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Applications of Mathematics, Vol. 59 (2014), No. 6, 715–728

Persistent URL: `http://dml.cz/dmlcz/143996`

DETERMINATION OF THE UNKNOWN SOURCE TERM IN A LINEAR PARABOLIC PROBLEM FROM THE MEASURED DATA AT THE FINAL TIME

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(Received March 19, 2013)

Abstract. The problem of determining the source term $F(x,t)$ in the linear parabolic equation $u_t = (k(x)u_x(x,t))_x + F(x,t)$ from the measured data at the final time $u(x,T) = \mu(x)$ is formulated. It is proved that the Fréchet derivative of the cost functional $J(F) = \|\mu_T(x) - u(x,T)\|_0^2$ can be formulated via the solution of the adjoint parabolic problem. Lipschitz continuity of the gradient is proved. An existence result for a quasi solution of the considered inverse problem is proved. A monotone iteration scheme is obtained based on the gradient method. Convergence rate is proved.

Keywords: inverse parabolic problem; unknown source; adjoint problem; Fréchet derivative; Lipschitz continuity

MSC 2010: 35R30

1. Introduction

We study the inverse source problem associated with the following linear parabolic problem:

$$
\begin{cases}
u_t = (k(x)u_x)_x + F(x,t), \quad (x,t) \in \Omega_T := (0,l) \times (0,T], \\
u(x,0) = \mu_0(x), \quad x \in (0,l), \\
u(0,t) = 0, \quad u(l,t) = 0, \quad t \in (0,T].
\end{cases} \tag{1.1}
$$

The inverse problem we investigate consists of determining the source term $F(x,t)$ from the measured data at the final time

$$
u(x,T) = \mu(x). \tag{1.2}
$$

The function $\mu(x)$ is assumed to be the measured output data and the fuction F will be defined as the input data. The inverse source problem (1.1)–(1.2) will be referred as the problem (ISP). In this context for a given $F(x,t)$, the parabolic problem (1.1) will be referred to as the direct problem.

For the Robin boundary conditions $u_x(0,t) = 0$, $-k(l)u_x(l,t) = \nu[u(l,t) - T_0(t)]$ the problem of simultaneous determination of the unknown pair $\langle F(x,t), T_0(t)\rangle$ of sources has been considered in [4]. Note that the mathematical model (1.1)–(1.2) arises in various physical and engineering problems, see [1], [2], [4], [5], [7], [9], [11], [12] and references there in.

In this paper, based on the methodology given in [4], [5], we will apply the adjoint problem approach given in [3], [6] to (ISP) (1.1)–(1.2). To this aim, the auxiliary functional

$$J(F) = \int_0^l [u(x,T;F) - \mu(x)]^2\,\mathrm{d}x \tag{1.3}$$

is introduced and (ISP) is reformulated as a minimization problem for this functional. It is shown that the gradient $J'(F)$ of the cost functional (1.3) is Lipschitz continuous. Moreover, an explicit formula for this gradient is obtained via the solution of the corresponding adjoint problem. Based on these results, monotonicity of the sequence $\{J(F^{(n)})\}$ where $F^{(n)}$ is the sequence of iterations obtained by the gradient method, is proved. The paper is organized as follows. In Section 2, we define a quasi-solution of the inverse source problem (1.1)–(1.2), based on the weak solution of the direct problem (1.1). In Section 3, we introduce an adjoint parabolic problem and prove an explicit relationship between the weak solution of this problem and the gradient of the cost functional (1.3). Lipschitz continuity of the gradient is obtained in Section 4. This permits to construct a gradient type iteration process for the sequence of approximate solutions $F^{(n)} \subset \mathcal{F}$ of the inverse problem and prove monotonicity of the sequence of functionals $\{J(F^{(n)}\} \subset \mathbb{R}_+$. In Section 5, convexity of the Fréchet derivative is studied.

