta u$, $g := -\Delta_\Gamma u_\gamma + \partial_n u$. Assume $u \in C^\infty(\overline{\Omega})$ such that the following estimates hold:

$$\|u\|_{H^2(\Omega)} \leqslant C(\|f\|_{L^2(\Omega)} + \|u\|_{H^2(\Gamma)}), \tag{6.3}$$

$$\|u\|_{H^2(\Gamma)} \leqslant C(\|g\|_{L^2(\Gamma)} + \|\partial_n u\|_{L^2(\Gamma)}) \leqslant C(\|g\|_{L^2(\Gamma)} + \|u\|_{W_2^{3/2}(\Omega)}). \tag{6.4}$$

Ehrling's Lemma yields for every $\delta > 0$ a constant C_δ such that

$$\|u\|_{W_2^{3/2}(\Omega)} \leqslant \delta\|u\|_{H^2(\Omega)} + C_\delta\|u\|_{H^1}. \tag{6.5}$$

Combining (6.3)–(6.5), we get the second part of the lemma. □

In order to construct an entropy space in sense of Definition 4.1, we make the following choice of the triple of function spaces:

$$\mathcal{H}_0 := H^1_\Delta, \quad \widetilde{\mathcal{H}} := H^1_\Gamma, \quad \mathcal{H} := H^{-1}_{(0)}(\Omega) \times L^2(\Gamma).$$

With the additional space

$$\|u\|_\Gamma := \int_\Omega u_\omega^2 + \int_\Gamma u_\gamma^2, \quad L^2_\Gamma := L^2_{(0)}(\Omega) \times L^2(\Gamma)$$

the chain of dense embeddings $\mathcal{H}_0 \hookrightarrow \widetilde{\mathcal{H}} \hookrightarrow L^2_\Gamma \hookrightarrow \mathcal{H}$ holds with the first and second embedding being compact.

Corollary 6.3. *The triple* $(\mathcal{H}_0, \widetilde{\mathcal{H}}, \mathcal{H})$ *satisfies point* (1) *of Definition* 4.1.

Note that $\mathcal{H}^{-1} = H^1_{(0)}(\Omega) \times L^2(\Gamma)$ and on $\mathcal{H}$ we introduce the local scalar products

$$\begin{aligned} g_u(r_1, r_2) &:= \int_\Omega \nabla p^u_{1,\omega} A(u, \nabla u) \nabla p^u_{2,\omega} + \int_\Gamma p^u_{1,\gamma} A_\Gamma(u) p^u_{2,\gamma} = \langle r_1, p_2 \rangle_{\mathcal{H},\mathcal{H}^{-1}} \\ &= \int_\Omega r_{1,\omega} p^u_{2,\omega} + \int_\Gamma r_{1,\gamma} p^u_{2,\gamma} = \int_\Omega r_{2,\omega} p^u_{1,\omega} + \int_\Gamma r_{2,\gamma} p^u_{1,\gamma} \\ &= \langle r_2, p_1 \rangle_{\mathcal{H},\mathcal{H}^{-1}}, \end{aligned} \tag{6.6}$$

where $p^u_i = (p^u_{i,\omega}, p^u_{i,\gamma}) \in \mathcal{H}^{-1}$ satisfy the equations

$$\int_\Omega (A(u, \nabla u) \nabla p^u_{i,\omega}) \nabla \varphi_\omega + \int_\Gamma A_\Gamma(u) p^u_{i,\gamma} \varphi_\gamma = \langle r_i, \varphi \rangle_{\mathcal{H},\mathcal{H}^{-1}} \qquad \text{for } i = 1, 2 \text{ and } \forall\, \varphi \in \mathcal{H}^{-1}, \tag{6.7}$$

with the constraint

$$(A(u, \nabla u) \nabla p^u_{i,\omega}) \cdot \boldsymbol{n}_\Gamma = 0.$$

In other words, $p_i^u \in \mathcal{H}^{-1}$ solves

$$-\operatorname{div}(A(u,\nabla u)\nabla p_{i,\omega}^u) = r_{i,\omega} \quad \text{on } \Omega \quad \text{and} \quad A(u,\nabla u)\nabla p_{i,\omega}^u \cdot \boldsymbol{n}_\Gamma = 0 \quad \text{on } \Gamma,$$
$$A_\Gamma(u) p_{i,\gamma}^u = r_{i,\gamma} \quad \text{on } \Gamma.$$

Note that in general $p_{i,\gamma}^u \neq E_\Gamma(p_{i,\omega}^u)$.

Corollary 6.4. $g_\bullet\colon \widetilde{\mathcal{H}} \to B(\mathcal{H})$ *satisfies point* (2) *of Definition* 4.1.

Proof. For fixed r_2 consider $r_{1,m}$ and $p_{1,m} = (p_{1,m,\omega}, p_{1,m,\gamma})$, solutions of (6.7) for $r_{1,m}$, such that $r_{1,m} \rightharpoonup r_1$ weakly in $\mathcal{H}$ and $(u_m)_{m\in\mathbb{N}} \subset \widetilde{\mathcal{H}}$ with $u_m \to u$. We check that $p_{1,m} \rightharpoonup \tilde{p}_1$ and $\tilde{p}_1$ solves (6.7) for u and r_1. Thus, from the representation in (6.6), we conclude

$$g_{u_m}(r_{1,m}, r_2) \to g_u(r_1, r_2).$$

□

6.2. The entropy functional and existence of solutions. In this part, we shall rigorously use notation announced in Remark 6.1 for functions $u \in \widetilde{\mathcal{H}} = V$. Note that this notation is not applicable to L^2_Γ, $\mathcal{H}$ or $\mathcal{H}^{-1}$, which is why we still use full notation in those spaces.

Definition 6.5. Let $\mathcal{S}$ be a proper functional $\mathcal{S}\colon \mathcal{H} \to (-\infty, \infty]$. Then we consider the restriction $\widetilde{\mathcal{S}} := \mathcal{S}\big|_{L^2_\Gamma}$ of $\mathcal{S}$ to L^2_Γ and define the set valued L^2-subdifferential $(\delta_\Gamma \mathcal{S}/\delta u)(u) \subset L^2_\Gamma$ at $u \in D(\widetilde{\mathcal{S}})$ through

$$\delta \in \frac{\delta_\Gamma \mathcal{S}}{\delta u}(u) \Leftrightarrow \langle \delta, v\rangle_{L^2_\Gamma} \leqslant \liminf_{h\searrow 0} \frac{\widetilde{\mathcal{S}}(u+hv) - \widetilde{\mathcal{S}}(u)}{h} \quad \forall\, v \in L^2_\Gamma.$$

Remark 6.6. Comparing with Section 5, due to the Riesz isomorphism $-\Delta_N\colon H^1_{(0)}(\Omega) \to H^{-1}_{(0)}(\Omega)$, we find

$$\mathrm{d}\mathcal{S}(u) = \Big\{(s_\omega, s_\gamma)\colon\ (-\Delta_N^{-1} s_\omega, s_\gamma) \in \frac{\delta_\Gamma \mathcal{S}}{\delta u}(u)\Big\}. \tag{6.8}$$

We introduce the following functional on L^2_Γ or $\mathcal{H}$:

$$\mathcal{S}(u) := \begin{cases} \int_\Omega (s(u_\omega) + \frac{1}{2}|\nabla u_\omega|^2) + \int_{\partial\Omega} (s_\Gamma(u_\gamma) + \frac{1}{2}|\nabla_\Gamma u_\gamma|^2) & \text{for } u \in H^1_\Gamma(\Omega), \\ \infty \quad \text{otherwise} \end{cases} \tag{6.9}$$

with s, s_Γ as introduced in Subsection 1.3.

