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DYNAMICS IN A DISCRETE PREDATOR-PREY SYSTEM WITH INFECTED PREY

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Abstract. In this paper, a discrete version of continuous non-autonomous predator-prey model with infected prey is investigated. By using Gaines and Mawhin's continuation theorem of coincidence degree theory and the method of Lyapunov function, some sufficient conditions for the existence and global asymptotical stability of positive periodic solution of difference equations in consideration are established. An example shows the feasibility of the main results.

Keywords: predator-prey model; periodic solution; topological degree; global asymptotic stability

MSC 2010: 34K20, 34C25

1. Introduction

Recently, qualitative research and analysis of population dynamical models, especially predator-prey models, has been an important and interesting problem which has attracted a great deal of attention. Many authors have explored the dynamics of predator-prey systems [1], [2], [5], [6], [9], [14], [18], [21], [23], [31]. For example, Liu and Chen [8] discussed complex dynamics of Holling type II Lotka-Volterra predator-prey model with impulsive perturbations on the predator. Song and Li [21] studied the linear stability of the trivial periodic solution and semi-trivial periodic solutions and the permanence of the periodic predator-prey model with modified Leslie-Gower

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Holling-type II schemes and impulsive effect. Liu and Xu [16] investigated the existence of periodic solutions for a delay one-predator and two-prey system with Holling type-II functional response. Agiza et al. [1] considered the chaotic phenomena of a discrete prey-predator model of Holling type II. Pei et al. [19] analysed the extinction and permanence for one-prey multi-predators of Holling type II function response system with impulsive biological control. Zhang and Luo [30] gave a theoretical study on the existence of multiple positive periodic solutions for a delayed predator-prey system with stage structure for the predator. Zhang and Hou [29] established the existence of at least four positive periodic solutions for a ratio-dependent predator-prey system with multiple exploited (or harvesting) terms. Ko and Ryu [13] focused on the coexistence states of a nonlinear Lotka-Volterra type predator-prey model with cross-diffusion. Jian et al. [10] considered the prey-extinction periodic solution of a biological management model with impulsive stocking juvenile predators and continuous harvesting adult predators. In detail, one can see [2], [11], [12], [15], [17], [18]. In 2006, Hilker and Malchow [9] dealt with the local dynamics and strange periodic attractor of the predator-prey model with infected prey

$$
\tag{1.1}
\begin{cases}
\dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = r_1 S(t)(1 - S(t) - I(t)) - \dfrac{aS(t)Z(t)}{1 + b(S(t) + I(t))} \\
\qquad\qquad - \dfrac{\alpha S(t)I(t)}{S(t) + I(t)}, \\
\dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = r_2 I(t)(1 - S(t) - I(t)) - \dfrac{aI(t)Z(t)}{1 + b(S(t) + I(t))} \\
\qquad\qquad + \dfrac{\alpha S(t)I(t)}{S(t) + I(t)} - m_2 I(t), \\
\dfrac{\mathrm{d}Z(t)}{\mathrm{d}t} = \dfrac{a(S(t) + I(t))Z(t)}{1 + b(S(t) + I(t))} - m_3 Z(t),
\end{cases}
$$

where $S(t)$ and $I(t)$ are the susceptible phytoplankton population and the infected phytoplankton population, $Z(t)$ grazes on both the susceptible and infected phytoplankton. Frequency-dependent transmissions rate $\alpha > 0$ as well as an additional disease-induced mortality of infected prey (virulence) with rate m_2 is assumed. Then r_1 and r_2 are the intrinsic growth rates of susceptible and infected population, respectively. Rate m_3 represents the natural mortality rate of zooplankton. Futher a, b are positive constants. In detail, on can see [9], [20].

Noting that any biological or environmental parameters are naturally subject to fluctuation in time, we think that it is necessary and important to consider models with periodic ecological parameters. Thus the assumption of periodicity of the parameters is a way of incorporating the periodicity of the environment. Then the system (1.1) can be modified to the form

$$
(1.2) \qquad \begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = r_1(t)S(t)(1 - S(t) - I(t)) - \dfrac{a(t)S(t)Z(t)}{1 + b(t)(S(t) + I(t))} \\ \qquad\qquad - \dfrac{\alpha(t)S(t)I(t)}{S(t) + I(t)}, \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = r_2(t)I(t)(1 - S(t) - I(t)) - \dfrac{a(t)I(t)Z(t)}{1 + b(t)(S(t) + I(t))} \\ \qquad\qquad + \dfrac{\alpha(t)S(t)I(t)}{S(t) + I(t)} - m_2(t)I(t), \\ \dfrac{\mathrm{d}Z(t)}{\mathrm{d}t} = \dfrac{a(t)(S(t) + I(t))Z(t)}{1 + b(t)(S(t) + I(t))} - m_3(t)Z(t). \end{cases}
$$

Many authors have argued that discrete time models governed by difference equations are more appropriate for describing the dynamics relationship among populations than the continuous ones when the populations have non-overlapping generations. Moreover, discrete time models can also provide efficient models of continuous ones for numerical simulation. Therefore, it is reasonable to study time predator-prey systems governed by difference equations.

In implementing the continuous-time predator-prey model for simulation or computational purposes, in this paper we formulate a discrete-time system which is an analogue of the continuous-time model (1.2). We know that the discrete-time analogue when derived as a numerical approximation of (1.2) can preserve the dynamical behavior of the continuous-time model. Once this is established, the discrete-time analogue can be used without loss of functional similarity to the continuous-time model and it preserves any biological reality that the continuous-time model has. There are several schemes how to obtain the discrete-time analogues of continuous-time models.

We consider the autonomous differential equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = f(u(t)), \quad t \geqslant 0.$$

The first-order derivative is approximated by the forward Euler expression

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} \to \frac{u(k+1) - u(k)}{\varphi},$$

with the denominator function φ such that

$$\varphi(h) = h + O(h^2),$$

where $h = 1/m$ denotes the step-size and $u(k)$ denotes the approximate value to $u(kh)$. Then we get

$$u(k+1) - u(k) = \varphi(h)f(u(k)).$$

This method can be seen as the forward Euler method.

The principle object of this article is to propose a discrete analogue system (1.2) by using the Euler's method and explore its dynamics. That is, following the methods in [5], [24], we derive a discrete analogue of (1.2), apply Mawhin's continuous theorem [8] and the method of Lyapunov function to study the existence and globally asymptotic stability of positive periodic solutions of the discrete analogue of (1.2). There are some papers which deal with this topic, see [3], [4], [9], [22], [23], [25], [26], [27], [28].

The paper is organized as follows: in Section 2, by means of differential equations with piecewise constant arguments, we first propose a discrete analogue of system (1.2). In Section 3, based on the coincidence degree and the related continuation theorem, some sufficient conditions for the existence of positive periodic solution of difference equations are established. Using the method of Lyapunov function, some sufficient conditions for the globally asymptotical stability of the system under consideration are obtained in Section 4. The paper ends with an example which shows the feasibility of the main results.

2. Discrete analogue of system (1.2)

In the following, we will discretize the system (1.2). Following the lines of [5], [24], we assume that the average growth rates in system (1.2) change at regular intervals of time. We can incorporate this aspect in (1.2) and obtain the following modified system:

$$
\tag{2.1}
\begin{cases}
\dfrac{1}{S(t)}\dot{S}(t) = r_1([t])(1 - S([t]) - I([t])) - \dfrac{a([t])Z([t])}{1 + b([t])(S([t]) + I([t]))} \\
\qquad\qquad - \dfrac{\alpha([t])I([t])}{S([t]) + I([t])}, \\
\dfrac{1}{I(t)}\dot{I}(t) = r_2([t])(1 - S([t]) - I([t])) - \dfrac{a([t])Z([t])}{1 + b([t])(S([t]) + I([t]))} \\
\qquad\qquad + \dfrac{\alpha([t])S([t])}{S([t]) + I([t])} - m_2([t]), \\
\dfrac{1}{Z(t)}\dot{Z}(t) = \dfrac{a([t])(S([t]) + I([t]))}{1 + b([t])(S([t]) + I([t]))} - m_3([t]),
\end{cases}
$$

where $t \neq 0, 1, 2, \ldots$, $[t]$ denotes the integer part of t, $t \in (0, \infty)$. Equations of type (2.1) are known as differential equations with piecewise constant arguments and they occupy a position midway between differential equations and difference equations. By a solution of (2.1), we mean a function $\widetilde{x} = (S, I, Z)^{\mathrm{T}}$ which is defined for $t \in [0, \infty)$ and has the following properties:

1. $\bar{x}$ is continuous on $[0, \infty)$.

