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FIXED POINT RESULTS ON A METRIC SPACE ENDOWED WITH A FINITE NUMBER OF GRAPHS AND APPLICATIONS

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Abstract. In this paper, we consider self-mappings defined on a metric space endowed with a finite number of graphs. Under certain conditions imposed on the graphs, we establish a new fixed point theorem for such mappings. The obtained result extends, generalizes and improves many existing contributions in the literature including standard fixed point theorems, fixed point theorems on a metric space endowed with a partial order and fixed point theorems for cyclic mappings.

Keywords: fixed point; graph; metric space; order; cyclic map

MSC 2010: 47H10, 05C40, 06A06

1. Introduction

Given a nonempty set X and a mapping $T\colon X \to X$, a point $x \in X$ is said to be fixed under T if $Tx = x$; the set of all these points will be denoted as $\mathrm{Fix}(T)$. Metric fixed point theory is the branch of analysis which focuses on the existence, uniqueness and localization of fixed points under metric conditions on both the domain of the mapping and the mapping itself. Banach's contraction principle is considered a fundamental result of this theory. It guarantees existence and uniqueness of fixed points for mappings $T\colon X \to X$, where (X,d) is a complete metric space and T is a (metrical) contraction, i.e.,

$$d(Tx,Ty) \leqslant kd(x,y),$$

for all $x, y \in X$, and some $k \in (0,1)$. This principle has been generalized and

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extended by many authors in several directions (see, for example, [6], [7], [8], [11], [13], [14], [15], [22]).

In [10], Espinola and Kirk provided useful results by combining fixed point theory and graph theory. In [13], Jachymski developed this idea from a different perspective; and in [4], Beg, Butt and Radojević extended some of Jachymski's results to the case of set-valued mappings. Very recently, Aleomraninejad, Rezapour and Shahzad [2] presented some iterative scheme results for G-contractive and G-nonexpansive mappings on graphs.

In this paper, we establish fixed point results for self-mappings defined on a metric space endowed with a finite number of graphs. Our statements generalize and improve many existing fixed point theorems in the literature including theorems on ordered metric spaces and theorems for cyclic mappings.

2. Preliminaries and notation

In this section, we introduce some concepts and give some examples.

Definition 2.1. A graph G is defined as a pair of sets $G = (V, E)$ with $E \subseteq V \times V$. We say that V is the vertex set and E is the edge set.

Definition 2.2. Let $G = (V, E)$ be a graph and D be a subset of V. We say that D is G-directed if for every $x, y \in D$ there exists $z \in V$ such that (x, z) and (y, z) are edges of G.

Example 2.1. Let $V = \mathcal{F}([0,1], \mathbb{R})$ be the set of functions $f\colon [0,1] \to \mathbb{R}$. Define $E \subset V \times V$ by

$$(u, v) \in E \iff u(t) \leqslant v(t), \quad \text{for all } t \in [0,1].$$

Then $G = (V, E)$ is a graph. Let $D = C([0,1], \mathbb{R})$ be the set of continuous functions $f\colon [0,1] \to \mathbb{R}$. Then D is G-directed. Indeed, for every $u, v \in D$, $z = \max\{u, v\}$ satisfies $(u, z) \in E$ and $(v, z) \in E$.

Let (V, d) be a metric space. We consider a family $G = \{G_i\}_{i=1}^{p}$ of p graphs ($p \geqslant 1$) such that $G_i = (V, E_i)$, $E_i \subseteq V \times V$, $i = 1, 2, \ldots, p$.

Definition 2.3. Let $T\colon V \to V$ be a given mapping. We say that T is G-monotone if for all $i = 1, 2, \ldots, p$ we have

$$(x, y) \in E_i \Longrightarrow (Tx, Ty) \in E_{i+1},$$

with $E_{p+1} = E_1$.

Remark 2.1. If $p = 1$ ($G = G_1$), we say that T preserves edges of G (see [13]).