Lemma 6.7. *The functional $\mathcal{S}$ is lower semicontinuous on $\mathcal{H}$ and L^2_Γ. If $s_1 \equiv s_2 \equiv 0$, $\mathcal{S}$ is convex on both the spaces.*

P r o o f. If $s_1 \equiv s_2 \equiv 0$, convexity is trivial. Furthermore, for any sequence $u_n \in \mathcal{H}$ with a constant $C > 0$ such that $\mathcal{S}(u_n) < C$, we find u_n to be bounded in $\widetilde{\mathcal{H}}$, i.e. due to the particular structure of $s(\cdot)$, a short calculation yields

$$\mathcal{S}(u) \leqslant \liminf_{n\to\infty} \mathcal{S}(u_n).$$

In case $s_1, s_2 \not\equiv 0$, note that up to a minimizing subsequence $u_n \to u$ strongly in L^2_Γ and the statement follows from the Lipschitz continuity of s_1 and s_2. $\square$

Lemma 6.8. *Let $\mathcal{S}$ be given through* (6.9). *Then*

$$D\Big(\frac{\delta_\Gamma \mathcal{S}}{\delta u}\Big) = \{c \in H^1_\Gamma(\Omega)\colon\ s'(c) \in L^2_\Gamma,\ s''(c)|\nabla c|^2 \in L^1(\Omega),\ s''(c)|\nabla_\Gamma c|^2 \in L^1(\Gamma)\} \tag{6.10}$$

and

$$\|\tilde u\|_{H^1_\Delta(\Omega)} \leqslant C\Big(\Big\|\frac{\delta_\Gamma \mathcal{S}}{\delta u}(\tilde u)\Big\|^2_{L^2_\Gamma} + \|\tilde u\|^2_{L^2(\Omega)} + 1\Big). \tag{6.11}$$

Furthermore, $\tilde u \in D(\mathrm{d}\mathcal{S})$ implies $\frac{\delta \mathcal{S}}{\delta_\Gamma u}(\tilde u) \in H^1_{(0)} \times L^2(\Gamma)$,

$$\|\tilde u\|_{H^1_\Delta(\Omega)} \leqslant C(\|\,\mathrm{d}\mathcal{S}_\Gamma(\tilde u)\|^2_{\mathcal{H}} + \|\tilde u\|^2_{L^2(\Omega)} + 1), \tag{6.12}$$

and for any $u \in D(\mathcal{S})$, the L^2_Γ-subdifferential is given through

$$\begin{aligned}\Big\langle \frac{\delta_\Gamma \mathcal{S}}{\delta u}, \psi\Big\rangle_{L^2_\Gamma} &= \langle P_0(s'(u)), \psi_\omega\rangle_{L^2(\Omega)} - \langle P_0(\Delta u), \psi_\omega\rangle_{L^2(\Omega)} \\ &\quad + \langle \nabla u \cdot \boldsymbol{n}_\Gamma + s'_\Gamma(u) - \Delta_\Gamma u, \psi_\gamma\rangle_{L^2(\Gamma)}\end{aligned} \tag{6.13}$$

for all $\psi = (\psi_\omega, \psi_\gamma) \in L^2_\Gamma$.

We postpone the proof to Subsection 6.3.

R e m a r k 6.9. Thus, as the last lemma yields $\|u\|_{\mathcal{H}_0} \leqslant (\mathcal{S}(u) + 1 + \|\,\mathrm{d}\mathcal{S}\|_{\mathcal{H}})$, we have shown that $\mathcal{S}$ satisfies all claims of Definition 4.4.

The proof of the next lemma follows the proof of Lemma 5.4 and is left to the reader.

Lemma 6.10. $\mathrm{d}\mathcal{S}$ *is single valued and strong-weak closed.*

Now, for any $u \in D(\mathrm{d}\mathcal{S}_\Gamma)$, $r \in L^2_\Gamma$ with p from (6.7), $\gamma \in AC(0,T;L^2_\Gamma)$ with $\gamma(0) = \tilde{u}$, $\gamma'(0) = r$ we formally write

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(\gamma(t))\Big|_0 \geqslant \int_\Omega \Big(\frac{\delta_\Gamma \mathcal{S}_\Gamma}{\delta u}(u)\Big)_\omega r_\omega + \int_\Gamma \Big(\frac{\delta_\Gamma \mathcal{S}_\Gamma}{\delta u}(u)\Big)_\gamma r_\gamma.$$

In particular, the last inequality holds for $r \in \mathcal{H}$ and we thus find

$$\begin{aligned}\lim_{h\to 0}\frac{\mathcal{S}(u+hv)-\mathcal{S}(u)}{h} &\geqslant \int_\Omega -\operatorname{div}(A(u,\nabla u)\nabla(s_0'(u)-\Delta u))p_\omega \\ &\quad + \int_\Gamma \Big(\frac{\delta_\Gamma \mathcal{S}_\Gamma}{\delta u}(u)\Big)_\gamma A_\Gamma(u)p_\gamma = \langle \nabla\mathcal{S}, r\rangle_{g(u)},\end{aligned}$$

where p is the solution for r in (6.7). Similarly to Section 5 we deduce that the gradient flow (4.10) is equivalent to

$$\begin{aligned}\langle \partial_t u, p\rangle_{\mathcal{H},\mathcal{H}^{-1}} &= \int_\Omega \operatorname{div}(A(u,\nabla u)\nabla(s'(u)-\Delta u))p_\omega \\ &\quad - \int_\Gamma \Big(\frac{\delta_\Gamma \mathcal{S}_\Gamma}{\delta u}(u)\Big)_\gamma A_\Gamma(u)p_\gamma \quad \forall\, p \in L^2_\Gamma,\end{aligned} \tag{6.14}$$

or, as p_ω and p_γ are independent, the last equation is also equivalent to (1.6). Theorem 1.3 is then a consequence of Theorem 4.5.