2. The derivatives $\mathrm{d}S(t)/\mathrm{d}t$, $\mathrm{d}I(t)/\mathrm{d}t$, $\mathrm{d}Z(t)/\mathrm{d}t$ exist at each point $t \in [0, \infty)$ with the possible exception of the points $t \in \{0, 1, 2, \ldots\}$, where left-sided derivatives exist.

3. The equations in (2.1) are satisfied on each interval $[k, k+1)$ with $k = 0, 1, 2, \ldots$

We integrate (2.1) on any interval of the form $[k, k+1)$, $k = 0, 1, 2, \ldots$, and obtain for $k \leqslant t < k+1$, $k = 0, 1, 2, \ldots$

$$
(2.2) \qquad \begin{cases} S(t) = S(k) \exp\Big\{\Big[r_1(k)(1 - S(k) - I(k)) - \dfrac{a(k)Z(k)}{1 + b(k)(S(k) + I(k))} \\ \qquad\qquad - \dfrac{\alpha(k)I(k)}{S(k) + I(k)}\Big](t - k)\Big\}, \\ I(t) = I(k) \exp\Big\{\Big[r_2(k)(1 - S(k) - I(k)) - \dfrac{a(k)Z(k)}{1 + b(k)(S(k) + I(k))} \\ \qquad\qquad + \dfrac{\alpha(k)S(k)}{S(k) + I(k)} - m_2(t)\Big](t - k)\Big\}, \\ Z(t) = Z(k) \exp\Big\{\Big[\dfrac{a(k)(S(k) + I(k))}{1 + b(k)(S(k) + I(k))} - m_3(k)\Big](t - k)\Big\}. \end{cases}
$$

Let $t \to k + 1$, then (2.2) takes the following form:

$$
(2.3) \qquad \begin{cases} S(k+1) = S(k) \exp\Big[r_1(k)(1 - S(k) - I(k)) \\ \qquad\qquad - \dfrac{a(k)Z(k)}{1 + b(k)(S(k) + I(k))} - \dfrac{\alpha(k)I(k)}{S(k) + I(k)}\Big], \\ I(k+1) = I(k) \exp\Big[r_2(k)(1 - S(k) - I(k)) \\ \qquad\qquad - \dfrac{a(k)Z(k)}{1 + b(k)(S(k) + I(k))} + \dfrac{\alpha(k)S(k)}{S(k) + I(k)} - m_2(k)\Big], \\ Z(k+1) = Z(k) \exp\Big[\dfrac{a(k)(S(k) + I(k))}{1 + b(k)(S(k) + I(k))} - m_3(k)\Big], \end{cases}
$$

which is a discrete time analogue of system (1.2). Here $k = 0, 1, 2, \ldots$

3. Existence of positive periodic solutions

For convenience and simplicity in our discussion, we use the following notation throughout the paper:

$$
I_\omega := \{0, 1, 2, \ldots, \omega - 1\}, \ \overline{f} := \frac{1}{\omega} \sum_{k=0}^{\omega-1} f(k), \ f^L := \min_{k \in \mathbb{Z}} \{f(k)\}, \ f^M := \max_{k \in \mathbb{Z}} \{f(k)\},
$$

where $f(k)$ is an ω-periodic sequence of real numbers defined for $k \in \mathbb{Z}$. We always assume that

(H1) $\quad a, b, \alpha, m_2, m_3, r_1, r_2\colon\ \mathbb{Z} \to \mathbb{R}^+$ are ω-periodic, i.e.,

$$a(k+\omega) = a(t),\ b(k+\omega) = b(k),\ \alpha(k+\omega) = \alpha(k),\ m_2(k+\omega) = m_2(k),$$
$$m_3(k+\omega) = m_3(k),\ r_1(k+\omega) = r_1(k),\ r_2(k+\omega) = r_2(k) \quad \text{for any } k \in \mathbb{Z}.$$

In order to explore the existence of positive periodic solutions of (2.3) and for the reader's convenience, we shall first introduce a few concepts and results without proof, borrowing them from [8].

Let X, Y be normed vector spaces, $L\colon \operatorname{Dom} L \subset X \to Y$ a linear mapping, $N\colon X \to Y$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projectors $P\colon X \to X$ and $Q\colon Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I-Q)$, it follows that $L : \operatorname{Dom} L \cap \operatorname{Ker} P\colon (I-P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N\colon \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 3.1 ([8], Continuation theorem)**.** *Let L be a Fredholm mapping of index zero and let N be L-compact on $\overline{\Omega}$. Suppose*
(a) *for each $\lambda \in (0,1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
(b) *$QNx \neq 0$ for each $x \in \operatorname{Ker} L \cap \partial\Omega$, and $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.*
Then the equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \overline{\Omega}$.

Lemma 3.2 ([5])**.** *Let $g\colon Z \to \mathbb{R}$ be ω-periodic, i.e., $g(k+\omega) = g(k)$, then for any fixed $k_1, k_2 \in I_\omega$ and any $k \in \mathbb{Z}$, one has*

$$g(k) \leqslant g(k_1) + \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|,$$
$$g(k) \geqslant g(k_2) - \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|.$$

Define

$$l_3 = \{y = \{y(k)\}\colon\ y(k) = (y_1(k), y_2(k), y_3(k))^{\mathrm{T}} \in \mathbb{R}^3,\ k \in \mathbb{Z}\}.$$

Let $l^\omega \subset l_3$ denote the subspace of all ω-periodic sequences equipped with the usual supremum norm $\|\cdot\|$, i.e., $\|y\| = |y_1(k)| + |y_2(k)| + |y_3(k)|$ for any $y = \{y(k)\colon\ k \in \mathbb{Z}\} \in l^\omega$. It is easy to show that l^ω is a finite-dimensional Banach space.

Let

$$l_0^\omega = \Big\{ y = \{y(k)\} \in l^\omega \colon \sum_{k=0}^{\omega-1} y(k) = 0 \Big\}, \tag{3.1}$$

$$l_c^\omega = \{ y = \{y(k)\} \in l^\omega \colon\ y(k) = h \in \mathbb{R}^3,\ k \in \mathbb{Z} \}. \tag{3.2}$$

Then it follows that both l_0^ω and l_c^ω are closed linear subspaces of l^ω and

$$l^\omega = l_0^\omega + l_c^\omega,\ \dim l_c^\omega = 3.$$

Now we are ready to establish our result.

Theorem 3.1. *Let* S_1, $\widetilde{S}_1$, S_2 *and* $\widetilde{S}_2$ *be defined by* (3.16), (3.33), (3.20) *and* (3.29), *respectively, and set*

$$\begin{aligned}
K_1 &= \bar{r}_1(\exp\{S_1\} + \exp\{S_2\}),\\
K_2 &= \bar{r}_2(\exp\{-S_1\} + \exp\{-S_2\}) - \bar{a}\exp\{S_1\},\\
K_3 &= \bar{r}_2(\exp\{\widetilde{S}_1\} + \exp\{\widetilde{S}_2\}),\\
K_4 &= \bar{r}_2(\exp\{-\widetilde{S}_1\} + \exp\{-\widetilde{S}_2\}) - \bar{a}\exp\{\widetilde{S}_1\}.
\end{aligned}$$

Suppose that conditions

$$\overline{m}_3 > \max\{\bar{a}\exp\{S_1\}, \bar{a}\exp\{\widetilde{S}_1\}\}, \tag{H2}$$

$$\bar{r}_2 - \overline{m}_3 > \max\{K_1, K_2, K_3, K_4\} \tag{H3}$$

hold. Then system (2.3) *has at least one* ω*-periodic solution.*

P r o o f. First of all, we make the change of variables

$$x_1(k) = \exp\{u_1(k)\}, \quad x_2(k) = \exp\{u_2(k)\}, \quad z(k) = \exp\{u_3(k)\},$$

then (2.3) can be reformulated as

$$\begin{cases}
u_1(k+1) - u_1(k) = r_1(k)(1 - \exp\{u_1(k)\} - \exp\{u_2(k)\}) \\
\qquad\qquad - \dfrac{a(k)\exp\{u_3(k)\}}{1 + b(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})} \\
\qquad\qquad - \dfrac{\alpha(k)\exp\{u_2(k)\}}{\exp\{u_1(k)\} + \exp\{u_2(k)\}}, \\
u_2(k+1) - u_2(k) = r_2(k)(1 - \exp\{u_1(k)\} - \exp\{u_2(k)\}) \\
\qquad\qquad - \dfrac{a(k)\exp\{u_3(k)\}}{1 + b(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})} \\
\qquad\qquad + \dfrac{\alpha(k)\exp\{u_1(k)\}}{\exp\{u_1(k)\} + \exp\{u_2(k)\}} - m_2(k), \\
u_3(k+1) - u_3(k) = \dfrac{a(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})}{1 + b(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})} - m_3(k).
\end{cases} \tag{3.3}$$