2. Quasi-solution of the inverse problem and the gradient

Let us denote by $\mathcal{F} := \{F \in H^0(\Omega_T)\colon 0 \leqslant m_* \leqslant F(x,t) \leqslant m^* < \infty$ a.e., $(x,t) \in \Omega_T\}$, where $H^0 \equiv L_2$, the set of admissible unknown sources F. Evidently, the set $\mathcal{F}$ is a closed convex set in $H^0(\Omega_T)$. We will assume that

$$k(x) \in L_\infty[0,l], \quad 0 < c_0 \leqslant k(x) \leqslant c_1, \quad \mu(x) \in H^0[0,l]. \tag{2.1}$$

The weak solution of the direct problem (1.1) will be defined as the function $u \in V^{1,0}$ satisfying the integral identity

$$-\iint_{\Omega_T} (uv_t - ku_x v_x)\,\mathrm{d}x\,\mathrm{d}t = \iint_{\Omega_T} Fv\,\mathrm{d}x\,\mathrm{d}t \quad \forall\, v \in^0 V^{1,0}(\Omega_T), \tag{2.2}$$

where $V^{1,0}(\Omega_T)$ is the Banach space of functions (see [11]) with the norm

$$\|u\|_{V^{1,0}} := \sup_{t\in[0,T]} |v(x,t)| + \|v_x\|_{H^0(\Omega_T)}$$

and ${}^0V^{1,0}(\Omega_T) = \{v \in V^{1,0}(\Omega_T)\colon\ v(0) = v(T) = 0\}$. Under the above conditions with respect to the given data, the weak solution $u \in V^{1,0}(\Omega_T)$ of the direct problem (1.1) exists and unique [10].

We denote the solution of the parabolic problem (1.1) by $u(x,t;F)$ corresponding to a given $F \in \mathcal{F}$. If this function satisfies the additional condition (1.2), then it must satisfy the nonlinear equation

$$u(x,t;F)|_{t=T} = \mu(x), \quad x \in (0,l). \tag{2.3}$$

However, in practice the measured data $\mu(x)$ is usually given with some measurement errors and the exact fulfilment of the condition (1.2) may not be possible. For this reason, we define a quasi-solution of the inverse problem as a solution of the minimization problem for the cost functional $J(F)$, given by (1.3):

$$J(F_*) = \inf_{F\in\mathcal{F}} J(F). \tag{2.4}$$

Clearly, if $J(F_*) = 0$, then the quasi-solution $F_* \in \mathcal{F}$ is also a strict solution of the inverse problem (1.1)–(1.2), since $F_* \in \mathcal{F}$ satisfies the functional equation (2.3). Further, in view of the weak solution theory for parabolic problems, one can prove that if the sequence $\{F^{(n)}\} \subset \mathcal{F}$ weakly converges to the function $F \in \mathcal{F}$, then the sequence of traces $\{u(x,T;F^{(n)})\}$ of the corresponding solutions of the direct problem (1.1) converges in the H^0-norm to the solution $\{u(x,T;F)\}$, which means $J(F^{(n)}) \to J(F)$, as $n \to \infty$. This means that the functional $J(F)$ is weakly continuous on $\mathcal{F}$, hence due to the Weierstrass existence theorem [13], the set of solutions

$$\mathcal{F}_* := \{F \in \mathcal{F}\colon\ J(F_*) = J_* = \inf_{F\in\mathcal{F}} J(F)\}$$

of the minimization problem (2.4) is not an empty set.

Let F and $F+\Delta F \in \mathcal{F}$ be source functions. We denote by $u(x,t;F)$ and $u(x,t,F+\Delta F)$ the corresponding solutions of the problem (1.1). Then $\Delta u(x,t;F)$ is the solution of the parabolic problem

$$\tag{3.1} \begin{cases} \Delta u_t = (k(x)\Delta u_x)_x + \Delta F(x,t), & (x,t)\in\Omega_T,\\ \Delta u(x,0)=0, \quad x\in(0,l), \\ \Delta u(0,t)=0, \quad \Delta u(l,t)=0, \quad t\in(0,T]. \end{cases}$$

The first variation $\Delta J(F)$ of the cost functional $J(F)$ is

$$\tag{3.2} \begin{aligned} \Delta J(F) &:= J(F+\Delta F) - J(F) \\ &= 2\int_0^l [u(x,t;F)-\mu(x)]\Delta u(x,T;F)]\,\mathrm{d}x + \int_0^l [\Delta u(x,T;F)]^2\,\mathrm{d}x, \end{aligned}$$

where $\Delta u(x,t;F)$ is the solution of (3.1).