R e m a r k 6.11. Even though $(\partial_t u)_\omega$ and $(\partial_t u)_\gamma$ are not directly related to each other, note that still the condition $u \in L^2(0,T;H^1(\Omega))$ relates the values on Γ with those in Ω.

6.3. Proof of Lemma 6.8. The proof mostly follows the lines of the proof of Theorem 4.3 in [1]. The idea is the following. For any $u \in D(\delta S/\delta u)$, we know that for $w \in (\delta S/\delta u)(u)$

$$\langle w, v\rangle_{L^2_\Gamma} \geqslant \lim_{t\to 0}\frac{1}{t}(\mathcal{S}(u)-\mathcal{S}(u+tv)) \quad \forall\, v \in L^2_\Gamma.$$

Supposing that all calculations involved are valid, we get

$$\lim_{t\to 0}\frac{1}{t}(\mathcal{S}(u)-\mathcal{S}(u+tv)) = \int_\Omega (s_0'(u)v) + \int_\Omega \nabla u\cdot\nabla v + \int_\Gamma s_0'(u)v + \int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma v. \tag{6.15}$$

However, we do not know if $s_0'(u) \in L^2_\Gamma$, but we have to prove this. The a priori estimates in (6.10) suggest to take $v = s_0'(u)$ as a test function in (6.15). This would lead to

$$\langle w, s_0'(u)\rangle_{L^2_\Gamma} \geqslant \|s_0'(u)\|_{L^2_\Gamma} + \int_\Omega s_0''(u)|\nabla u|^2 + \int_\Gamma s_0''(u)|\nabla_\Gamma u|^2$$

and thus give the apriori estimates. However, the singular behavior of s_0' in a and b makes it necessary to use a linearization of s_0' given by $f_n^{\pm}$ below. Also, we have to make sure that the values of the test function $u + tv$ lie in the interval (a, b) almost surely. This will be achieved by the correcting term $m(f_n^+(\tilde{u}_t))\psi_u$ below.

Now turning to the proof, let us recall $0 \in (a, b)$ and assume without loss of generality $s_0'(0) = s_0(0) = 0$ (shift s_0, s_1 and s_2 by affine functions) and define $s_0^+(x) := \max\{0, s_0(x)\}$, $s_0^-(x) := \min\{0, s_0(x)\}$. Furthermore, assume for the moment $s_1 \equiv s_2 \equiv 0$. Due to the assumptions on s_0, for any $n \in \mathbb{N}$ large enough there exist $a_n \in (a, \frac{1}{2}a)$ with $s_0'(a_n) = -n$ and $b_n \in (\frac{1}{2}b, b)$ with $s_0'(b_n) = n$ and we introduce the functions

$$f_n^+(u) := \begin{cases} s_0'(u) & \text{for } c \in (\frac{1}{2}b, b_n), \\ n + s_0''(b_n)(u - b_n) & \text{for } c \geqslant b_n, \\ 0 & \text{for } c \leqslant 0, \end{cases}$$

$$f_n^-(u) := \begin{cases} s_0'(u) & \text{for } c \in (a_n, \frac{1}{2}a), \\ n + s_0''(a_n)(u - a_n) & \text{for } c \leqslant a_n, \\ 0 & \text{for } c \leqslant 0, \end{cases}$$

and extend $f_n^+(\cdot)$, $f_n^-(\cdot)$ respectively to $(0, \frac{1}{2}b)$ and $(\frac{1}{2}a, 0)$, monotone and $C^2(\mathbb{R})$, so that they are approximating $(s_0^+)'$ and $(s_0^-)'$. Note that also $y \mapsto y + f_n^+(y)$ is strictly monotone and we introduce $M_n := \sup\limits_{c \in [a,b]} |f_n^+(u)'|$.

Now, let $u \in D(\mathcal{S}_\Gamma)$, i.e. $u \in H_\Gamma^1$ and $0 < t \leqslant 2/M_n$. By continuity and strict monotonicity we get unique existence of

$$\tilde{u}_t(x) = u(x) - t f_n^+(\tilde{u}_t(x))$$

and the theorem on the inverse function yields $\tilde{u}_t(x) = F_t^n(u(x))$, where $F_t^n \colon [a, b] \to [a, b]$ is a continuously monotone differentiable mapping with

$$F_t^n(x) \to x, \quad (F_t^n)'(x) \to 1, \quad \text{as } t \to 0 \text{ uniformly on } [a, b].$$

Thus, we see that for $u \in H_\Gamma^1(\Omega)$, also $\tilde{u}_t \in H^1(\Omega) \times L^2(\Gamma)$. Furthermore, the properties of F_t^n yield $\tilde{u}_t \to u$ in $H^1(\Omega) \times L^2(\Gamma)$ as $t \to 0$. Finally, monotonicity of $f_n^+(\cdot)$ yields $0 < \tilde{u}_t < u$ if $u > b_n$.

For $\varphi \in C^2(\mathbb{R})$ being monotone decreasing with $\varphi(x) = 1$ for $x < 0$, $\varphi(x) = 0$ for $\varphi > b/2$ and $\varphi' \geqslant -4/b$ define $\psi_u(x) := \varphi(u(x))/m(\varphi(u(x)))$, where $m(\varphi(u(x))) =$

$\fint_\Omega \varphi(u(x))$ such that

$$\int_\Omega \nabla\psi_u \cdot \nabla u = \int_\Omega \frac{\varphi'(u)}{m(\varphi(u))}|\nabla u|^2 \leqslant 0, \quad \int_\Gamma \nabla_\Gamma\psi_u \cdot \nabla_\Gamma u = \int_\Gamma \frac{\varphi'(u)}{m(\varphi(u))}|\nabla_\Gamma u|^2 \leqslant 0$$

and $u_t := \tilde{u}_t + tm(f_n^+(\tilde{u}_t))\psi_u \in H^1_\Gamma \cap D(\mathcal{S})$ for t small enough, i.e. $\int_\Omega u_t = 0$.