Let $X = Y = l^\omega$,

$$(3.4) \qquad (Lu)(k) = u(k+1) - u(k) = \begin{pmatrix} u_1(k+1) - u_1(k) \\ u_2(k+1) - u_2(k) \\ u_3(k+1) - u_3(k) \end{pmatrix},$$

$$(3.5) \qquad (Nu)(k) = \begin{pmatrix} f_1(u_1(k), u_2(k), u_3(k)) \\ f_2(u_1(k), u_2(k), u_3(k)) \\ f_3(u_1(k), u_2(k), u_3(k)) \end{pmatrix},$$

where $u \in X, k \in \mathbb{Z}$ and

$$\begin{aligned} f_1(u_1(k), u_2(k), u_3(k)) &= r_1(k)(1 - \exp\{u_1(k)\} - \exp\{u_2(k)\}) \\ &\quad - \frac{a(k)\exp\{u_3(k)\}}{1 + b(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})} \\ &\quad - \frac{\alpha(k)\exp\{u_2(k)\}}{\exp\{u_1(k)\} + \exp\{u_2(k)\}}, \\ f_2(u_1(k), u_2(k), u_3(k)) &= r_2(k)(1 - \exp\{u_1(k)\} - \exp\{u_2(k)\}) \\ &\quad - \frac{a(k)\exp\{u_3(k)\}}{1 + b(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})} \\ &\quad + \frac{\alpha(k)\exp\{u_1(k)\}}{\exp\{u_1(k)\} + \exp\{u_2(k)\}} - m_2(k), \\ f_3(u_1(k), u_2(k), u_3(k)) &= \frac{a(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})}{1 + b(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})} - m_3(k). \end{aligned}$$

Then it is trivial to see that L is a bounded linear operator and

$$\operatorname{Ker} L = l_c^\omega, \quad \operatorname{Im} L = l_0^\omega,$$

and

$$\dim \operatorname{Ker} L = 3 = \operatorname{codim} \operatorname{Im} L.$$

Then it follows that L is a Fredholm mapping of index zero. Define

$$Py = \frac{1}{\omega}\sum_{s=0}^{\omega-1} y(s), \quad y \in X, \quad Qz = \frac{1}{\omega}\sum_{s=0}^{\omega-1} z(s), \quad z \in Y.$$

It is not difficult to show that P and Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \quad \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q).$$

Furthermore, the generalized inverse (to L) $K_P\colon \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ exists and is given by

$$K_P(z) = \sum_{s=0}^{\omega-1} z(s) - \frac{1}{\omega}\sum_{s=0}^{\omega-1}(\omega - s)z(s).$$

Obviously, QN and $K_P(I-Q)N$ are continuous. Since X is a finite-dimensional Banach space, using the Arzelà-Ascoli theorem it is not difficult to show that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$.

Now we are at the point to search for an appropriate open, bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation $Lu = \lambda Nu, \lambda \in (0,1)$, we have

$$
\begin{cases}
u_1(k+1) - u_1(k) = \lambda f_1(u_1(k), u_2(k), u_3(k)), \\
u_2(k+1) - u_2(k) = \lambda f_2(u_1(k), u_2(k), u_3(k)), \\
u_3(k+1) - u_3(k) = \lambda f_3(u_1(k), u_2(k), u_3(k)).
\end{cases} \tag{3.6}
$$

Suppose that $u(k) = (u_1(k), u_2(k), u_3(k))^{\mathrm{T}} \in X$ is an arbitrary solution of system (3.6) for a certain $\lambda \in (0,1)$. Summing both sides of (3.6) from 0 to $\omega - 1$ with respect to k, we obtain

$$
\begin{cases}
\sum\limits_{k=0}^{\omega-1} \Big[r_1(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\}) + \dfrac{a(k)\exp\{u_3(k)\}}{1 + b(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})} \\
\qquad\qquad + \dfrac{\alpha(k)\exp\{u_2(k)\}}{\exp\{u_1(k)\} + \exp\{u_2(k)\}} \Big] = \bar{r}_1\omega, \\
\sum\limits_{k=0}^{\omega-1} \Big[r_2(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\}) + \dfrac{a(k)\exp\{u_3(k)\}}{1 + b(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})} \\
\qquad\qquad - \dfrac{\alpha(k)\exp\{u_1(k)\}}{\exp\{u_1(k)\} + \exp\{u_2(k)\}} \Big] = (\bar{r}_2 - \overline{m}_3)\omega, \\
\sum\limits_{k=0}^{\omega-1} \Big[\dfrac{a(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})}{1 + b(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})} \Big] = \overline{m}_3\omega.
\end{cases} \tag{3.7}
$$

From (3.7) it follows that

$$
\sum_{k=0}^{\omega-1} |u_1(k+1) - u_1(k)| \leqslant 2\bar{r}_1\omega, \tag{3.8}
$$

$$
\sum_{k=0}^{\omega-1} |u_2(k+1) - u_2(k)| \leqslant 2(\bar{r}_2 - \overline{m}_3)\omega, \tag{3.9}
$$

$$
\sum_{k=0}^{\omega-1} |u_3(k+1) - u_3(k)| \leqslant 2\overline{m}_3\omega. \tag{3.10}
$$

In view of the hypothesis that $u = \{u(k)\} \in X$, there exist $\xi_i, \eta_i \in I_\omega$ such that

$$
u_i(\xi_i) = \min_{k \in I_\omega}\{u_i(k)\}, \quad u_i(\eta_i) = \max_{k \in I_\omega}\{u_i(k)\} \ (i = 1,2,3). \tag{3.11}
$$

From the first equation of (3.7), we have

$$\bar{r}_1\omega > \sum_{k=0}^{\omega-1} r_1(k)\exp\{u_1(k)\} \geqslant \bar{r}_1\omega\exp(u_1(\xi_1)),$$
$$\bar{r}_1\omega > \sum_{k=0}^{\omega-1} r_1(k)\exp\{u_2(k)\} \geqslant \bar{r}_1\omega\exp(u_2(\xi_2)),$$

which leads to

$$u_1(\xi_1) < 0, \quad u_2(\xi_2) < 0. \tag{3.12}$$

In the sequel, we consider two cases.

Case (a). If $u_1(\eta_1) \geqslant u_2(\eta_2)$, then it follows from the third equation of (3.7) that

$$\begin{aligned}\overline{m}_3\omega &\leqslant \sum_{k=0}^{\omega-1}[a(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})]\\ &\leqslant \sum_{k=0}^{\omega-1}[2a(k)\exp\{u_1(\eta_1)\}] = 2\bar{a}\omega\exp\{u_1(\eta_1)\}.\end{aligned}$$

Then

$$u_1(\eta_1) \geqslant \ln\left[\frac{\overline{m}_3}{2\bar{a}}\right]. \tag{3.13}$$

Therefore

$$u_1(k) \leqslant u_1(\xi_1) + \sum_{s=0}^{\omega-1}|u_1(s+1) - u_1(s)| \leqslant 2\bar{r}_1\omega := M_1, \tag{3.14}$$

$$u_1(k) \geqslant u_1(\eta_1) - \sum_{s=0}^{\omega-1}|u_1(s+1) - u_1(s)| \geqslant \ln\left[\frac{\overline{m}_3}{2\bar{a}}\right] - 2\bar{r}_1\omega := m_1. \tag{3.15}$$

Thus

$$\max_{k\in I_\omega}\{u_1(k)\} \leqslant \max\{|m_1|, |M_1|\} := S_1. \tag{3.16}$$

By the third equation of (3.7) again, we get

$$\overline{m}_3\omega \leqslant \sum_{k=0}^{\omega-1}[a(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})] \leqslant \sum_{k=0}^{\omega-1}[a(k)(\exp\{S_1\} + \exp\{u_2(\eta_2)\})].$$

Then

$$u_2(\eta_2) \geqslant \ln\Big[\frac{\overline{m}_3 - \bar{a}\exp\{S_1\}}{\bar{a}}\Big]. \tag{3.17}$$

From Lemma 3.2, (3.12) and (3.17), we get

$$u_2(k) \leqslant u_2(\xi_2) + \sum_{s=0}^{\omega-1} |u_2(s+1) - u_2(s)| \leqslant 2(\bar{r}_2 - \overline{m}_3)\omega := M_2, \tag{3.18}$$

$$\begin{aligned} u_2(k) &\geqslant u_2(\eta_2) - \sum_{s=0}^{\omega-1} |u_2(s+1) - u_2(s)| \\ &\geqslant \ln\Big[\frac{\overline{m}_3 - \bar{a}\exp\{S_1\}}{\bar{a}}\Big] - 2(\bar{r}_2 - \overline{m}_3)\omega := m_2. \end{aligned} \tag{3.19}$$