Example 2.2. Consider the sets

$$V = [0, 2], \quad E_1 = [0, 1] \times [1, 2], \quad E_2 = [1, 2] \times [0, 1].$$

Let $G = \{G_i\}_{i=1}^{2}$ be the family of graphs $G_i = (V, E_i)$, $i = 1, 2$. Define the mapping $T\colon V \to V$ by

$$Tx = \begin{cases} x + 1, & \text{if } x \in [0, 1), \\ x - 1, & \text{if } x \in (1, 2], \\ 1, & \text{if } x = 1. \end{cases}$$

Observe that

$$(x, y) \in E_1 \Longrightarrow (Tx, Ty) \in E_2$$

and

$$(x, y) \in E_2 \Longrightarrow (Tx, Ty) \in E_1.$$

Then T is G-monotone.

Definition 2.4. We say that the pair (G, d) is regular if the following condition holds: if $\{x_n\}$ is a sequence in V such that for all $i \in \{1, 2, \dots, p\}$ there exists a subsequence $\{x_{m_{i,k}}\}$ of $\{x_n\}$ satisfying

$$(x_{m_{i,k}}, x_{m_{i,k}+1}) \in E_i, \quad \text{for all } k; \tag{2.1}$$

$$d(x_n, x) \to 0 \quad \text{as } n \to \infty \text{ for some } x \in V, \tag{2.2}$$

then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a rank $j \in \{1, 2, \dots, p\}$ such that

$$(x_{n_k}, x) \in E_j, \quad \text{for all } k.$$

Example 2.3. Let $V = C([0, 1], \mathbb{R})$ be the set defined by

$$V = \{f\colon [0, 1] \to \mathbb{R}\colon f \text{ is continuous}\}.$$

Define $E \subset V \times V$ by

$$(u, v) \in E \Longleftrightarrow u(t) \leqslant v(t), \quad \text{for all } t \in [0, 1].$$

Consider the graph $G = (V, E)$. We endow V with the metric d given by

$$d(u, v) = \max_{0 \leqslant t \leqslant 1} |u(t) - v(t)|, \quad \text{for all } u, v \in V.$$

Let $\{x_n\}$ be a sequence in V and x be a point in V satisfying conditions (2.1) and (2.2), i.e.,

(a) there exists a subsequence $\{x_{m_k}\}$ of $\{x_n\}$ such that

$$x_{m_k}(t) \leqslant x_{m_k+1}(t), \quad \text{for all } k, \ \text{for all } t \in [0,1];$$

(b) $\max_{0\leqslant t\leqslant 1} |x_n(t) - x(t)| \to 0 \quad \text{as } n \to \infty.$

Then $(x_{m_k}, x) \in E$ for all k, which implies that the pair (G, d) is regular.

Let Φ be the set of functions $\varphi\colon [0,\infty) \to [0,\infty)$ satisfying

(P_1) φ is nondecreasing;

(P_2) $\sum\limits_{n=1}^{\infty} \varphi^n(t) < \infty$ for each $t > 0$, where φ^n is the n-th iterate of φ.

The following auxiliary fact is immediate, so we omit its proof.

Lemma 2.1. *Let φ be a function in Φ. Then $\varphi(t) < t$ for all $t > 0$.*

3. Main result

Let (V, d) be a metric space. We consider a family $G = \{G_i\}_{i=1}^{p}$ of p graphs ($p \geqslant 1$) such that $G_i = (V, E_i)$, $E_i \subseteq V \times V$, $i = 1, 2, \ldots, p$. Let $T\colon V \to V$ be a given mapping. Our main result is the following.

Theorem 3.1. *Suppose that the following conditions hold:*

(a) *(V, d) is complete;*

(b) *T is G-monotone;*

(c) *there exists $x_0 \in V$ such that $(x_0, Tx_0) \in E_1$;*

(d) *(G, d) is regular;*

(e) *there exists $\varphi \in \Phi$ such that*

$$d(Tx, Ty) \leqslant \varphi(d(x,y)), \quad \text{for all } (x,y) \in E_i, \ i = 1, 2, \ldots, p. \tag{3.1}$$

Then T has a fixed point. Moreover, if there exists $i \in \{1, 2, \ldots, p\}$ such that $\mathrm{Fix}(T)$ *is G_i-directed, we obtain uniqueness of the fixed point.*

Proof. From (c), there exists $x_0 \in V$ such that $(x_0, Tx_0) \in E_1$. Define the sequence $\{x_n\}$ in V by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots$$