Thus, we can easily calculate

$$\begin{aligned}\mathcal{S}(u) - \mathcal{S}(u_t) \geqslant{}& \int_\Omega (s_0(u) - s_0(u_t)) + t\int_\Omega \nabla u \cdot \nabla f_n^+(u_t) - \frac{t^2 m(f_n^+(\tilde{u}_t))^2}{2}\int_\Omega |\nabla\psi_u|^2 \\ &+ \int_\Gamma (s_0(u) - s_0(u_t)) + t\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma f_n^+(u_t) - \frac{t^2 m(f_n^+(\tilde{u}_t))^2}{2}\int_\Gamma |\nabla\psi_u|^2.\end{aligned}$$

For the first part of the above expression we get

$$\begin{aligned}&\int_\Omega (s_0(u) - s_0(u_t)) + t\int_\Omega \nabla u \cdot \nabla f_n^+(u_t) - \frac{t^2 m(f_n^+(\tilde{u}_t))^2}{2}\int_\Omega |\nabla\psi_u|^2 \\ &\qquad\geqslant \int_{\Omega\cap\{u>b/2\}} ts_0'(u_t)f_n^+(u_t) + \int_{\Omega\cap\{a/2<u<b/2\}} (s_0(u) - s_0(\tilde{u}_t + d_n)) \\ &\qquad\quad - \frac{t^2 m(f_n^+(\tilde{u}_t))^2}{2}\int_\Omega |\nabla\psi_u|^2 \\ &\qquad\quad + \int_{\Omega\cap\{u<a/2\}} (s_0(u) - s_0(\tilde{u}_t + d_n)) + t\int_\Omega \nabla u \cdot \nabla f_n^+(u_t) \\ &\qquad\geqslant \int_{\Omega\cap\{u>b/2\}} ts_0'(u_t)f_n^+(u_t) + \int_{\Omega\cap\{a/2<u<b/2\}} (s_0(u) - s_0(\tilde{u}_t + d_n)) \\ &\qquad\quad + t\int_\Omega \nabla u \cdot \nabla f_n^+(u_t) - \frac{t^2 m(f_n^+(\tilde{u}_t))^2}{2}\int_\Omega |\nabla\psi_u|^2,\end{aligned}$$

where we have used $s_0(u(x)) - s_0(u_t(x)) \geqslant s_0'(u_t(x))(u(x) - u_t(x))$ and $u_t(x) < u(x)$ if $u(x) > b/2$, $s_0'(u(x)) \geqslant f_n^+(u_t(x))$ as well as $s_0(u(x)) - s_0(u(x) + td_n(x)) \geqslant 0$ if $u(x) \leqslant a/2$ and $t \leqslant a/(2M_n)$. We similarly conclude

$$\begin{aligned}&\int_\Gamma (s_0(u) - s_0(u_t)) + t\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma f_n^+(u_t) - \frac{t^2 m(f_n^+(\tilde{u}_t))^2}{2}\int_\Gamma |\nabla\psi_u|^2 \\ &\qquad\geqslant \int_{\Gamma\cap\{u>b/2\}} ts_0'(u_t)f_n^+(u_t) + \int_{\Gamma\cap\{a/2<u<b/2\}} (s_0(u) - s_0(\tilde{u}_t + d_n)) \\ &\qquad\quad + t\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma f_n^+(u_t) - \frac{t^2 m(f_n^+(\tilde{u}_t))^2}{2}\int_\Gamma |\nabla\psi_u|^2.\end{aligned}$$

Now, let $w \in (\delta_\Gamma \mathcal{S}/\delta u)(u)$, we then get by definition (note that $\mathcal{S}$ is convex in case $s_1 \equiv s_2 \equiv 0$)

$$\begin{aligned}
\langle w, f_n^+(\tilde{u}_t) - d_n\rangle_{L^2_\Gamma} &\geqslant \frac{1}{t}(\mathcal{S}(u) - \mathcal{S}(\tilde{u}_t)) \\
&= \int_{\Omega\cap\{u>b/2\}} s_0'(u_t) f_n^+(u_t) + \frac{1}{t}\int_{\Omega\cap\{a/2<u<b/2\}} (s_0(u) - s_0(\tilde{u}_t + d_n)) \\
&\quad + \int_\Omega \nabla u \cdot \nabla f_n^+(u_t) \\
&\quad + \int_{\Gamma\cap\{u>b/2\}} s_0'(u_t) f_n^+(u_t) + \frac{1}{t}\int_{\Gamma\cap\{a/2<u<b/2\}} (s_0(u) - s_0(\tilde{u}_t + d_n)) \\
&\quad + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma f_n^+(u_t) \\
&\quad - \frac{tm(f_n^+(\tilde{u}_t))^2}{2}\int_\Omega |\nabla\psi_u|^2 - \frac{tm(f_n^+(\tilde{u}_t))^2}{2}\int_\Gamma |\nabla\psi_u|^2,
\end{aligned}$$

which yields for $t \to 0$:

$$\begin{aligned}
\langle w, f_n^+(u) - d_n\rangle_{L^2_\Gamma} \geqslant &\int_{\Omega\cap\{u>b/2\}} s_0'(u) f_n^+(u) + \int_{\Omega\cap\{a/2<u<b/2\}} s_0'(u)(f_n^+(u) - d_n) \\
&+ \int_\Omega \nabla u \cdot \nabla f_n^+(u) \int_{\Gamma\cap\{u>b/2\}} s_0'(u) f_n^+(u) \\
&+ \int_{\Gamma\cap\{a/2<u<b/2\}} s_0'(u)(f_n^+(u) - d_n) + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma f_n^+(u)
\end{aligned}$$

and

$$\begin{aligned}
\langle w, f_n^+(u) - d_n\rangle_{L^2_\Gamma} \geqslant &\int_{\Omega\cap\{u>b/2\}} f_n^+(u)^2 + \int_{\Omega\cap\{a/2<u<b/2\}} s_0'(u)(f_n^+(u) - d_n) \\
&+ t\int_\Omega (f_n^+)'(u) \nabla u \cdot \nabla u_t \int_{\Gamma\cap\{u>b/2\}} f_n^+(u)^2 \\
&+ \int_{\Gamma\cap\{a/2<u<b/2\}} s_0'(u)(f_n^+(u) - d_n) + t\int_\Gamma (f_n^+)'(u) \nabla_\Gamma u \cdot \nabla_\Gamma u_t.
\end{aligned}$$

We make use of the simple estimate $\|m(f_n^+(u))\|_{L^2_\Gamma} \leqslant C\|f_n^+(u)\|_{L^2_\Gamma}$, following directly from the definition of $m(f_n^+(u))$, yielding for $n \to \infty$

$$\|w\|^2_{L^2_\Gamma} \gtrsim \int_\Omega (s_0^+)'(u)^2 + \int_\Omega (s_0^+)''(u)|\nabla u|^2 + \int_\Gamma (s_0^+)'(u)^2 + \int_\Gamma (s_0^+)''(u)|\nabla_\Gamma u|^2.$$

Together with a similar calculation for f_n^-, this yields the estimate (6.10). In particular, $s_0'(u) \in L^2(\Omega) \times L^2(\Gamma)$ implies $u \in (a, b)$ almost surely with respect to L^2_Γ.

Thus, we find for some $\delta > 0$ that $|\{x\colon u(x) \in (a+\delta, b-\delta)\}| > 0$ and for some non negative $\varphi \in C_0^\infty((a+\delta, b-\delta))$, with $\operatorname{spt}\varphi = [a+\delta, b-\delta]$, define $\varphi_u := \varphi(u(x))/m(\varphi(u(x)))$, being in $H^1(\Omega) \times L^2(\Gamma)$.