Lemma 3.1. *Let F, $F+\Delta F$ be given elements. If $u=u(x,t;F)\in V^{1,0}(\Omega_T)$ is the corresponding solution of the direct problem* (1.1) *and $\psi(x,t;F)\in V^{1,0}(\Omega_T)$ is the solution of the backward parabolic problem*

$$\tag{3.3} \begin{cases} \psi_t = -(k(x)\psi_x)_x, & (x,t)\in\Omega_T,\\ \psi(x,T)=p(x), \quad x\in(0,l), \\ \psi(0,t)=0, \quad \psi(l,t)=0, \quad t\in(0,T], \end{cases}$$

then for all $F\in\mathcal{F}$ the following integral identity holds:

$$\tag{3.4} \int_0^l p(x)\Delta u(x,T;F)\,\mathrm{d}x = \iint_{\Omega_T} \Delta F(x,t)\psi(x,t;F)\,\mathrm{d}x\,\mathrm{d}t, \quad F\in\mathcal{F},$$

with an arbitrary data $p(x)\in H^0(0,l)$.

P r o o f. We multiply (3.1) by ψ and integrate over Ω_T to get

$$\tag{3.5} \begin{aligned} &\iint_{\Omega_T} \Delta u_t(x,t)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \iint_{\Omega_T} (k(x)\Delta u_x(x,t))_x\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega_T} \Delta F(x,t)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Applying integration by parts, the integral on the left hand side of (3.5) becomes

$$\tag{3.6} \begin{aligned} &\iint_{\Omega_T} \Delta u_t(x,t)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = -\iint_{\Omega_T} \Delta u(x,t)\psi_t(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^l \Delta u(x,T)\psi(x,T)\,\mathrm{d}x. \end{aligned}$$

The first integral on the right-hand side of (3.5) is

$$\begin{aligned}(3.7)\qquad \iint_{\Omega_T} &(k(x)\Delta u_x(x,t))_x \psi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T [k(l)\Delta u_x(l,t)\psi(l,t) - k(0)\Delta u_x(0,t)\psi(0,t)]\,\mathrm{d}t \\ &\quad - \iint_{\Omega_T} k(x)\Delta u_x(x,t)\psi_x(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\iint_{\Omega_T} k(x)\Delta u_x(x,t)\psi_x(x,t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The last integral in (3.7) can be calculated as

$$\begin{aligned}(3.8)\qquad -\iint_{\Omega_T} &k(x)\Delta u_x(x,t)\psi_x(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T [k(l)\psi_x(l,t)\Delta u(l,t) - k(0)\psi_x(0,t)\Delta u(0,t)]\,\mathrm{d}t \\ &\quad + \iint_{\Omega_T} (k(x)\psi_x(x,t))_x \Delta u(x,t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Using (3.8), (3.7) and (3.6) in (12), we get (cf. (3.1))

$$\begin{aligned}(3.9)\qquad \int_0^l &\Delta u(x,T)\psi(x,T)\,\mathrm{d}x \\ &= \iint_{\Omega_T} (\psi_t + (k\psi_x)_x)\Delta u\,\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega_T} \Delta F(x,t)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since $\psi(x,T) = p(x)$, thanks to (3.3), (3.9) we complete the proof. $\square$

Substituting in (3.4) $p(x) = 2[u(x,T) - \mu(x)]$, we obtain the integral identity

$$\begin{aligned}(3.10)\qquad 2\int_0^l &[u(x,T;F) - \mu(x)]\Delta u(x,T;F)\,\mathrm{d}x \\ &= \iint_{\Omega_T} \Delta F(x,t)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t \quad \forall\, F \in \mathcal{F}.\end{aligned}$$

Corollary 3.1. *Let us choose an arbitrary control function* $p(x)$ *in* (3.4) *as* $p(x) = \Delta u(x,T;F)/\|\Delta u(x,T;F)\|_{H^0(0,l)}$ *assuming that* $\Delta u(x,T;F) = u(x,T;F_1) - u(x,T;F_2)$, *where* $u_i(x,t) := u(x,t;F_i)$ *is the solution of the direct prolem corresponding to arbitrary* $F_i \in \mathcal{F}, i = 1,2$. *Then from* (3.4) *we obtain*

$$\|u(x,T;F_1) - u(x,T;F_2)\|_{H^0(0,l)} \leqslant \|\psi\|_{H^0(\Omega_T)}\|F_1 - F_2\|_{H^0(\Omega_T)}.$$