Then

$$\max_{k\in I_\omega}\{u_2(k)\} \leqslant \max\{|m_2|, |M_2|\} := S_2. \tag{3.20}$$

From the second equation of (3.7), we obtain

$$\sum_{k=0}^{\omega-1}[r_2(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\}) + a(k)\exp\{u_3(k)\}] > (\bar{r}_2 - \overline{m}_3)\omega$$

and

$$\begin{aligned} \sum_{k=0}^{\omega-1}\Big[&r_2(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\}) \\ &+ \frac{a(k)\exp\{u_3(k)\}}{1 + b(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})} - \alpha(k)\exp\{u_1(k)\}\Big] < (\bar{r}_2 - \overline{m}_3)\omega, \end{aligned}$$

hence

$$\sum_{k=0}^{\omega-1}[r_2(k)(\exp\{S_1\} + \exp\{S_2\}) + a(k)\exp\{u_3(\eta_3)\}] > (\bar{r}_2 - \overline{m}_3)\omega$$

and

$$\begin{aligned} \sum_{k=0}^{\omega-1}\Big[&r_2(k)(\exp\{-S_1\} + \exp\{-S_2\}) \\ &+ \frac{a(k)\exp\{u_3(\xi_3)\}}{1 + b^M(\exp\{S_1\} + \exp\{S_2\})} - \alpha(k)\exp\{S_1\}\Big] < (\bar{r}_2 - \overline{m}_3)\omega. \end{aligned}$$

Then

$$u_3(\xi_3) < \ln\Big[\frac{(\bar r_2 - \overline{m}_3) - \bar r_2(\exp\{S_1\} + \exp\{S_2\})}{\bar a}\Big], \tag{3.21}$$

$$u_3(\eta_3) > \ln\Big[\frac{\Theta_1\Theta_2}{\bar a}\Big], \tag{3.22}$$

where

$$\begin{aligned}\Theta_1 &= (\bar r_2 - \overline{m}_3) - \bar r_2(\exp\{-S_1\} + \exp\{-S_2\}) + \bar a \exp\{S_1\},\\ \Theta_2 &= 1 + b^M(\exp\{S_1\} + \exp\{S_2\}).\end{aligned}$$

From Lemma 3.2, (3.21) and (3.22), we derive

$$\begin{aligned}u_3(k) &\leqslant u_3(\xi_3) + \sum_{s=0}^{\omega-1}|u_3(s+1) - u_3(s)| \\ &\leqslant \ln\Big[\frac{(\bar r_2 - \overline{m}_3) - \bar r_2(\exp\{S_1\} + \exp\{S_2\})}{\bar a}\Big] + 2\overline{m}_3\omega := M_3^*,\end{aligned} \tag{3.23}$$

$$\begin{aligned}u_3(k) &\geqslant u_3(\eta_3) - \sum_{s=0}^{\omega-1}|u_3(s+1) - u_3(s)| \\ &\geqslant \ln\Big[\frac{\Theta_1\Theta_2}{\bar a}\Big] - 2\overline{m}_3\omega := m_3^*.\end{aligned} \tag{3.24}$$

Then

$$\max_{k\in I_\omega}\{u_3(k)\} \leqslant \max\{|m_3^*|, |M_3^*|\} := S_3. \tag{3.25}$$

Case (*b*). If $u_1(\eta_1) < u_2(\eta_2)$, then it follows from the second equation of (3.7) that

$$\begin{aligned}\overline{m}_3\omega &\leqslant \sum_{k=0}^{\omega-1}[a(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})] \\ &< \sum_{k=0}^{\omega-1}[2a(k)\exp\{u_2(\eta_2)\}] = 2\bar a\omega\exp\{u_2(\eta_2)\}.\end{aligned}$$

Then

$$u_2(\eta_2) \geqslant \ln\Big[\frac{\overline{m}_3}{2\bar a}\Big]. \tag{3.26}$$

Therefore

$$(3.27)\quad u_2(k) \leqslant u_2(\xi_2) + \sum_{s=0}^{\omega-1} |u_2(s+1) - u_2(s)| \leqslant 2(\bar{r}_2 - \overline{m}_3)\omega := \widetilde{M}_2,$$

$$(3.28)\quad u_2(k) \geqslant u_2(\eta_2) - \sum_{s=0}^{\omega-1} |u_2(s+1) - u_2(s)| \geqslant \ln\Big[\frac{\overline{m}_3}{2\bar{a}}\Big] - 2(\bar{r}_2 - \overline{m}_3)\omega := \widetilde{m}_2.$$

Thus

$$(3.29)\qquad \max_{k\in I_\omega}\{u_2(k)\} \leqslant \max\{|\widetilde{m}_2|, |\widetilde{M}_2|\} := \widetilde{S}_2.$$

By the third equation of (3.7) again, we get

$$\begin{aligned}\overline{m}_3\omega &\leqslant \sum_{k=0}^{\omega-1}[a(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\})]\\ &\leqslant \sum_{k=0}^{\omega-1}[a(k)(\exp\{u_1(\eta_1)\} + \exp\{\widetilde{S}_2\})].\end{aligned}$$

Then

$$(3.30)\qquad u_1(\eta_1) \geqslant \ln\Big[\frac{\overline{m}_3 - \bar{a}\exp\{\widetilde{S}_2\}}{\bar{a}}\Big].$$

From Lemma 3.2, (3.12) and (3.30), we get

$$(3.31)\; u_1(k) \leqslant u_1(\xi_1) + \sum_{s=0}^{\omega-1} |u_1(s+1) - u_1(s)| \leqslant 2\bar{r}_1\omega := \widetilde{M}_1,$$

$$(3.32)\; u_1(k) \geqslant u_1(\eta_1) - \sum_{s=0}^{\omega-1} |u_1(s+1) - u_1(s)| \geqslant \ln\Big[\frac{\overline{m}_3 - \bar{a}\exp\{\widetilde{S}_2\}}{\bar{a}}\Big] - \bar{r}_1\omega := \widetilde{m}_1.$$

Then

$$(3.33)\qquad \max_{k\in I_\omega}\{u_1(k)\} \leqslant \max\{|\widetilde{m}_1|, |\widetilde{M}_1|\} := \widetilde{S}_1.$$

From the second equation of (3.7), we obtain

$$\sum_{k=0}^{\omega-1}[r_2(k)(\exp\{u_1(k)\} + \exp\{u_2(k)\}) + a(k)\exp\{u_3(k)\}] > (\bar{r}_2 - \overline{m}_3)\omega$$

and

$$\sum_{k=0}^{\omega-1}\Big[r_2(k)(\exp\{u_1(k)\}+\exp\{u_2(k)\})$$
$$+\frac{a(k)\exp\{u_3(k)\}}{1+b(k)(\exp\{u_1(k)\}+\exp\{u_2(k)\})}-\alpha(k)\exp\{u_1(k)\}\Big]<(\bar r_2-\overline{m}_3)\omega,$$

hence

$$\sum_{k=0}^{\omega-1}[r_2(k)(\exp\{\widetilde S_1\}+\exp\{\widetilde S_2\})+a(k)\exp\{u_3(\eta_3)\}]>(\bar r_2-\overline{m}_3)\omega$$

and

$$\sum_{k=0}^{\omega-1}\Big[r_2(k)(\exp\{-\widetilde S_1\}+\exp\{-\widetilde S_1\})$$
$$+\frac{a(k)\exp\{u_3(\xi_3)\}}{1+b^M(\exp\{\widetilde S_1\}+\exp\{\widetilde S_1\})}-\alpha(k)\exp\{\widetilde S_1\}\Big]<(\bar r_2-\overline{m}_3)\omega.$$

Then

$$u_3(\xi_3)<\ln\Big[\frac{(\bar r_2-\overline{m}_3)-\bar r_2(\exp\{\widetilde S_1\}+\exp\{\widetilde S_2\})}{\bar a}\Big], \tag{3.34}$$

$$u_3(\eta_3)>\ln\Big[\frac{\widetilde\Theta_1\widetilde\Theta_2}{\bar a}\Big], \tag{3.35}$$

where

$$\widetilde\Theta_1=(\bar r_2-\overline{m}_3)-\bar r_2(\exp\{-\widetilde S_1\}+\exp\{-\widetilde S_2\})+\bar a\exp\{\widetilde S_1\},$$
$$\widetilde\Theta_2=1+b^M(\exp\{\widetilde S_1\}+\exp\{\widetilde S_2\}).$$