Now, let $M \in \mathbb{N}$ and $\psi_M \in C^\infty(\mathbb{R})$ be such that $\psi_M(x) = 0$ for $|x| > M+1$, $\psi_M(x) = 1$ for $|x| < M$ and $\psi_M'(x) \leqslant 2$ for all x. Note that by the properties of s_0, $u \in H^1_\Gamma$ implies $\chi_M := \psi_M(s_0'(u)) \in (H^1_\Gamma \oplus \mathbb{R})$ and $\chi_M = 0$ if $|s_0'(u)| > M+1$. Thus, for φ_u as above and any $\psi \in C^\infty_{(0)}(\overline{\Omega})$, $u \in D(\mathcal{S})$, we find some $t_0 > 0$ such that also $\tilde{u} := u + t\chi_M\psi - t\varphi_u m(\chi_M\psi) \in D(\mathcal{S})$ for all $0 < t < t_0$.

Thus, we find for $w \in (\delta_\Gamma\mathcal{S}/\delta u)(u)$

$$\begin{aligned}
\langle w, \chi_M\psi - \varphi_u m(\chi_M\psi)\rangle &\geqslant \lim_{t\to 0}\frac{1}{t}(\mathcal{S}(u) - \mathcal{S}(\tilde{u}))\\
&= \lim_{t\to 0}\Big(\int_\Omega \frac{1}{t}(s_0(u) - s_0(\tilde{u})) + \int_\Omega \nabla u\cdot\nabla(\chi_M\psi - \varphi_u m(\chi_M\psi))\Big)\\
&\quad + \lim_{t\to 0}\Big(\int_\Gamma \frac{1}{t}(s_0(u) - s_0(\tilde{u})) + \int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma(\chi_M\psi - \varphi_u m(\chi_M\psi))\Big)\\
&\geqslant \int_\Omega (s_0'(u)(\chi_M\psi - \varphi_u m(\chi_M\psi))) + \int_\Omega \nabla u\cdot\nabla(\chi_M\psi - \varphi_u m(\chi_M\psi))\\
&\quad + \int_\Gamma (s_0'(u)(\chi_M\psi - \varphi_u m(\chi_M\psi))) + \int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma(\chi_M\psi - \varphi_u m(\chi_M\psi)).
\end{aligned}$$

In order to investigate the behavior as $M\to\infty$, note that trivially $m(\chi_M\psi)) \to 0$ and $\chi_M \to 1$ pointwise and due to the boundedness by 1 also in $L^2(\Omega)\times L^2(\Gamma)$. Furthermore, as ψ_M' is bounded by 2 and $\psi_M'(s_0'(u)) \to 0$ pointwise for $M\to\infty$, it is straight forward to see

$$\begin{aligned}
\int_\Omega \nabla u\cdot\nabla(\psi_M(s_0'(u))\psi) &= \int_\Omega \nabla u\cdot(\chi_M\nabla\psi) + \int_\Omega s_0''(u)\psi_M'(s_0'(u))|\nabla u|^2\\
&\to \int_\Omega \nabla u\cdot\nabla\psi \quad \text{as } M\to\infty
\end{aligned}$$

and similarly for $\int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma(\psi_M(s_0'(u))\psi)$. Thus, we find

$$\langle w,\psi\rangle_{L^2_\Gamma} \geqslant \int_\Omega (s_0'(u)\psi) + \int_\Omega \nabla u\cdot\nabla\psi + \int_\Gamma s_0'(u)\psi + \int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma\psi.$$

Replacing ψ by $-\psi$, we find equality. Using partial integration, definition (6.1), and Lemma 6.2, we get

$$\langle w,\psi\rangle_{L^2_\Gamma} = \int_\Omega s_0'(u)\psi - \int_\Omega \Delta u\psi + \int_\Gamma (\nabla u\cdot\boldsymbol{n}_\Gamma + s_0'(u) - \Delta_\Gamma u)\psi$$

and hence $w_\omega = P_0(s_0'(u)) - P_0(\Delta u)$, $w_\gamma = (\nabla u \cdot \boldsymbol{n}_\Gamma + s_0'(u) - \Delta_\Gamma u)$ in the weak sense yielding (6.13) and $u \in \mathcal{H}_0$. Then (6.11) follows immediately from the calculation whereas (6.12) follows from (6.11) and (6.8).

It is elementary to verify that the statement still holds in case $s_1 \not\equiv s_2 \not\equiv 0$: To this aim, note that the domain $D(\mathrm{d}\mathcal{S})$ remains the same and that u is essentially bounded by $a < u < b$. In particular, calculating the $\delta_\Gamma/\delta u$-derivative of

$$\widehat{\mathcal{S}}(u) := \int_\Omega s_1(u) + \int_\Gamma s_2(Eu)$$

for $u \in D(\mathrm{d}\mathcal{S})$, it is easy to see that estimate (6.13) remains valid. Thus, having in mind the above estimates in case $s_1 \equiv s_2 \equiv 0$, it is easy to verify that (6.11) still holds.

7. Proof of Theorem 1.4

We will now prove Theorem 1.4 in four steps: First we will construct an approximate problem that can be directly solved using Theorem 4.5. Then we will show convergence of a subsequence of the approximate solutions as the approximation parameter tends to zero and demonstrate that the limit function solves the original problem. We then finally prove a technical lemma on the subdifferentials.

Sections 5 and 6 suggest that the correct choices for the three Hilbert spaces are

$$\mathcal{H} := H^{-1}_{(0)}(\Omega), \quad \widetilde{\mathcal{H}} := H^1_{(0)}(\Omega), \quad \mathcal{H}_0 := H^2(\Omega),$$

but in fact, we need a different choice. Note that one is tempted to directly consider the problem as a generalized gradient flow

$$\partial_t u = -\nabla \mathcal{S}(u),$$

where the gradient is with respect to the metric structure $g_\bullet(\cdot,\cdot)$ defined through

$$\begin{aligned} g_\bullet \colon \mathcal{H}_0 &\to B(\mathcal{H}), \\ u &\mapsto g_u(\cdot,\cdot) \end{aligned}$$

and using $w = -\Delta u + s'(u)$ we obtain

$$g_u(r_1, r_2) = \int_\Omega \nabla p_1^u A(u, \nabla u, w) \nabla p_2^u = \int_\Omega r_1 p_2^u = \int_\Omega r_2 p_1^u, \tag{7.1}$$

where p_i^u solves

$$-\operatorname{div}(A(u, \nabla u, w)\nabla p_i^u) = r_i \quad \text{for } i = 1, 2.$$

However, $g_\bullet$ then is defined on $\mathcal{H}_0$ instead of $\widetilde{\mathcal{H}}$ and Theorem 4.5 does not apply. Approximating the problem by a version that is smoothed in w and using a compactness property of w we will circumvent this problem.