The integral equality (3.10) yields that the first variation of the cost functional $J(F)$ may be written in the following form:

$$\Delta J(F) = \iint_{\Omega_T} \Delta F(x,t)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^l [\Delta u(x,T;F)]^2\,\mathrm{d}x. \tag{3.11}$$

Lemma 3.2. *If $F \in \mathcal{F}$ is a given source function and $u(x,t;F) \in V^{1,0}(\Omega_T)$ is the corresponding solution of the direct problem* (1.1), *then the inequality*

$$\|\Delta u(x,T;F)\|_{H^0(0,l)} \leqslant \mathrm{e}^T \|\Delta F\|_{H^0(\Omega_T)} \tag{3.12}$$

holds.

P r o o f. We multiply (3.1) by $\Delta u(x,t)$ and integrate over $(0,l)$ to get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l [\Delta u(x,t;F)]^2\,\mathrm{d}x &= k(l)\Delta u_x(l,t)\Delta u(l,t) \\ &\quad - \int_0^l k(\Delta u_x(x,t))^2\,\mathrm{d}x + \int_0^l F(x,t)\Delta u(x,t;F)\,\mathrm{d}x \\ &= -\nu[\Delta u(l,t)]^2 - \int_0^l k(\Delta u_x(x,t))^2\,\mathrm{d}x + \int_0^l F(x,t)\Delta u(x,t;F)\,\mathrm{d}x.\end{aligned} \tag{3.13}$$

If we use the Cauchy inequality for the last term in (3.13) together with the observation that $-\nu[\Delta u(l,t)]^2 - \int_0^l k(\Delta u_x(x,t))^2\,\mathrm{d}x$ is negative, the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l [\Delta u(x,t;F)]^2\,\mathrm{d}x \leqslant \int_0^l [\Delta u(x,t;F)]^2\,\mathrm{d}x + \int_0^l [\Delta F(x,t)]^2\,\mathrm{d}x \tag{3.14}$$

follows. Let us substitute

$$\int_0^l [\Delta u(x,t;F)]^2\,\mathrm{d}x = U(t) \quad \text{and} \quad \int_0^l [\Delta F(x,t)]^2\,\mathrm{d}x = f(t). \tag{3.15}$$

Then (3.14) becomes

$$U'(t) \leqslant U(t) + f(t), \quad t \in [0,T].$$

Applying Gronwall's Inequality, we get

$$U(t) \leqslant \mathrm{e}^{\int_0^t 1\,\mathrm{d}s}\Big[U(0) + \int_0^t f(s)\,\mathrm{d}s\Big] \quad \forall\, t \in [0,T].$$

Hence

$$U(t) \leqslant \mathrm{e}^t \int_0^t f(s)\,\mathrm{d}s \quad \forall\, t \in [0,T].$$

If we substitute t by T, then

$$U(T) \leqslant \mathrm{e}^T \int_0^T f(s)\,\mathrm{d}s.$$

Therefore, using (3.15), we get

$$\int_0^l [\Delta u(x,T)]^2\,\mathrm{d}x \leqslant \mathrm{e}^T \iint_{\Omega_T} [\Delta F(x,t)]^2\,\mathrm{d}x\,\mathrm{d}t, \tag{3.16}$$

which gives (3.12). □

By the definition of the Fréchet derivative, from (3.11) and (3.12) we conclude that the gradient of the cost functional $J(F)$ is the operator

$$J'(F) = (\psi(x,t;F)),$$

where ψ is the solution of (3.3).

By using Lemma 3.1 and Lemma 3.2, we prove the following theorem.

Theorem 3.1. *Let $F \in \mathcal{F}$. If the condition* (2.1) *holds, then the cost functional J is Fréchet-differentiable, $J(F) \in C^1(\mathcal{F})$. The Fréchet derivative at $F \in \mathcal{F}$ of the cost functional $J(F)$ is defined via the solution of the adjoint problem* (3.3) *as*

$$J'(F) = (\psi(x,t;F)). \tag{3.17}$$

Corollary 3.2. *Let $J(F) \in C^1(\mathcal{F})$ and let $\mathcal{F}_* \subset \mathcal{F}$ be the set of quasi-solutions of the inverse problem* (1.1)–(1.2)*. Then $F_* \in \mathcal{F}_*$ is a strict solution of the inverse problem* (1.1)–(1.2) *if and only if $\psi(x,t;F_*) \equiv 0$ on Ω_T.*