From Lemma 3.2, (3.21) and (3.22), we derive

$$u_3(k)\leqslant u_3(\xi_3)+\sum_{s=0}^{\omega-1}|u_3(s+1)-u_3(s)| \tag{3.36}$$
$$\leqslant\ln\Big[\frac{(\bar r_2-\overline{m}_3)-\bar r_2(\exp\{\widetilde S_1\}+\exp\{\widetilde S_2\})}{\bar a}\Big]+2\overline{m}_3\omega:=\widetilde{M}_3^*,$$

$$u_3(k)\geqslant u_3(\eta_3)-\sum_{s=0}^{\omega-1}|u_3(s+1)-u_3(s)| \tag{3.37}$$
$$\geqslant\ln\Big[\frac{\widetilde\Theta_1\widetilde\Theta_2}{\bar a}\Big]-2\overline{m}_3\omega:=\widetilde{m}_3^*.$$

Then

$$\max_{k\in I_\omega}\{u_3(k)\} \leqslant \max\{|\widetilde{m}_3^*|, |\widetilde{M}_3^*|\} := \widetilde{S}_3. \tag{3.38}$$

Obviously, m_i, M_i $(i=1,2)$, m_3^*, M_3^*, $\widetilde{m}_i^*$ $(i=1,2)$, $\widetilde{m}_3^*$ and $\widetilde{M}_3^*$ are independent of the choice of $\lambda \in (0,1)$. Take $M = \max\{S_1, \widetilde{S}_1\} + \max\{S_2, \widetilde{S}_2\} + \max\{S_3, \widetilde{S}_3\} + M_0$, where M_0 is taken sufficiently large such that

$$\max\{|\ln\{u_1^*\}|, |\ln\{u_2^*\}|, |\ln\{u_3^*\}|\} < M_0,$$

where $(u_1^*, u_2^*, u_3^*)^{\mathrm{T}}$ is the unique solution of the system of equations

$$\begin{cases} \bar{r}_1 - \bar{r}_1 \exp\{u_1(k)\} = 0, \\ \bar{r}_2 - \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1}\Big[\frac{a(k)\exp\{u_3(k)\}}{1+b(k)(\exp\{u_1(k)\}+\exp\{u_2(k)\})}\Big] = 0, \\ \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1}\Big[\frac{a(k)(\exp\{u_1(k)\}+\exp\{u_2(k)\})}{1+b(t)(\exp\{u_1(k)\}+\exp\{u_2(k)\})}\Big] - \overline{m}_3 = 0. \end{cases} \tag{3.39}$$

Now we will prove that any solution $u = \{u(k)\} = \{(u_1(k), u_2(k), u_3(k))^{\mathrm{T}}\}$ of (3.6) in X satisfies $\|u\| < M$, $k \in \mathbb{Z}$.

Let $\Omega := \{u = \{u(k)\} \in X\colon\ \|u\| < M\}$, then it is easy to see that Ω is an open, bounded set in X and verifies requirement (a) of Lemma 3.1. When $u \in \partial\Omega \cap \operatorname{Ker} L$, $u = \{(u_1(k), u_2(k), u_3(k))^{\mathrm{T}}\}$ is a constant vector in $\mathbb{R}^3$ with $\|u\| = |u_1| + |u_2| + |u_3| = M$. Then

$$QNu = \begin{pmatrix} \overline{f}_1 \\ \overline{f}_2 \\ \overline{f}_3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.40}$$

where

$$\begin{aligned} \overline{f}_1 &= \bar{r}_1(1 - \exp\{u_1(k)\} - \exp\{u_2(k)\}) - \frac{1}{\omega}\sum_{k=0}^{\omega-1}\Big[\frac{a(k)\exp\{u_3(k)\}}{1+b(k)(\exp\{u_1(k)\}+\exp\{u_2(k)\})}\Big] \\ &\quad - \sum_{k=0}^{\omega-1}\Big[\frac{\alpha(k)\exp\{u_2(k)\}}{\exp\{u_1(k)\}+\exp\{u_2(k)\}}\Big], \\ \overline{f}_2 &= \bar{r}_2(1 - \exp\{u_1(k)\} - \exp\{u_2(k)\}) - \frac{1}{\omega}\sum_{k=0}^{\omega-1}\Big[\frac{a(k)\exp\{u_3(k)\}}{1+b(k)(\exp\{u_1(k)\}+\exp\{u_2(k)\})}\Big] \\ &\quad + \frac{1}{\omega}\sum_{k=0}^{\omega-1}\Big[\frac{\alpha(k)\exp\{u_1(k)\}}{\exp\{u_1(k)\}+\exp\{u_2(k)\}}\Big] - \overline{m}_2, \\ \overline{f}_3 &= \frac{1}{\omega}\sum_{k=0}^{\omega-1}\Big[\frac{a(k)(\exp\{u_1(k)\}+\exp\{u_2(k)\})}{1+b(k)(\exp\{u_1(k)\}+\exp\{u_2(k)\})}\Big] - \overline{m}_3. \end{aligned}$$

Now let us consider the homotopy $\varphi(u_1,u_2,u_3,\mu)=\mu QNu+(1-\mu)Gu$, $\mu\in[0,1]$, where

$$Gu=\begin{pmatrix} \bar r_1-\bar r_1\exp\{u_1(k)\} \\ \bar r_2-\dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1}\Big[\frac{a(k)\exp\{u_3(k)\}}{1+b(k)(\exp\{u_1(k)\}+\exp\{u_2(k)\})}\Big] \\ \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1}\Big[\frac{a(k)(\exp\{u_1(k)\}+\exp\{u_2(k)\})}{1+b(k)(\exp\{u_1(k)\}+\exp\{u_2(k)\})}\Big]-\overline{m}_3 \end{pmatrix}.$$

Letting J be equal to the identity mapping I, by direct calculation we get

$$\begin{aligned}\deg\{JQN(u_1,u_2,u_3)^{\mathrm T};\ \Omega\cap\operatorname{Ker}L;\ 0\} &= \deg\{QN(u_1,u_2,u_3)^{\mathrm T};\ \Omega\cap\operatorname{Ker}L;\ 0\}\\ &= \deg\{\varphi(u_1,u_2,u_3,1);\ \Omega\cap\operatorname{Ker}L;\ 0\}\\ &= \deg\{\varphi(u_1,u_2,u_3,0);\ \Omega\cap\operatorname{Ker}L;\ 0\}\\ &= \operatorname{sign}\left\{\det\begin{pmatrix}\chi_{11} & 0 & 0\\ \chi_{21} & \chi_{22} & \chi_{23}\\ \chi_{21} & \chi_{32} & 0\end{pmatrix}\right\}\end{aligned}$$

where

$$\begin{aligned}\chi_{11}&=\bar r_1\exp\{u_1^*\}>0,\\ \chi_{21}&=\frac{1}{\omega}\sum_{k=0}^{\omega-1}\Big[\frac{a(k)b(k)\exp\{u_1^*+u_3^*\}}{[1+b(k)(\exp\{u_1^*\}+\exp\{u_2^*\})]^2}\Big]>0,\\ \chi_{22}&=\frac{1}{\omega}\sum_{k=0}^{\omega-1}\Big[\frac{a(k)\exp\{u_2^*+u_3^*\}}{[1+b(k)(\exp\{u_1^*\}+\exp\{u_2^*\})]^2}\Big]>0,\\ \chi_{23}&=-\frac{1}{\omega}\sum_{k=0}^{\omega-1}\Big[\frac{a(k)\exp\{u_3^*\}}{[1+b(k)(\exp\{u_1^*\}+\exp\{u_2^*\})]^2}\Big]<0,\\ \chi_{31}&=\frac{1}{\omega}\sum_{k=0}^{\omega-1}\Big[\frac{a(k)\exp\{u_1^*\}}{1+b(k)(\exp\{u_1^*\}+\exp\{u_2^*\})}\Big]>0,\\ \chi_{32}&=\frac{1}{\omega}\sum_{k=0}^{\omega-1}\Big[\frac{a(k)\exp\{u_2^*\}}{1+b(k)(\exp\{u_1^*\}+\exp\{u_2^*\})}\Big]>0.\end{aligned}$$

Then

$$\deg\{JQN(u_1,u_2,u_3)^{\mathrm T};\Omega\cap\operatorname{Ker}L;0\}=\operatorname{sign}\{\chi_{11}\chi_{23}\chi_{32}\}=-1\neq 0.$$

By now, we have proved that Ω verifies all requirements of Lemma 3.1, hence it follows that $Lu=Nu$ has at least one solution in $\operatorname{Dom}L\cap\overline{\Omega}$, that is to say, (3.3) has at least one ω-periodic solution in $\operatorname{Dom}L\cap\overline{\Omega}$, say $u^*=\{u^*(k)\}=\{(u_1^*(k),u_2^*(k),u_3^*(k))^{\mathrm T}\}$. Let $x^*(k)=(x_1^*(k),x_2^*(k),z^*(k))^{\mathrm T}=(\exp\{u_1^*(k)\},\exp\{u_2^*(k)\},\exp\{u_3^*(k)\})^{\mathrm T}$, then it follows that $x^*(k)$ is an ω-periodic solution of system (2.3) with strictly positive components. The proof is complete. □

4. Global asymptotic stability

In this section, we shall present sufficient conditions for the global asymptotic stability of system (2.3).