If $x_{n+1} = x_n$ for some n, then x_n is a fixed point of T and the existence of a fixed point is proved. Now, suppose that

$$x_{n+1} \neq x_n, \quad n = 0, 1, 2, \ldots \tag{3.2}$$

Since T is G-monotone, for all $n \geqslant 0$, there exists $i = i(n) \in \{1, 2, \dots, p\}$ such that $(x_n, x_{n+1}) \in E_i$. For all $n \geqslant 1$, applying the inequality (3.1) with $x = x_{n-1}$ and $y = x_n$, we obtain

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leqslant \varphi(d(x_{n-1}, x_n)). \tag{3.3}$$

Using (P_1), by induction, we get

$$d(x_n, x_{n+1}) \leqslant \varphi^n(d(x_0, x_1)), \quad \text{for all } n \geqslant 0. \tag{3.4}$$

Fix $\varepsilon > 0$ and let $h = h(\varepsilon)$ be a positive integer (given by (P_2)) such that

$$\sum_{n \geqslant h} \varphi^n(d(x_1, x_0)) < \varepsilon.$$

Let $m > n > h$. Using the triangular inequality and (3.4), we obtain

$$d(x_n, x_m) \leqslant \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leqslant \sum_{k=n}^{m-1} \varphi^k(d(x_0, x_1)) \leqslant \sum_{n \geqslant h} \varphi^n(d(x_0, x_1)) < \varepsilon.$$

Thus we proved that $\{x_n\}$ is a Cauchy sequence in the metric space (V, d). Since (V, d) is complete, there exists $x^* \in V$ such that

$$\lim_{n \to \infty} d(x_n, x^*) = 0. \tag{3.5}$$

Clearly the sequence $\{x_n\}$ satisfies conditions (2.1) and (2.2) with $x = x^*$. Since (G, d) is regular, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a $j \in \{1, 2, \dots, p\}$ such that $(x_{n_k}, x^*) \in E_j$, for all k. Applying (3.1) with $x = x_{n_k}$ and $y = x^*$, we obtain

$$d(x_{n_k+1}, Tx^*) = d(Tx_{n_k}, Tx^*) \leqslant \varphi(d(x_{n_k}, x^*)), \quad \text{for all } k. \tag{3.6}$$

Denote $\varrho = d(x^*, Tx^*)$. Suppose that $\varrho > 0$. By (3.5), there exists $p = p(\varrho)$ such that

$$d(x_{n_k}, x^*) \leqslant \frac{1}{2}\varrho, \quad \text{for all } k \geqslant p.$$

Substituting into (3.6) yields (as φ is nondecreasing)

$$d(x_{n_k+1}, Tx^*) \leqslant \varphi\Big(\frac{1}{2}\varrho\Big), \quad \text{for all } k \geqslant p.$$

So, passing to limit as k tends to infinity, one derives from Lemma 2.1

$$\varrho \leqslant \varphi\Big(\frac{1}{2}\varrho\Big) < \frac{1}{2}\varrho < \varrho,$$

which is a contradiction. Then, $\varrho = 0$, i.e., $d(x^*, Tx^*) = 0$, hence $x^* \in V$ is a fixed point of T.

Now, suppose that there exists $i \in \{1, 2, \ldots, p\}$ such that $\mathrm{Fix}(T)$ is G_i-directed. We shall prove that x^* is the unique fixed point of T. Suppose that $y^* \in V$ is another fixed point of T. Then there is $z \in V$ such that (x^*, z) and (y^*, z) are edges of G_i. Define the sequence $\{z_n\}$ in V by $z_0 = z$ and $z_{n+1} = Tz_n$ for all $n \geqslant 0$. Since T is G-monotone, for all $n \geqslant 0$, there exists $i = i(n) \in \{1, 2, \ldots, p\}$ such that $(x^*, z_n) \in E_i$. Applying (3.1), for all $n \geqslant 0$, we have

$$d(z_{n+1}, x^*) = d(Tz_n, Tx^*) \leqslant \varphi(d(z_n, x^*)). \tag{3.7}$$

Now, we shall prove that

$$\lim_{n\to\infty} d(z_n, x^*) = 0. \tag{3.8}$$

From (3.7), by induction, we get

$$d(z_n, x^*) \leqslant \varphi^n(d(z_0, x^*)), \quad \text{for all } n.$$

Letting $n \to \infty$ in the above inequality yields (3.8). Similarly, we can prove that

$$\lim_{n\to\infty} d(z_n, y^*) = 0. \tag{3.9}$$

It follows from (3.8) and (3.9) that $x^* = y^*$. This concludes the proof. □

Remark 3.1. It is easy to observe that condition (c) of Theorem 3.1 can be replaced by: there exists $x_0 \in V$ such that $(x_0, Tx_0) \in E_i$ for some $i \in \{1, 2, \ldots, p\}$.