4. Lipschitz continuity of the gradient and the monotone iteration scheme

It is well known that any gradient type iteration algorithm for minimization of problem (2.4) has the form (see [8])

$$F^{(n+1)} = F^{(n)} - \alpha_n J'(F^{(n)}), \quad n = 0,1,2,\ldots, \tag{4.1}$$

where $F^{(0)} \in \mathcal{F}$ is a given initial iteration. The choice of α_n defines different gradient methods; in many cases, the estimation of this parameter is difficult but in the case of Lipschitz continuity of the gradient $J'(F)$ of the cost functional, the iteration parameter α_n can be estimated via the Lipschitz constant $C := \mathrm{e}^T$ in (3.12) as

$$0 < \delta_0 \leqslant \alpha_n \leqslant 1/(2C\sqrt{T} + \delta_1), \tag{4.2}$$

where $\delta_0, \delta_1 > 0$ are arbitrary parameters.

Now we will prove the Lipschitz continuity of the cost functional (1.3).

Lemma 4.1. *Let the conditions of Theorem* 3.1. *hold. Then the functional* $J(F)$ *is of Hölder class* $C^1(\mathcal{F})$ *and*

$$\|J'(F + \Delta F) - J'(F)\|_{H^0(\Omega_T)} \leqslant 2C\sqrt{T}\|\Delta F\|_{H^0(\Omega_T)} \quad \forall\, F, F + \Delta F \in \mathcal{F}, \tag{4.3}$$

where

$$\|J'(F + \Delta F) - J'(F)\|^2_{H^0(\Omega_T)} := \iint_{\Omega_T} [\Delta\psi(x, t; F)]^2 \,\mathrm{d}x. \tag{4.4}$$

P r o o f. Let the functions $\psi(x, t; F)$ and $\psi(x, t; F + \Delta F)$ be the solutions to the problem (3.3) with $p(x) = 2[u(x, T; F) - \mu(x)]$ and $p(x) = 2[u(x, T; F + \Delta F) - \mu(x)]$, respectively. Then the function $\Delta\psi(x, t; F) := \psi(x, t; F + \Delta F) - \psi(x, t; F) \in V^{1,0}(\Omega_T)$ is the solution of the backward parabolic problem

$$\begin{cases} \Delta\psi_t = -(k(x)\Delta\psi_x)_x, & (x, t) \in \Omega_T, \\ \Delta\psi(x, T) = 2\Delta u(x, T; F), & x \in (0, l), \\ \Delta\psi(0, t) = 0, \Delta\psi(l, t) = 0, & t \in (0, T]. \end{cases} \tag{4.5}$$

For the proof, we multiply (4.5) by $\Delta\psi(x, t; F)$ and integrate over $(0, l)$ to get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l [\Delta\psi(x, t; F)]^2 \,\mathrm{d}x &= -k(l)\Delta\psi_x(l, t)\Delta\psi(l, t) + k(0)\Delta\psi_x(0, t)\Delta\psi(0, t) \\ &\quad + \int_0^l k(x)[\Delta\psi_x(x, t; F)]^2 \,\mathrm{d}x = \int_0^l k(x)[\Delta\psi_x(x, t; F)]^2 \,\mathrm{d}x. \end{aligned} \tag{4.6}$$

Let

$$\Phi(t) := \int_0^l [\Delta\psi(x, t; F)]^2 \,\mathrm{d}x.$$

Since $\Phi'(t) \geqslant 0$, $\Phi(t)$ is increasing on $(0, T]$, hence

$$\Phi(t) \leqslant \Phi(T), \quad t \in (0, T]. \tag{4.7}$$

By (4.7) we have that

$$\int_0^l [\Delta\psi(x, t; F)]^2 \,\mathrm{d}x \leqslant \int_0^l [\Delta\psi(x, T; F)]^2 \,\mathrm{d}x = 4\int_0^l [\Delta u(x, T; F)]^2 \,\mathrm{d}x.$$

By Lemma 3.2, we get the inequality (4.3). □

We will prove the monotonicity, convergence and convergence properties of the sequence $J(F^n)$, where $F^n \in \mathcal{F}$, $n = 0, 1, 2 \ldots$, are defined by (4.1).