Theorem 4.1. *Let A_1, A_2 and A_3 be defined by* (4.7), (4.8) *and* (4.9), *respectively. Assume that* (H1), (H2) *and* (H3) *are satisfied and furthermore suppose that there exist positive constants v, θ_1, θ_2 and θ_3 such that*

$$\text{(i)}\qquad 1-\max\{M_1,\widetilde{M}_1\}+\frac{b(k)\min\{m_3,\widetilde{m}_3\}\min\{m_1,\widetilde{m}_1\}}{(1+b(k)(\max\{M_1,\widetilde{M}_1\}+\max\{M_2,\widetilde{M}_2\}))^2} + \frac{\alpha(k)\min\{m_1,\widetilde{m}_1\}\min\{m_2,\widetilde{m}_2\}}{(\max\{M_1,\widetilde{M}_1\}+\max\{M_2,\widetilde{M}_2\})^2} > v,$$

$$\text{(ii)}\qquad \Big[1-\max\{M_2,\widetilde{M}_2\}+\frac{b(k)\min\{m_3,\widetilde{m}_3\}}{(1+b(k)(\max\{M_1,\widetilde{M}_1\}+\max\{M_2,\widetilde{M}_2\}))^2} + \frac{a(k)\min\{m_2,\widetilde{m}_2\}}{(\max\{M_1,\widetilde{M}_1\}+\max\{M_2,\widetilde{M}_2\})^2}\Big] > v,$$

and $A_i > 0$ $(i = 1,2,3)$. Then the positive ω-periodic solution of system (2.3) *is globally asymptotically stable.*

Proof. In view of Theorem 3.1, there exists a positive periodic solution $\{S^*(k), I^*(k), Z^*(k)\}$ of system (2.3). Now we prove that it is uniformly asymptotically stable. First, we make the change of variable

$$N_1(k)=S(k)-S^*(k),\quad N_2(k)=I(k)-I^*(k),\quad N_3(k)=Z(k)-Z^*(k). \tag{4.1}$$

It follows from (2.3) that

$$\begin{aligned} N_1(k+1) &= S(k+1)-S^*(k+1) \\ &= S(k)\exp\Big[r_1(k)(1-S(k)-I(k))-\frac{a(k)Z(k)}{1+b(k)(S(k)+I(k))} \\ &\quad -\frac{\alpha(k)I(k)}{S(k)+I(k)}\Big]-S^*(k)\exp\Big[r_1(k)(1-S^*(k)-I^*(k)) \\ &\quad -\frac{a(k)Z^*(k)}{1+b(k)(S^*(k)+I^*(k))}-\frac{\alpha(k)I^*(k)}{S^*(k)+I^*(k)}\Big] \\ &= \Big\{S(k)\exp\Big[-N_1(k)-N_2(k) \\ &\quad -\Big(\frac{a(k)Z(k)}{1+b(k)(S(k)+I(k))}-\frac{a(k)Z^*(k)}{1+b(k)(S^*(k)+I^*(k))}\Big) \\ &\quad \times\Big(\frac{\alpha(k)I(k)}{S(k)+I(k)}-\frac{\alpha(k)I^*(k)}{S^*(k)+I^*(k)}\Big)-S^*(k)\Big\}\frac{S^*(k+1)}{S^*(k)} \end{aligned} \tag{4.2}$$

$$
\begin{aligned}
&= \Big\{\Big[1 - S^*(k) + \frac{b(k)Z^*(k)S^*(k)}{(1+b(k)(S^*(k)+I^*(k)))^2} + \frac{\alpha(k)I^*(k)S^*(k)}{(S^*(k)+I^*(k)))^2}\Big]\\
&\quad \times \frac{N_1(k)}{S^*(k)} + \Big[\frac{b(k)Z^*(k)}{(1+b(k)(S^*(k)+I^*(k)))^2} - \frac{\alpha(k)S^*(k)}{(S^*(k)+I^*(k))^2} - 1\Big]\\
&\quad \times N_2(k) + \frac{a(k)}{1+b(k)(S^*(k)+I^*(k))}N_3(k) + f_1\Big\}S^*(k+1),
\end{aligned}
$$

$$
\begin{aligned}
(4.3)\quad N_2(k+1) &= I(k+1) - I^*(k+1)\\
&= I(k)\exp\Big[r_2(k)(1-S(k)-I(k)) - \frac{a(k)Z(k)}{1+b(k)(S(k)+I(k))}\\
&\quad + \frac{\alpha(k)S(k)}{S(k)+I(k)} - m_2(k)\Big] - I^*(k)\exp\Big[r_2(k)(1-S^*(k)-I^*(k))\\
&\quad - \frac{a(k)Z^*(k)}{1+b(k)(S^*(k)+I^*(k))} + \frac{\alpha(k)S^*(k)}{S^*(k)+I^*(k)} - m_2(k)\Big]\\
&= \Big\{I(k)\exp\Big[-N_1(k) - N_2(k) - \Big(\frac{a(k)Z(k)}{1+b(k)(S(k)+I(k))}\\
&\quad - \frac{a(k)Z^*(k)}{1+b(k)(S^*(k)+I^*(k))}\Big) + \Big(\frac{\alpha(k)S(k)}{S(k)+I(k)}\\
&\quad - \frac{\alpha(k)S^*(k)}{S^*(k)+I^*(k)}\Big) - I^*(k)\Big\}\frac{I^*(k+1)}{I^*(k)}\\
&= \Big\{\Big[1 - I^*(k) + \frac{b(k)Z^*(k)}{(1+b(k)(S^*(k)+I^*(k)))^2} + \frac{a(k)I^*(k)}{(S^*(k)+I^*(k))^2}\Big]\\
&\quad \times \frac{N_2(k)}{I^*(k)} + \Big[\frac{b(k)Z^*(k)}{(1+b(k)(S^*(k)+I^*(k)))^2} + \frac{a(k)S^*(k)}{(S^*(k)+I^*(k))^2} - 1\Big]\\
&\quad \times N_1(k) - \frac{a(k)}{1+b(k)(S^*(k)+I^*(k))}N_3(k) + f_2\Big\}I^*(k+1),
\end{aligned}
$$

$$
\begin{aligned}
(4.4)\quad N_3(k+1) &= Z(k+1) - Z^*(k+1)\\
&= Z(k)\exp\Big[\frac{a(k)(S(k)+I(k))}{1+b(k)(S(k)+I(k))} - m_3(k)\Big]\\
&\quad - Z^*(k)\exp\Big[\frac{a(k)(S^*(k)+I^*(k))}{1+b(k)(S^*(k)+I^*(k))} - m_3(k)\Big]\\
&= \Big\{Z(k)\exp\Big[\frac{a(k)(S(k)+I(k))}{1+b(k)(S(k)+I(k))} - \frac{a(k)(S^*(k)+I^*(k))}{1+b(k)(S^*(k)+I^*(k))}\Big]\\
&\quad - z^*(k)\Big\}\frac{Z^*(k+1)}{Z^*(k)}\\
&= \Big\{\frac{a(k)(S^*(k)+I^*(k))}{1+b(k)(S^*(k)+I^*(k))}\frac{N_3(k)}{Z^*(k)} + \frac{a(k)}{(1+b(k)(S^*(k)+I^*(k)))^2}\\
&\quad \times N_1(k) + \frac{a(k)}{(1+b(k)(S^*(k)+I^*(k)))^2}N_2(k) + f_3\Big\}Z^*(k+1),
\end{aligned}
$$

where $f_i/\|N_i\|$ $(i=1,2,3)$ converges, uniformly with respect to $k \in \mathbb{Z}^+$, to zero as $\|N\| \to 0$.