Taking $\varphi(t) = kt$ with $k \in (0, 1)$ in Theorem 3.1, we obtain

Corollary 3.1. *Suppose that the following conditions hold:*

(a) (V, d) *is complete;*
(b) T *is G-monotone;*
(c) *there exists $x_0 \in V$ such that $(x_0, Tx_0) \in E_1$;*
(d) (G, d) *is regular;*
(e) *there exists $k \in (0, 1)$ such that*

$$d(Tx, Ty) \leqslant kd(x, y), \quad \text{for all } (x, y) \in E_i,\ i = 1, 2, \ldots, p.$$

Then T has a fixed point. Moreover, if there exists $i \in \{1, 2, \dots, p\}$ such that $\mathrm{Fix}(T)$ *is G_i-directed, we obtain uniqueness of the fixed point.*

Remark 3.2. Taking $V = X$, $p = 1$ and $E_1 = X \times X$ in Corollary 3.1, we recover the Banach contraction principle.

4. Applications

We now derive further consequences of our main result.

4.1. Fixed point theorems on a metric space with a partial order. Recently, there have been many exciting developments in the field of existence of fixed points in partially ordered metric spaces. For more details, we refer the reader to the papers by Turinici [23], Ran and Reurings [19], Nieto and López [16], [17], Agarwal et al. [1], Ćirić et al. [9], Harjani and Sadarangani [12], Altun and Simsek [3], Jachymski [13], Bhaskar and Lakshmikantham [5], Petruşel and Rus [18], Samet et al. [20], [21], and the references therein.

Let $(X, \preceq)$ be a partially ordered metric space. We say that that the pair $(X, \preceq)$ satisfies the property (H) if

(H) $\forall\, (x, y) \in X \times X,\ \exists\, z \in X$ such that $x \preceq z$ and $y \preceq z$.

Definition 4.1. We say that $(X, \preceq, d)$ is regular if the following condition holds: if $\{x_n\}$ is a sequence in X whose consecutive terms are comparable and $d(x_n, x) \to 0$ as $n \to \infty$ for some $x \in X$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that every term is comparable to the limit x.

Definition 4.2. Let $T\colon\ X \to X$ be a given mapping. We say that T is a comparative mapping if T maps comparable elements into comparable elements, that is,

$$x, y \in X, \quad x \preceq y \Longrightarrow Tx \preceq Ty \quad \text{or} \quad Ty \preceq Tx.$$

We have the following result.

Corollary 4.1. *Let $(X, \preceq, d)$ be a partially ordered metric space satisfying* (H), *such that (X, d) is complete. Let $T\colon\ X \to X$ be a given mapping. Suppose that the following conditions hold:*

(i) *T is a comparative mapping;*
(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ or $Tx_0 \preceq x_0$;*
(iii) *$(X, \preceq, d)$ is regular;*

(iv) *there exists* $\varphi \in \Phi$ *such that*

$$d(Tx, Ty) \leqslant \varphi(d(x, y)), \quad \text{for all } x, y \in X,\ x \preceq y.$$

Then T *has a unique fixed point.*

Proof. It follows from Theorem 3.1 by taking $V = X$, $p = 1$ and $G = \{G_1\}$, where $G_1 = (V, E_1)$ with

$$E_1 = \{(x, y) \in X \times X;\ x \preceq y \text{ or } y \preceq x\}.$$

$\square$

Remark 4.1. Taking $\varphi(t) = kt$ with $k \in (0, 1)$ in Corollary 4.1, we obtain a fixed point result of Nieto and Lopez [17] (see also [16]).

4.2. Fixed point theorems for cyclic contractive mappings. The following notion was introduced in [15].

Definition 4.3. Let X be a nonempty set, p a positive integer and $T\colon X \to X$ an operator. By definition, $X = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of X with respect to T if

(c_1) A_i, $i = 1, 2, \ldots, p$ are nonempty sets;
(c_2) $T(A_1) \subseteq A_2, \ldots, T(A_{p-1}) \subseteq A_p, T(A_p) \subseteq A_1$.