The basic formal idea behind the following proof is to identify a set $A \subset L^2(0,T;\mathcal{H}_0)$ that is not compact in $L^2(0,T;\widetilde{\mathcal{H}})$ but still has sufficiently nice properties in order to guaranty (4.3) and (3.3).

7.1. An approximate problem. We start by considering the following problem: Like in Section 5, we choose

$$\mathcal{H}_0 := H^2(\Omega) \cap H^1_{(0)}(\Omega), \quad \widetilde{\mathcal{H}} := H^1_{(0)}, \quad \text{and} \quad \mathcal{H} = H^{-1}_{(0)}(\Omega).$$

We extend w to $\mathbb{R}^n$ by 0 and for any $\eta > 0$ we consider $w * \varphi_\eta$, where φ_η is the standard mollifier.

For any $u \in H^1_{(0)}(\Omega) \cap H^2(\Omega)$ we then consider the following scalar product on $\mathcal{H}$: for $r_1, r_2 \in \mathcal{H}$ we define

$$g^\eta_u(r_1, r_2) = \int_\Omega \nabla p^u_1 A(u, \nabla u, w * \varphi_\eta) \nabla p^u_2 = \int_\Omega r_1 p^u_2 = \int_\Omega r_2 p^u_1, \tag{7.2}$$

where p^u_i solves

$$-\operatorname{div}(A(u, \nabla u, w * \varphi_\eta) \nabla p^u_i) = r_i \quad \text{for } i = 1, 2. \tag{7.3}$$

It is immediate to check that g is a densely defined metric in the sense of Definition 4.1. For convenience of notation, we write the gradient with respect to g^η as ∇_η, i.e.

$$g^\eta_u(\nabla_\eta \mathcal{S}(u), \psi) = \langle \mathrm{d}\mathcal{S}(u), \psi \rangle_{\mathcal{H}} \quad \forall\, \psi \in \mathcal{H},$$

and denote by $\nabla_{\eta,l}$ the corresponding limiting subgradient with respect to ∇_η according to Definition 4.2.

This time, instead of Lemma 5.3, we consider

Lemma 7.1. *Let $\mathcal{S}$ and s be as introduced in Subsection* 1.4. *Then for the L^2-subdifferential we have*

$$D\Big(\frac{\delta^0 \mathcal{S}}{\delta u}\Big) = \big\{ c \in H^2(\Omega) \cap L^2_{(0)}(\Omega) \colon\ s'(c) \in L^2(\Omega), \\ s''(c)|\nabla c|^2 \in L^1(\Omega),\ \partial_n c\big|_{\partial\Omega} = 0 \big\} \tag{7.4}$$

and

$$\frac{\delta^0 \mathcal{S}}{\delta u}(\tilde{u}) = -\Delta \tilde{u} + P_0 s'(\tilde{u}). \tag{7.5}$$

Moreover,

$$\|\tilde u\|^2_{H^2(\Omega)}+\|s'(\tilde u)\|^2_{L^2(\Omega)}+\int_\Omega s''(\tilde u)|\nabla\tilde u|^2\leqslant C\Big(\Big\|\frac{\delta^0\mathcal{S}}{\delta u}(\tilde u)\Big\|^2_{L^2(\Omega)}+\|\tilde u\|^2_{L^2(\Omega)}+1\Big)\tag{7.6}$$

for some constant C independent of $\tilde u$.

For the $\mathcal{H}$-subdifferential we have

$$D(\mathrm{d}\mathcal{S})=\Big\{c\in D\Big(\frac{\delta^0\mathcal{S}}{\delta u}\Big)\colon\ \frac{\delta^0\mathcal{S}}{\delta u}(c)\in H^1_{(0)}(\Omega)\Big\},\tag{7.7}$$

$$\mathrm{d}\mathcal{S}(\tilde u)=\Delta(-\Delta\tilde u+P_0s'(\tilde u)),\tag{7.8}$$

i.e. $\mathrm{d}\mathcal{S}(\tilde u)$ *is single valued and*

$$\|\tilde u\|^2_{H^2(\Omega)}\leqslant C(\|\,\mathrm{d}\mathcal{S}(\tilde u)\|^2_{\mathcal{H}}+\|\tilde u\|^2_{L^2(\Omega)}+1).\tag{7.9}$$

Furthermore, we find:

Lemma 7.2. $\mathrm{d}\mathcal{S}$ *is strongly-weakly closed.*

Similar to Section 5, we observe that $g^\eta_\bullet$ and $\mathcal{S}$ satisfy all conditions of Theorem 4.5, so we get the existence of a solution $u_\eta\in H^1(0,T;\mathcal{H})\cap L^2(0,T;\mathcal{H}_0)$ to the equation

$$\int_0^T g^\eta_{u_\eta}(\partial_t u_\eta,\psi)=-\int_0^T\langle\mathrm{d}\mathcal{S}(u_\eta),\psi\rangle_{\mathcal{H}}\quad\forall\,\psi\in L^2(0,T;\mathcal{H}),\tag{7.10}$$

or

$$\partial_t u_\eta=-\nabla_\eta\mathcal{S}(u_\eta),\tag{7.11}$$

where $u(0)=u_0$ for $t=0$. This is a weak formulation to the problem

$$\begin{gathered}\partial_t u_\eta-\operatorname{div}(A(u_\eta,\nabla u_\eta,w_\eta*\varphi_\eta)\nabla w_\eta)\ni 0\quad\text{on }(0,T]\times U,\\ w_\eta+\Delta u_\eta-s'(u_\eta)=0\quad\text{on }(0,T]\times U,\\ (A(u_\eta,\nabla u_\eta,w_\eta*\varphi_\eta)\nabla w_\eta)\cdot\boldsymbol{n}_\Gamma=\nabla u_\eta\cdot\boldsymbol{n}_\Gamma=0\quad\text{on }(0,T]\times\partial U,\\ u_\eta(0)=u_0\quad\text{for }t=0.\end{gathered}$$

Note that the solution satisfies the a priori estimate

$$\begin{aligned}\frac12\int_0^t\|u'_\eta\|^2_{g^\eta(u_\eta)}+\frac12\int_0^t\|\nabla_{\eta,l}\mathcal{S}(u_\eta)\|^2_{g^\eta(u_\eta)}+\mathcal{S}(u_\eta(t))&\\ \leqslant\mathcal{S}(u(0))\quad\text{for a.e. }t\in(0,T).&\end{aligned}\tag{7.12}$$

However, we wish to study the behavior of solutions as $\eta \to 0$. In this context, note that we cannot decide whether $w_\eta * \varphi_\eta \to w$ in $L^2(0,T;L^2(\Omega))$ as we do not know whether $w_\eta \to w$ in $L^2(0,T;L^2(\Omega))$. (As w_η depends nonlinearly on u_η and s' is not Lipschitz in $\mathbb{R}$.)