Define a function V by

$$V(N(k)) = \theta_1\Big|\frac{N_1(k)}{S^*(k)}\Big| + \theta_2\Big|\frac{N_2(k)}{I^*(k)}\Big| + \theta_3\Big|\frac{N_3(k)}{Z^*(k)}\Big|, \tag{4.5}$$

where θ_1, θ_2 and θ_3 are positive constants given by (4.7), (4.8) and (4.9), respectively. Calculating the difference of V along the solution of system (4.2)–(4.4), in view of (i) and (ii) we get

$$\begin{aligned}
\triangle V = {} & \theta_1\Big(\Big|\frac{N_1(k+1)}{S^*(k+1)} - \frac{N_1(k)}{S^*(k)}\Big|\Big) + \theta_2\Big(\Big|\frac{N_2(k+1)}{I^*(k+1)} - \frac{N_2(k)}{I^*(k)}\Big|\Big) \\
& + \theta_3\Big(\Big|\frac{N_3(k+1)}{Z^*(k+1)} - \frac{N_3(k)}{z^*(k)}\Big|\Big) \\
\leqslant {} & -\theta_1\Big[S^*(k) - \frac{b(k)Z^*(k)S^*(k)}{(1+b(k)(S^*(k)+I^*(k)))^2} - \frac{\alpha(k)S^*(k)I^*(k)}{(S^*(k)+I^*(k))^2}\Big]|N_1(k)| \\
& + \theta_1\Big[1 + \frac{b(k)Z^*(k)S^*(k)}{(1+b(k)(S^*(k)+I^*(k)))^2} + \frac{\alpha(k)S^*(k)}{(S^*(k)+I^*(k))^2}\Big]|N_2(k)| \\
& + \theta_1\frac{a(k)}{1+b(k)(S^*(k)+I^*(k))}|N_3(k)| + \theta_1 f_1 \\
& - \theta_2\Big[I^*(k) - \frac{b(k)Z^*(k)}{(1+b(k)(S^*(k)+I^*(k)))^2} - \frac{a(k)I^*(k)}{(S^*(k)+I^*(k))^2}\Big]|N_2(k)| \\
& + \theta_2\Big[1 + \frac{b(k)Z^*(k)}{(1+b(k)(S^*(k)+I^*(k)))^2} + \frac{\alpha(k)S^*(k)}{(S^*(k)+I^*(k))^2}\Big]|N_1(k)| \\
& + \theta_2\frac{a(k)}{1+b(k)(S^*(k)+I^*(k))}|N_3(k)| + \theta_2 f_2 \\
& - \theta_3\Big[1 - \frac{a(k)(S^*(k)+I^*(k))}{1+b(k)(S^*(k)+I^*(k))}\Big]|N_3(k)| + \theta_3\frac{\alpha(k)}{(S^*(k)+I^*(k))^2}|N_1(k)| \\
& + \theta_3\frac{a(k)}{(1+b(k)(S^*(k)+I^*(k)))^2}|N_2(k)| \\
\leqslant {} & -\theta_1\Big[\min\{m_1,\widetilde{m}_1\} - \frac{b(k)\max\{M_3,\widetilde{M}_3\}\max\{M_1,\widetilde{M}_1\}}{(1+b(k)(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\}))^2} \\
& - \frac{\alpha(k)\max\{M_1,\widetilde{M}_1\}\max\{M_1,\widetilde{M}_1\}}{(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\})^2}\Big]|N_1(k)| \\
& + \theta_1\Big[1 + \frac{b(k)\max\{M_1,\widetilde{M}_1\}\max\{M_3,\widetilde{M}_3\}}{(1+b(k)(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\}))^2} \\
& + \frac{\alpha(k)\max\{M_1,\widetilde{M}_1\}}{(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\})^2}\Big]|N_2(k)|
\end{aligned} \tag{4.6}$$

$$
\begin{aligned}
&+\theta_1 \frac{a(k)}{1+b(k)(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\})}|N_3(k)|+\theta_1 f_1 \\
&-\theta_2\Big[\min\{m_2,\widetilde{m}_2\}-\frac{b(k)\max\{M_3,\widetilde{M}_3\}}{(1+b(k)(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\}))^2} \\
&-\frac{a(k)\max\{M_2,\widetilde{M}_2\}}{(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\})^2}\Big]|N_2(k)| \\
&+\theta_2\Big[1+\frac{b(k)\max\{M_3,\widetilde{M}_3\}}{(1+b(k)(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\}))^2} \\
&+\frac{\alpha(k)\max\{M_1,\widetilde{M}_1\}}{(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\})^2}\Big]|N_1(k)| \\
&+\theta_2 \frac{a(k)}{1+b(k)(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\})}|N_3(k)|+\theta_2 f_2 \\
&-\theta_3\Big[1-\frac{a(k)(\max\{M_1,\widetilde{M}_1\}+\max\{M_2,\widetilde{M}_2\})}{1+b(k)(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\})}\Big]|N_3(k)| \\
&+\theta_3 \frac{\alpha(k)}{(S^*(k)+I^*(k))^2}|N_1(k)| \\
&+\theta_3 \frac{a(k)}{(1+b(k)(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\}))^2}|N_2(k)|+\theta_3 f_3 \\
=\ &-A_1|N_1(k)|-A_2|N_2(k)|-A_3|N_3(k)|+\sum_{i=1}^{3}\theta_i f_i,
\end{aligned}
$$

where

$$
\begin{aligned}
(4.7)\qquad A_1=\ &\theta_1\Big[\frac{b(k)\max\{M_3,\widetilde{M}_3\}\max\{M_1,\widetilde{M}_1\}}{(1+b(k)(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\}))^2} \\
&+\frac{\alpha(k)\max\{M_1,\widetilde{M}_1\}\max\{M_1,\widetilde{M}_1\}}{(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\})^2}-\min\{m_1,\widetilde{m}_1\}\Big] \\
&-\theta_2\Big[1+\frac{b(k)\max\{M_3,\widetilde{M}_3\}}{(1+b(k)(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\}))^2} \\
&+\frac{\alpha(k)\max\{M_1,\widetilde{M}_1\}}{(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\})^2}\Big]-\theta_3\frac{\alpha(k)}{(S^*(k)+I^*(k))^2}, \\
(4.8)\qquad A_2=\ &\theta_2\Big[I^*(k)-\frac{b(k)Z^*(k)}{(1+b(k)(S^*(k)+I^*(k)))^2}-\frac{a(k)I^*(k)}{(S^*(k)+I^*(k))^2}\Big] \\
&-\theta_1\Big[1+\frac{b(k)Z^*(k)S^*(k)}{(1+b(k)(S^*(k)+I^*(k)))^2}+\frac{\alpha(k)S^*(k)}{(S^*(k)+I^*(k))^2}\Big] \\
&-\theta_3\frac{a(k)}{(1+b(k)(\min\{m_1,\widetilde{m}_1\}+\min\{m_2,\widetilde{m}_2\}))^2},
\end{aligned}
$$

$$A_3 = \theta_3\Big[1 - \frac{a(k)(\max\{M_1, \widetilde{M}_1\} + \max\{M_2, \widetilde{M}_2\})}{1 + b(k)(\min\{m_1, \widetilde{m}_1\} + \min\{m_2, \widetilde{m}_2\})}\Big] - \theta_1 \frac{a(k)}{1 + b(k)(\min\{m_1, \widetilde{m}_1\} + \min\{m_2, \widetilde{m}_2\})} - \theta_2 \frac{a(k)}{1 + b(k)(\min\{m_1, \widetilde{m}_1\} + \min\{m_2, \widetilde{m}_2\})}. \tag{4.9}$$

It follows from conditions (i) and (ii) that there exists a positive constant α such that, if k is sufficiently large and $\|N\| < \alpha$, then

$$\Delta V \leqslant -\frac{v}{3}\{|N_1(k)| + |N_2(k)| + |N_3(k)|\} < -\frac{v}{3}\|N(k)\|. \tag{4.10}$$

In view of Freedman [7], we can see that the trivial solution of (4.2)–(4.4) is uniformly asymptotically stable and so is the solution $x^* = \{x^*(k)\} = \{(x_1^*(k), x_2^*(k), z^*(k))^{\mathrm{T}}\}$ of (2.3). Thus we can conclude that the positive periodic solution of (2.3) is globally asymptotically stable. The proof is complete. $\square$

5. An example

In this section, we give an example which shows the feasibility of the main results Theorem 3.1 and Theorem 4.1 of this paper. Let us consider the following special form of system (2.3):

$$\begin{cases} S(k+1) = S(k)\exp\Big\{[0.0005 + 0.0005\sin(k\pi/2)][1 - S(k) - I(k)]) \\ \qquad - \dfrac{[0.0002 + 0.0002\sin(k\pi/2)]Z(k)}{1 + [0.0005 + 0.0005\sin(k\pi/2)][S(k) + I(k)]} \\ \qquad - \dfrac{[0.0002 + 0.0002\cos(k\pi/2)]I(k)}{S(k) + I(k)}\Big\}, \\ I(k+1) = I(k)\exp\Big\{[0.0008 + 0.0008\cos(k\pi/2)][1 - S(k) - I(k)] \\ \qquad - \dfrac{[0.0002 + 0.0002\sin(k\pi/2)]Z(k)}{1 + [0.0005 + 0.0005\sin(k\pi/2)][S(k) + I(k)]} \\ \qquad + \dfrac{\alpha(k)S(k)}{S(k) + I(k)} - m_2(k)\Big\}, \\ Z(k+1) = Z(k)\exp\Big\{\dfrac{[0.0002 + 0.0002\sin(k\pi/2)][S(k) + I(k)]}{1 + [0.0005 + 0.0005\sin(k\pi/2)][S(k) + I(k)]} \\ \qquad - m_3(k)\Big\}. \end{cases} \tag{5.1}$$