Let (X, d) be a metric space. We denote by $P_{\rm cl}(X)$ the collection of nonempty closed subsets of X.

Definition 4.4. Let (X, d) be a metric space, p a positive integer, $A_1, A_2, \ldots, A_p \in P_{\rm cl}(X)$, $Y := \bigcup_{i=1}^{p} A_i$ and $T\colon Y \to Y$ an operator. If

(I) $\bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to T;
(II) there exists $\varphi \in \Phi$ such that

$$d(Tx, Ty) \leqslant \varphi(d(x, y)), \quad \text{for all } x \in A_i,\ y \in A_{i+1}, \text{ where } A_{p+1} = A_1,$$

then T is a cyclic φ-contraction.

We have the following result.

Corollary 4.2. *Let (X, d) be a complete metric space, p a positive integer, $A_1, \ldots, A_p \in P_{\mathrm{cl}}(X)$, $Y := \bigcup_{i=1}^{p} A_i$, $\varphi \in \Phi$ and $T\colon Y \to Y$ an operator. Assume that*

(i) $\bigcup_{i=1}^{p} A_i$ *is a cyclic representation of Y with respect to T;*

(ii) T *is a cyclic φ-contraction.*

Then T has a unique fixed point in $\bigcap_{i=1}^{p} A_i$.

P r o o f. Let $V = Y$ and consider the family of graphs $G = \{G_i\}_{i=1}^{p}$, where

$$G_i = (V, E_i), \quad E_i = A_i \times A_{i+1}, \quad A_{p+1} = A_1, \quad i = 1, 2, \ldots, p.$$

Since $A_1, A_2, \ldots, A_p \in P_{\mathrm{cl}}(X)$ and (X, d) is complete, (V, d) is a complete metric space. Let $(x, y) \in E_i$ for some $i \in \{1, 2, \ldots, p\}$. From condition (i), we have $T(A_i) \subseteq A_{i+1}$ and $T(A_{i+1}) \subseteq A_{i+2}$. This yields $(Tx, Ty) \in A_{i+1} \times A_{i+2}$, i.e., $(Tx, Ty) \in E_{i+1}$. Then T is G-monotone. Let $x_0 \in A_1$ (such a point exists since $A_1 \neq \emptyset$). Since $T(A_1) \subseteq A_2$, we have $Tx_0 \in A_2$, which implies that $(x_0, Tx_0) \in E_1$. Now, we shall prove that (G, d) is regular. Let $\{x_n\}$ be a sequence in V and x be a point in V satisfying (2.1) and (2.2). For $i \in \{1, ..., p\}$, there exists a subsequence $\{x_{m_{i,k}}\}$ of $\{x_n\}$ such that $(x_{m_{i,k}}, x_{m_{i,k}+1}) \in E_i$, for all k, which yields $x_{m_{i,k}} \in A_i$, for all k. Since A_i is closed and $d(x_n, x) \to 0$ as $n \to \infty$, we obtain $x \in A_i$; so, as $i \in \{1, \ldots, p\}$ was arbitrarily chosen, we get that $x \in \bigcap_{i=1}^{p} A_i$. This yields, by the choice of our subsequences, $(x_{m_{i.k}}, x) \in E_i$, for all $i \in \{1, \ldots, p\}$ and all k. Thus we proved that (G, d) is regular. Finally, we shall prove that $\mathrm{Fix}(T)$ is G_1-directed. Let $x, y \in \mathrm{Fix}(T)$. From (i), we have $x, y \in \bigcap_{i=1}^{p} A_i$, which implies that $(x, y) \in E_1$ and $(y, y) \in E_1$, and our claim is proved. Now, all the hypotheses of Theorem 3.1 being satisfied, we deduce that T has a unique fixed point in V. Moreover, from (i), this fixed point belongs to $\bigcap_{i=1}^{p} A_i$ □

Remark 4.2. Taking in Corollary 4.2 $\varphi(t) = kt$ with $k \in (0, 1)$ in Corollary 4.2, we obtain Theorem 1.3 in [15].

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