7.2. Convergence of the approximate problem. It is thus necessary to repeat some of the steps in [17]. First, as $n \leqslant 3$, we find $\mathcal{H}_0 \hookrightarrow\hookrightarrow C(\overline{\Omega})$ compactly and thus $u_\eta \in L^2(0,T;C(\overline{\Omega}))$.

We find a subsequence $(u_{\eta_k})_{k\in\mathbb{N}}$ with $\eta_k \to 0$ as $k \to \infty$ such that there is $u \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{H}_0)$ with

$$\begin{aligned} u_{\eta_k} &\rightharpoonup u \quad \text{weakly in } H^1(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{H}_0),\\ u_{\eta_k} &\to u \quad \text{strongly in } L^2(0,T;\widetilde{\mathcal{H}}) \cap L^2(0,T;C(\overline{\Omega})),\\ u_{\eta_k}(t) &\to u(t) \quad \text{in } C(\overline{\Omega}) \cap H^1(\Omega) \text{ for a.e. } t \in (0,T). \end{aligned}$$

Now, let $\varepsilon > 0$. By Egorov's theorem, there is a compact set $K_0 \subset (0,T)$ with $\mathcal{L}((0,T) \setminus K_0) < \varepsilon/2$ such that uniformly for all $t \in K_0$ we find $u_{\eta_k}(t) \to u(t)$ strongly in $C(\overline{\Omega}) \cap H^1(\Omega)$. For each $k \in \mathbb{N}\setminus\{0\}$, Lusin's theorem yields the existence of a compact set $K_k \subset (0,T)$ with $\mathcal{L}((0,T)\setminus K_k) \leqslant 2^{-k-1}\varepsilon$ and $u_{\eta_k} \in C(K_k;C(\overline{\Omega}))$. Defining $K_\varepsilon := \bigcap\limits_{k=0}^{\infty} K_k$, we find $\mathcal{L}((0,T)\setminus K_\varepsilon) \leqslant \varepsilon$, $u_{\eta_k} \in C(K_\varepsilon;C(\overline{\Omega}))$ for all k and by the pointwise convergence also $u_{\eta_k} \to u$ uniformly in $C(K_\varepsilon;C(\overline{\Omega}))$ and strongly in $L^2(0,T;H^1(\Omega))$. In particular, we find $|u(t,x)| \leqslant C_\varepsilon$, $|u_{\eta_k}(t,x)| \leqslant C_\varepsilon$ for all k for some constant $C_\varepsilon > 0$ for all $(t,x) \in K \times \overline{\Omega}$. Now, it is evident that $s_0'(u_{\eta_k}) \to s_0'(u)$ strongly in $L^2(K_\varepsilon;L^2(\Omega))$as well as $\Delta u_{\eta_k} \rightharpoonup \Delta u$ weakly in $L^2(0,T;L^2(\Omega))$, implying $w_{\eta_k} \rightharpoonup w = -\Delta u + s_0(u)$ weakly in $L^2(K_\varepsilon;H^1_{(0)}(\Omega))$.

Thus, we may perform the following calculation:

$$\begin{aligned} \lim_{k\to\infty}\int_{K_\varepsilon}\int_\Omega w_{\eta_k}^2 &= -\lim_{k\to\infty}\int_{K_\varepsilon}\int_\Omega \Delta u_{\eta_k} w_{\eta_k} + \lim_{k\to\infty}\int_{K_\varepsilon}\int_\Omega s'(u_{\eta_k})w_{\eta_k}\\ &= \lim_{k\to\infty}\int_{K_\varepsilon}\int_\Omega \nabla u_{\eta_k}\nabla w_{\eta_k} + \lim_{k\to\infty}\int_{K_\varepsilon}\int_\Omega s'(u_{\eta_k})w_{\eta_k}\\ &= \int_{K_\varepsilon}\int_\Omega \nabla u\nabla w + \int_{K_\varepsilon}\int_\Omega s'(u)w\\ &= \int_{K_\varepsilon}\int_\Omega w^2, \end{aligned}$$

where we have used boundedness of u_{η_k} to get local Lipschitz continuity of $s'(\cdot)$. In particular, we find for fixed ε a further subsequence $w^\varepsilon_{\eta_k}$ such that $w^\varepsilon_{\eta_k}(t) \to w^\varepsilon(t)$ in $L^2(\Omega)$ for a.e. $t \in K^\varepsilon$. A standard diagonalization argument yields the existence of a subsequence such that $w_{\eta_k}(t) \to w(t)$ in $L^2(\Omega)$ for a.e. $t \in (0,T)$.

We consider the space $\widehat{\mathcal{H}} := H^1_{(0)}(\Omega) \times L^2(\Omega)$ and

$$\begin{aligned} \hat{g}_\bullet \colon \widehat{\mathcal{H}} &\to B(\mathcal{H}), \\ (u,w) &\mapsto \hat{g}_{(u,w)}(\cdot,\cdot), \end{aligned}$$

where

$$\hat{g}_{u,w}(r_1,r_2) = \int_\Omega \nabla p_1^u A(u,\nabla u, w) \nabla p_2^u = \int_\Omega r_1 p_2^u = \int_\Omega r_2 p_1^u$$

and p_i^u solves

$$-\operatorname{div}(A(u,\nabla u,w)\nabla p_i^u) = r_i \quad \text{for } i = 1,2,$$

and we immediately check with (7.1) and (7.2) that

$$g_u^\eta(\cdot,\cdot) = \hat{g}_{(u,w*\varphi_\eta)}(\cdot,\cdot), \quad g_u(\cdot,\cdot) = \hat{g}_{(u,w)}(\cdot,\cdot).$$

We find with the above estimates and Theorem 3.5 two Young measures $\boldsymbol{\mu},\boldsymbol{\nu} \in \mathcal{Y}(0,T;\mathcal{H})$ associated with u'_{η_k} and $\nabla_{\eta_k}\mathcal{S}(u_{\eta_k})$ such that $u'_{\eta_k} \rightharpoonup \int_{\mathcal{H}} \xi \,\mathrm{d}\mu_t(\xi)$ and $\nabla_{\eta_k}\mathcal{S}(u_{\eta_k}) \rightharpoonup \int_{\mathcal{H}} \xi \,\mathrm{d}\nu_t(\xi)$ weakly in $L^2(0,T;\mathcal{H})$. Our final aim is now to identify the sets of concentration of $\boldsymbol{\mu},\boldsymbol{\nu}$:

We find with help of Theorem 3.5 and Corollary 3.4 that

$$\begin{aligned} \liminf_{k\to\infty} \int_0^T \hat{g}_{(u_{\eta_k},w_{\eta_k})}(\partial_t u_{\eta_k}, \partial_t u_{\eta_k}) &\geqslant \int_0^T\!\!\int_{\mathcal{H}} \hat{g}_{(u,w)}(\xi,\xi)\,\mathrm{d}\mu_t(\xi) \quad \text{and} \\ \liminf_{k\to\infty} \int_0^T (\nabla_{\eta_k}\mathcal{S}(u_{\eta_k}))^2 &\geqslant \int_0^T\!\!\int_{\mathcal{H}} \hat{g}_{(u,w)}(\xi,\xi)\,\mathrm{d}\nu_t(\xi). \end{aligned}$$

Also, with help of (7.11) as well as Corollary 7.3 below, arguing as in the proof of Theorem 4.5 in [17], we find that μ_t, ν_t are concentrated on $(\tilde{\hat{g}}_{u,w})^{-1}(\mathrm{d}_l\mathcal{S}(u)) = \tilde{g}_u^{-1}(\mathrm{d}_l\mathcal{S}(u))$ for $t \in K_\varepsilon$ for all $\varepsilon > 0$. As $\mathrm{d}_l\mathcal{S}(u)$ is convex for all u and ε was arbitrary, the theorem is proved. $\square$

Corollary 7.3 ([17]). *For a bounded sequence $\varphi_n \in \mathcal{H}$ and $u_n \to u$ strongly in $\widetilde{\mathcal{H}}$, we have $\varphi_n \rightharpoonup \varphi$ weakly in $\mathcal{H}$ if and only if $\tilde{g}_{u_n}(\varphi_n) \rightharpoonup \tilde{g}_u(\varphi)$ weakly in $\mathcal{H}$, where $\tilde{g}_u$ is defined through* (4.4).

7.3. Proof of Lemma 7.1. The proof is similar to Subsection 6.3: This time, $s_0'(0) = s_0(0) = 0$ and we define $s_0^+(x) := \max\{0, s_0(x)\}$, $s_0^-(x) := \min\{0, s_0(x)\}$. For $a_0 \in (s_0')^{-1}(-1/2)$, $b_0 \in (s_0')^{-1}(1/2)$, there are for any $n \in \mathbb{N}$ $a_n \in (-\infty, a_0)$ with $s_0'(a_n) = -n$ and $b_n \in (b_0,\infty)$ with $s_0'(b_n) = n$ and we introduce f_n^+ and f_n^- similarly to Subsection 6.3, so that both $f_n^+(\cdot)$, $f_n^-(\cdot)$ are monotone and $C^2(\mathbb{R})$ with $y \mapsto y + f_n^+(y)$ being strictly monotone and $C^2(\mathbb{R}^n)$, too.

Now, let $u \in D(\mathcal{S})$, i.e. $u \in H^1_\Gamma$ and define $\tilde{u}_t := u - f_n^+(\tilde{u}_t)$

$$u_t := \tilde{u}_t + tm(f_n^+(\tilde{u}_t))/\mathcal{L}^n(\Omega) \in H^1_\Gamma \cap D(\mathcal{S})$$

for t small enough.

Using the notation $d_n := m(f_n^+(\tilde{u}_t))/\mathcal{L}^n(\Omega)$ and following the outline of Section 6.3 or the proof of Theorem 4.3 in [1], we calculate for $w \in (\delta_\Gamma \mathcal{S}/\delta u)(u)$

$$\begin{aligned}
&\langle w, f_n^+(\tilde{u}_t) - d_n\rangle_{L^2_\Gamma} \geqslant \frac{1}{t}(\mathcal{S}(u) - \mathcal{S}(u_t)) \\
&\geqslant \int_{\Omega\cap\{u>b/2\}} f_n^+(\tilde{u}_t)^2 + \frac{1}{t}\int_{\Omega\cap\{a/2<u<b/2\}} (s_0(u) - s_0(\tilde{u}_t + d_n)) + \int_\Omega \nabla u \cdot \nabla f_n^+(u_t),
\end{aligned}$$

which yields for $t \to 0$

$$\begin{aligned}
\langle w, f_n^+(u) - d_n\rangle_{L^2_\Gamma} \geqslant &\int_{\Omega\cap\{u>b/2\}} f_n^+(u)^2 \\
&+ \int_{\Omega\cap\{a/2<u<b/2\}} s_0'(u) f_n^+(u) + t\int_\Omega (f_n^+)'(u_t)\nabla u \cdot \nabla u_t
\end{aligned}$$

and for $n \to \infty$ by monotone convergence together with a similar calculation for f_n^- we get

$$1 + \|w\|^2_{L^2_\Gamma} \gtrsim \int_\Omega s_0'(u)^2 + \int_\Omega s_0''(u)|\nabla u|^2,$$

which is (6.10).

We find for any $\psi \in C^\infty_{(0)}(\overline{\Omega})$ and $u \in D(\mathrm{d}\mathcal{S})$ some $t_0 > 0$ such that $\tilde{u} := u + t\psi \in D(\mathcal{S})$ for all $0 < t < t_0$.

Thus, for $w \in (\delta^0 \mathcal{S}/\delta u)(u)$ we find

$$\begin{aligned}
\langle w, \psi\rangle &\geqslant \lim_{t\to 0} \frac{1}{t}(\mathcal{S}(u) - \mathcal{S}(\tilde{u})) \\
&= \lim_{t\to 0}\left(\int_\Omega \frac{1}{t}\,(s_0(u) - s_0(\tilde{u})) + \int_\Omega \nabla u \cdot \nabla \psi\right) \geqslant \int_\Omega s_0'(u)\psi + \int_\Omega \nabla u \cdot \nabla \psi.
\end{aligned}$$

Replacing ψ by $-\psi$, we find equality. Using partial integration, we get

$$\langle w, \psi\rangle_{L^2_\Gamma} = \int_\Omega P_0(s_0'(u))\psi - \int_\Omega \Delta u \psi \quad \forall\, \psi \in C^\infty(\overline{\Omega}),$$

and hence, the standard theory of elliptic equations tells us that u solves $w_\omega - P_0(s_0'(u)) = -\Delta u$ with $\partial_\nu u = 0$, implying $u \in H^2(\Omega)$ and $\|u\|_{H^2(\Omega)} \leqslant C\|w\|_{L^2}$ (see also Abels and Wilke [1], Section 2).

If $s_1 \equiv 0$, $\mathcal{S}$ is convex and the graph of $(\mathrm{d}\mathcal{S}, \mathcal{S})$ is strongly-weakly closed in the sense of (4.11), and $\mathrm{d}\mathcal{S}(u)$ is single valued for all $u \in D(\mathrm{d}\mathcal{S})$. These properties remain even in the case $s_1 \not\equiv 0$, since $(\delta/\delta u)(s_1(u)) = s_1'(u)$, whereas the subdifferentials remain in the form (7.5) and (7.8).

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