Lemma 4.2. *Let $\mathcal{F}$ be a closed convex set in a Hilbert space and $J(F) \in C^1(\mathcal{F})$. Then*

$$|J(F_1) - J(F_2) - (J'(F_2), F_1 - F_2)| \leqslant C\sqrt{T}\|F_1 - F_2\|^2_{H^0(\Omega_T)}, \quad F_1, F_2 \in \mathcal{F}. \tag{4.8}$$

P r o o f. If we substitute $F_1 = F + h$, $F_2 = F \in \mathcal{F}$ in the formula

$$J(F + h) - J(F) = \int_0^1 (J'(F + \theta h), h)\,\mathrm{d}\theta$$

for the functional $J(F)$, we get

$$J(F_1) - J(F_2) = \int_0^1 (J'(F_2 + \theta(F_1 - F_2)), F_1 - F_2)\,\mathrm{d}\theta.$$

If we use Lemma 4.1 in this equality, we have

$$\begin{aligned}
&|J(F_1) - J(F_2) - (J'(F_2), F_1 - F_2)| \\
&= \left| \int_0^1 (J'(F_2 + \theta(F_1 - F_2), F_1 - F_2)) - (J'(F_2), F_1 - F_2))\,\mathrm{d}\theta \right| \\
&\leqslant 2C\sqrt{T} \int_0^1 \theta \|F_1 - F_2\|^2_{H^0(\Omega_T)}\,\mathrm{d}\theta = C\sqrt{T}\|F_1 - F_2\|^2_{H^0(\Omega_T)}.
\end{aligned}$$

This completes the proof. □

Lemma 4.3. *Let $F^n \in \mathcal{F}$, $n = 0, 1, 2, \ldots$, be iterations defined by* (4.1) *with $\alpha_n = \alpha = const. > 0\ \forall\, n$. Then for all $n = 0, 1, 2, \ldots$,*

$$J(F^{(n)}) - J(F^{(n+1)}) \geqslant \frac{1}{4C\sqrt{T}}\|J'(F^{(n)})\|^2_{H^0(\Omega_T)}. \tag{4.9}$$

P r o o f. In (4.8), we put $F_1 = F^{(n+1)}$, $F_2 = F^{(n)}$ and use the formula (4.1) for $F^{(n+1)}$. We get

$$J(F^{(n+1)}) - J(F^{(n)}) - (J'(F^{(n)}), -\alpha J'(F^{(n)})) \leqslant \alpha^2 C\sqrt{T}\|J'(F^{(n)})\|^2_{H^0(\Omega_T)}.$$

Then we have for $\alpha > 0$

$$J(F^{(n+1)}) - J(F^{(n)}) \geqslant \alpha - \alpha^2 C\sqrt{T}\|J'(F^{(n)})\|^2_{H^0(\Omega_T)}.$$

The function $q(\alpha) = \alpha - \alpha^2 C\sqrt{T}$ takes its maximum at $\alpha_* = 1/\bigl(2C\sqrt{T}\bigr)$ and $q(\alpha_*) = 1/\bigl(4C\sqrt{T}\bigr)$. We substitute these values in the above inequality and obtain (4.9) □

Notice that the optimal value $\alpha_* = 1/\bigl(2C\sqrt{T}\bigr)$ for α is obtained corresponding to the vaules $\delta_0 = 1/\bigl(2C\sqrt{T}\bigr)$ and $\delta_1 = 1/C\sqrt{T}$.

We denote

$$J_* := J(F_*) = \lim_{n\to\infty} J(F^{(n)}), \quad F_* \in \mathcal{F},$$

as the limit of the sequence $\{J(F^{(n))}\}$.

Corollary 4.1. *Let $\mathcal{F}$ be a closed convex set and $J(F) \in C^{1,1}(\mathcal{F})$. If $\{F^{(n)}\} \subset \mathcal{F}$ is the sequence of iterations defined by*

$$F^{(n+1)} = F^{(n)} - \alpha_* J'(F^{(n)}), \quad \alpha_* = 1/2,\ n = 0, 1, 2, \ldots, \tag{4.10}$$

then $\{J(F^{(n)})\}$ is a monotone decreasing convergent sequence and

$$\lim_{n\to\infty} \|J'(F^{(n)})\|_0 = 0.$$

Moreover,

$$\|F^{(n+1)} - F^{(n)}\|_0 \leqslant \frac{1}{C\sqrt{T}}[J(F^{(n)}) - J(F^{(n+1)})], \quad n = 0, 1, 2, \ldots \tag{4.11}$$