Here

$$a(k) = 0.0002 + 0.0002\sin(k\pi/2),\ b(k) = 0.0005 + 0.0005\sin(k\pi/2),$$
$$\alpha(k) = 0.0002 + 0.0002\cos(k\pi/2),\ r_1(k) = 0.0005 + 0.0005\sin(k\pi/2),$$
$$r_2(k) = 0.0008 + 0.0008\cos(k\pi/2),\ m_3(k) = 0.0004 + 0.0004\cos(k\pi/2),$$
$$m_2(k) = 0.0003 + 0.0003\cos(k\pi/2),\ m_3(k) = 0.0004 + 0.0004\cos(k\pi/2).$$

By direct computation by Matlab 7.0 software, we get the following values:

$$\begin{aligned}&\bar{r}_1 = 0.0005,\ \bar{r}_2 = 0.0008,\ \overline{m}_3 = 0.0004,\ \bar{a} = 0.0002,\ m_1 = -0.0040,\\ &M_1 = 0.0040,\ S_1 = 0.0040,\ m_2 = -0.0032,\ M_2 = 0.0008,\ S_2 = 0.0032,\\ &m_3^* = -0.0028,\ M_3^* = 0.0036,\ S_3 = 0.0028,\ \widetilde{m}_2 = -0.0032,\ \widetilde{M}_2 = 0.0032,\\ &\widetilde{S}_2 = 0.0032,\ \widetilde{m}_1 = -0.0004,\ \widetilde{M}_1 = 0.0004,\ \widetilde{S}_1 = 0.0004,\ \widetilde{S}_3 = 0.0041,\\ &\widetilde{m}_3^* = -0.0038,\ \widetilde{M}_3^* = 0.0041,\ K_1 = 0.0002,\ K_2 = 0.00016,\ K_3 = 0.0004,\\ &K_4 = 0.00008.\end{aligned}$$

Let $v = 0.00003$, $\theta_1 = 0.0004$, $\theta_2 = 0.0003$ and $\theta_3 = 0.0007$. Then we can verify that all the assumptions in Theorem 3.1 and Theorem 4.1 are satisfied. Thus system (2.3) has a 4-periodic solution which is globally asymptotically stable.

References

[1] *H. N. Agiza, E. M. Elabbasy, H. El-Metwally, A. A. Elsadany*: Chaotic dynamics of a discrete prey-predator model with Holling type II. Nonlinear Anal., Real World Appl. *10* (2009), 116–129.

[2] *N. Apreutesei, G. Dimitriu*: On a prey-predator reaction-diffusion system with Holling type III functional response. J. Comput. Appl. Math. *235* (2010), 366–379.

[3] *B. Dai, J. Zou*: Periodic solutions of a discrete-time diffusive system governed by backward difference equations. Adv. Difference Equ. *2005* (2005), 263–274.

[4] *X. Ding, C. Lu*: Existence of positive periodic solution for ratio-dependent N-species difference system. Appl. Math. Modelling *33* (2009), 2748–2756.

[5] *M. Fan, K. Wang*: Periodic solutions of a discrete time nonautonomous ratio-dependent predator-prey system. Math. Comput. Modelling *35* (2002), 951–961.

[6] *M. Fazly, M. Hesaaraki*: Periodic solutions for a discrete time predator-prey system with monotone functional responses. C. R., Math., Acad. Sci. Paris *345* (2007), 199–202.

[7] *H. I. Freedman*: Deterministic Mathematical Models in Population Ecology. Monographs and Textbooks in Pure and Applied Mathematics 57, Marcel Dekker, New York, 1980.

[8] *R. E. Gaines, J. L. Mawhin*: Coincidence Degree, and Nonlinear Differential Equations. Lecture Notes in Mathematics 568, Springer, Berlin, 1977.

[9] *F. M. Hilker, H. Malchow*: Strange periodic attractors in a prey-predator system with infected prey. Math. Popul. Stud. *13* (2006), 119–134.

[10] *J. Jiao, S. Cai, L. Chen*: Dynamical behaviors of a biological management model with impulsive stocking juvenile predators and continuous harvesting adult predators. J. Appl. Math. Comput. *35* (2011), 483–495.

[11] *J. Jiao, X. Meng, L. Chen*: Harvesting policy for a delayed stage-structured Holling II predator-prey model with impulsive stocking prey. Chaos Solitons Fractals *41* (2009), 103–112.

[12] *T. K. Kar, S. Misra, B. Mukhopadhyay*: A bioeconomic model of a ratio-dependent predator-prey system and optimal harvesting. J. Appl. Math. Comput. *22* (2006), 387–401.

[13] *W. Ko, K. Ryu*: Coexistence states of a nonlinear Lotka-Volterra type predator-prey model with cross-diffusion. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *71* (2009), e1109–e1115.

[14] *Y. Li*: Positive periodic solutions of a discrete mutualism model with time delays. Int. J. Math. Math. Sci. *2005* (2005), 499–506.

[15] *Y. Li, K. Zhao, Y. Ye*: Multiple positive periodic solutions of n species delay competition systems with harvesting terms. Nonlinear Anal., Real World Appl. *12* (2011), 1013–1022.

[16] *Q. Liu, R. Xu*: Periodic solutions for a delayed one-predator and two-prey system with Holling type-II functional response. Ann. Differential Equations *21* (2005), 14–28.

[17] *Z. Liu, S. Zhong, X. Liu*: Permanence and periodic solutions for an impulsive reaction-diffusion food-chain system with Holling type III functional response. J. Franklin Inst. *348* (2011), 277–299.

[18] *A. F. Nindjin, M. A. Aziz-Alaoui, M. Cadivel*: Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay. Nonlinear Anal., Real World Appl. *7* (2006), 1104–1118.

[19] *Y. Pei, L. Chen, Q. Zhang, C. Li*: Extinction and permanence of one-prey multi-predators of Holling type II function response system with impulsive biological control. J. Theoret. Biol. *235* (2005), 495–503.

[20] *M. Scheffer*: Fish and nutrients interplay determines algal biomass: a minimal model. Oikos *62* (1991), 271–282.

[21] *X. Song, Y. Li*: Dynamic behaviors of the periodic predator-prey model with modified Leslie-Gower Holling-type II schemes and impulsive effect. Nonlinear Anal., Real World Appl. *9* (2008), 64–79.

[22] *G.-Q. Wang, S. S. Cheng*: Positive periodic solutions for nonlinear difference equations via a continuation theorem. Adv. Difference Equ. *2004* (2004), 311–320.

[23] *L.-L. Wang, W.-T. Li, P.-H. Zhao*: Existence and global stability of positive periodic solutions of a discrete predator-prey system with delays. Adv. Difference Equ. *2004* (2004), 321–336.

[24] *J. Wiener*: Differential equations with piecewise constant delays. Trends in Theory and Practice of Nonlinear Differential Equations (V. Lakshmikantham, ed.). Proc. Int. Conf., Arlington/Tex. 1982. Lecture Notes in Pure and Appl. Math. 90, Dekker, New York, 1984, pp. 547–552.

[25] *R. Xu, L. Chen, F. Hao*: Periodic solutions of a discrete time Lotka-Volterra type food-chain model with delays. Appl. Math. Comput. *171* (2005), 91–103.

[26] *J. Zhang, H. Fang*: Multiple periodic solutions for a discrete time model of plankton allelopathy. Adv. Difference Equ. (electronic only) *2006* (2006), Article ID 90479, 14 pages.

[27] *R. Y. Zhang, Z. C. Wang, Y. Chen, J. Wu*: Periodic solutions of a single species discrete population model with periodic harvest/stock. Comput. Math. Appl. *39* (2000), 77–90.

[28] *W. Zhang, D. Zhu, P. Bi*: Multiple positive periodic solutions of a delayed discrete predator-prey system with type IV functional responses. Appl. Math. Lett. *20* (2007), 1031–1038.

[29] *Z. Zhang, Z. Hou*: Existence of four positive periodic solutions for a ratio-dependent predator-prey system with multiple exploited (or harvesting) terms. Nonlinear Anal., Real World Appl. *11* (2010), 1560–1571.

[30] *Z. Zhang, J. Luo*: Multiple periodic solutions of a delayed predator-prey system with stage structure for the predator. Nonlinear Anal., Real World Appl. *11* (2010), 4109–4120.

[31] *K. Zhuang, Z. Wen*: Dynamics of a discrete three species food chain system. Int. J. Comput. Math. Sci. *5* (2011), 13–15.

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