Theorem 4.1. *Let condition* (2.1) *hold. Then for any initial source $F^{(0)} \in \mathcal{F}$ the sequence of iterations $\{F^{(n)}\} \subset \mathcal{F}$, given by* (4.1), *weakly converges in $H^0(\Omega_T)$ to a quasisolution $F_* \in \mathcal{F}_*$ of the inverse problem* (1.1)–(1.2). *Moreover, for the rate of convergence of the sequence $\{J(F^{(n)})\}$ the following estimate holds:*

$$0 \leqslant J(F^{(n)}) - J(F_*) \leqslant (4C\sqrt{T}d^2)n^{-1}, \quad d > 0,\ n = 0, 1, 2, \ldots \tag{4.12}$$

P r o o f. It is well known that, if $\mathcal{F}$ is a closed convex set in the Hilbert space $H^0(\Omega_T)$ and $\mathcal{F}_* \subset \mathcal{F}$ is a closed convex and bounded set of solutions of the minimization problem (2.4), then every minimizing sequence $\{F^{(n)}\} \subset \mathcal{F}$ weakly converges to an element $F_* \in \mathcal{F}_*$. Hence, for the sequence $\{F^{(n)}\} \subset \mathcal{F}$ defined by (4.1) we have $F^{(n)} \rightharpoonup F_* \in \mathcal{F}_*$, as $n \to \infty$.

To prove the rate of convergence denote

$$a_n = J(F^{(n)}) - J(F_*), \quad F^{(n)} \in \mathcal{F},\ F_* \in \mathcal{F}_*,\ n = 0, 1, 2, \ldots$$

Since $J(F)$ is convex, we have

$$J(F^{(n)}) - J(F_*) \leqslant (J'(F^{(n)}), F^{(n)} - F_*).$$

If we apply the Cauchy inequality to the right-hand side of the above one, we get

$$a_n = J(F^{(n)}) - J(F_*) \leqslant \|J'(F^{(n)})\|_0 \|F^{(n)} - F_*\|_0 \leqslant d\|J'(F^{(n)})\|_0, \quad F_* \in \mathcal{F},$$

where

$$\begin{aligned} d &:= \sup \|F^{(n)} - F_*\|, \quad \forall\, F^{(n)} \in \mathcal{F}, \\ a_n^2 &\leqslant d^2 \|J'(F^{(n)})\|_0^2 \leqslant 4C\sqrt{T} d^2 (J(F^{(n)}) - J(F^{(n+1)})) \\ &= 4C\sqrt{T} d^2 (a_n - a_{n+1}), \quad n = 0, 1, 2, \dots \end{aligned} \tag{4.13}$$

Thus, we obtained a monotone decreasing sequence $\{a_n\} \subset \mathbb{R}_+$ with

$$a_n^2 > 0, \quad a_n - a_{n+1} \geqslant \frac{1}{4C\sqrt{T}d^2} a_n^2.$$

Now,

$$\frac{1}{a_{k+1}} - \frac{1}{a_k} = \frac{a_k - a_{k+1}}{a_{k+1} a_k} \geqslant \frac{(4C\sqrt{T}d^2)^{-1}}{a_{k+1} a_k} > (4C\sqrt{T}d^2)^{-1}. \tag{4.14}$$

Taking the sum from 0 to $(n-1)$

$$\sum_{k=0}^{n-1} \Big(\frac{1}{a_{k+1}} - \frac{1}{a_k} \Big) := \frac{1}{a_n} - \frac{1}{a_0} \geqslant (4C\sqrt{T}d^2)^{-1} n,$$

(4.14) yields that $a_n \leqslant (4C\sqrt{T}d^2) n^{-1}$. This completes the proof. □

5. Convexity of the Fréchet derivative

We will study the convexity of the cost functional $J(F)$.

Lemma 5.1. *Let the conditions of Lemma* 4.1 *hold. Assume that* F, $\Delta F \in \mathcal{F}$, *then*

$$\begin{aligned} (J'(F + \Delta F) - J'(F), \Delta F)_{\mathcal{F}} &= \iint_{\Omega_T} \Delta\psi(x,t) \Delta F(x,t) \,\mathrm{d}x \,\mathrm{d}t \\ &= 2 \int_0^l [\Delta u(x,T)]^2 \,\mathrm{d}x. \end{aligned} \tag{5.1}$$

P r o o f. Using (3.1), we can write that

$$\begin{aligned}(5.2)\quad &\iint_{\Omega_T} \Delta\psi(x,t)\Delta F(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &= \iint_{\Omega_T} (\Delta u_t(x,t) - (k(x)\Delta u_x(x,t))_x)\Delta\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &= \iint_{\Omega_T} \Delta u_t(x,t)\Delta\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t - \iint_{\Omega_T} (k(x)\Delta u_x(x,t))_x\Delta\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The first integral on the second line of (5.2) is

$$\begin{aligned}(5.3)\quad &\iint_{\Omega_T} \Delta u_t(x,t)\Delta\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^l [\Delta u(x,T)\Delta\psi(x,T) - \Delta u(x,0)\Delta\psi(x,0)]\,\mathrm{d}x\\ &\quad - \iint_{\Omega_T} \Delta u(x,t)\Delta\psi_t(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &= 2\int_0^l [\Delta u(x,T)]^2\,\mathrm{d}x - \iint_{\Omega_T} \Delta u\Delta\psi_t\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The second integral on the second line of (5.2) (recall also (4.5)) is

$$\begin{aligned}(5.4)\quad &-\iint_{\Omega_T} (k(x)\Delta u_x(x,t))_x\Delta\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T [k(0)\Delta u_x(0,t)\Delta\psi(0,t) - k(l)\Delta u_x(l,t)\Delta\psi(l,t)]\,\mathrm{d}t\\ &\quad + \iint_{\Omega_T} k(x)\Delta u_x(x,t)\Delta\psi_x(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &= \iint_{\Omega_T} k(x)\Delta u_x(x,t)\Delta\psi_x(x,t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The last integral in (5.4) is

$$\begin{aligned}(5.5)\quad &\iint_{\Omega_T} k(x)\Delta u_x(x,t)\Delta\psi_x(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T [k(l)\Delta u(l,t)\Delta\psi_x(l,t) - k(0)\Delta u(0,t)\Delta\psi_x(0,t)]\,\mathrm{d}t\\ &\quad - \iint_{\Omega_T} \Delta u(x,t)(k(x)\Delta\psi_x(x,t))_x\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Substituting (5.5) in (5.4), we get

$$-\iint_{\Omega_T} (k(x)\Delta u_x(x,t))_x \Delta\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\iint_{\Omega_T} \Delta u(x,t)(k(x)\Delta\psi_x(x,t))_x\,\mathrm{d}x\,\mathrm{d}t. \tag{5.6}$$

If we use (5.3) and (5.6) in (5.2) (and recall (4.5)) we conclude that

$$\begin{aligned}\iint_{\Omega_T} \Delta\psi(x,t)\Delta F(x,t)\,\mathrm{d}x\,\mathrm{d}t &= 2\int_0^l [\Delta u(x,T)]^2\,\mathrm{d}x - \iint_{\Omega_T} (k(x)\Delta u_x(x,t))_x \Delta\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \iint_{\Omega_T} \Delta u(x,t)\Delta\psi_t(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &= 2\int_0^l [\Delta u(x,T)]^2\,\mathrm{d}x - \iint_{\Omega_T} \Delta u(\Delta\psi_t + (k\Delta\psi_x)_x)\,\mathrm{d}x\,\mathrm{d}t \\ &= 2\int_0^l [\Delta u(x,T)]^2\,\mathrm{d}x.\end{aligned} \tag{5.7}$$

This completes the proof. □

Lemma 5.1 proves that the cost functional $J(F) \in C^1(\mathcal{F})$ is convex. If the condition

$$\int_0^l [\Delta u(x,T)]^2\,\mathrm{d}x > 0 \quad \forall\, F \in \mathcal{F} \tag{5.8}$$

holds, then $J(F)$ is strictly convex.

Theorem 5.1. *If the condition of Lemma* 5.1 *and the condition* (5.8) *hold, then the inverse source problem* (1.1)–(1.2) *has at most one solution.*

Acknowledgement. We would like to thank Alemdar Hasanoğlu (Hasanov) for formulation of the problem and helpful suggestions.

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