Alán Tapia; Raymundo Márquez; Miguel Bernal; Joaquín Cortez

Sliding subspace design based on linear matrix inequalities

Kybernetika, Vol. 50 (2014), No. 3, 436–449

Persistent URL: `http://dml.cz/dmlcz/143884`

KYBERNETIKA — VOLUME *50* (2014), NUMBER 3, PAGES 436–449

SLIDING SUBSPACE DESIGN BASED ON LINEAR MATRIX INEQUALITIES

Alán Tapia, Raymundo Márquez, Miguel Bernal and Joaquín Cortez

In this work, an alternative for sliding surface design based on linear and bilinear matrix inequalities is proposed. The methodology applies for reduced and integral sliding mode control, both continuous- and discrete-time; it takes advantage of the Finsler's lemma to provide a greater degree of freedom than existing approaches for sliding subspace design. The sliding surfaces thus constructed are systematically found via convex optimization techniques, which are efficiently implemented in commercially available software. Examples are provided to illustrate the effectiveness of the proposed approach.

Keywords: sliding mode control, variable structure, sliding subspace design, linear matrix inequalities

Classification: 93B12, 90C25, 51M16

1. INTRODUCTION

Nowadays, variable structure control is among one of the most popular control techniques available due to its insensitiveness to matched disturbances and its finite-time convergence properties [26]. There are two stages of reduced order sliding mode control design: the definition of an appropriate sliding surface in order to guarantee a reduced-order sliding motion with prescribed dynamics, then the design of a control law to keep the system in the aforementioned motion [15]. This paper is concerned with the first stage. Its goal is to provide the sliding subspace to which the system states will be attracted. An appropriate construction of this subspace provides desired dynamics, disturbance rejection, optimal behavior, and robust stability in sliding mode.

On the other hand, in integral sliding mode control the size of the controlled system is augmented with a number of integrators connected to it, and then, it defines the sliding subspace for the augmented model, having its state as a concatenation of system states and integrator outputs. One of its important advantages is that a proper choice of integrator's initial conditions eliminates the reaching phase [17].

Several works have appeared for sliding mode motion design, most of them based on transformations having a form which allows desired pole placement or quadratic cost function minimization [12, 13, 21]; manipulation of the right eigenvector, spanning the sliding subspace in continuous- [7] and discrete-time case [6]; closed-form formulas

DOI: 10.14736/kyb-2014-3-0436

based on the Ackermann's pole-placement procedure [2, 19], and integral continuous control subspace design [27] as well as a closed form for discrete-time systems [1]. In [14] continuous- and discrete-time, both reduced and full order (integral) sliding subspace design are considered and solved through explicit formulas and algorithms; this paper reformulates these results under the linear matrix inequality (LMI) framework or in terms of bilinear matrix inequalities (BMI) when the first choice is not possible.

The LMI-based control field has had an impressive growth. Once a problem is stated in LMI terms, it can be efficiently solved by convex optimization techniques which are implemented in commercially available software [4], making it possible to solve traditional control problems using a systematic design procedure with a software implementation; a similar outcome can be found for solving BMIs [22]. Linear parameter varying (LPV) as well as quasi-LPV control systems have been treated first under the LMI approach in [25]. Later, several authors made use of LMIs in sliding mode control: for pole placement under the sliding motion [3], for first-order sliding surface design [8, 10], for integral sliding surface design [5, 9], for discrete-time sliding surface design [23], for linear quadratic-based methods [18], for simultaneous sliding surface and control law design [16], for asymptotic high-order sliding mode control design [24].

This paper is organized as follows: Section 2 provides basic notation introduced first in [21] allowing to treat continuous- and discrete-time sliding subspace design in a single frame, both for reduced- and integral sliding control; this is followed by problem statements. Section 3 contains the main contribution of this paper: the use of the Finsler's lemma to obtain LMI conditions for reduced-order sliding subspace design and BMI conditions for integral (full) order one. Section 4 exemplifies the developed techniques. Finally, Section 5 draws some conclusions and perspectives.

2. PRELIMINARIES AND PROBLEM STATEMENT

Consider the following uncertain linear system:

$$\delta x = Ax(t) + B\big(u(t) + d(t, x, u)\big), \tag{1}$$

where the operator δ represents either the differential operator d/dt in continuous-time models (thus $\delta x = \dot{x}$) or the forward shift operator in discrete-time models (i. e., $\delta x = x(t+1)$) [21], $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $d : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ represents matched uncertainty satisfying $\|d(t, x, u)\| \leq \rho(t, x, u)$ with $\rho(t, x, u)$ being a known function, and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ are the nominal system matrices. Note that the uncertainty $d(t, x, u)$ will play no role in designing the sliding surface; it is only included for completeness [16].

In the sequel, a zero (0) inside a matrix will be a zero matrix block of appropriate dimensions; similarly, I will denote an identity matrix of appropriate size. For in-line expressions, an asterisk $(*)$ represents the transpose of the terms on its left-hand side; inside a matrix, it denotes the transpose of its symmetric terms. For matrix expressions, symbols $<$ and $>$ stand for negative- and positive-definite relations, respectively; whereas $\prec$ and $\succ$ represent element-wise negative and positive relations, respectively.

The following assumptions are made:

A1) Matrix B is of full rank, i. e., rank $B = m$.

A2) The pair (A, B) is controllable.

Recalling the concept of equivalent control, a matrix K can be designed such that the closed-loop model

$$\delta x = Ax - BKx = (A - BK)x \tag{2}$$

has some desired dynamic features; therefore, pole placement comes at hand. Since this paper presents a full LMI/BMI methodology, pole placement will be achieved under the same framework. To that end, let J be a diagonal matrix containing the n desired poles in its diagonal entries. The following lemma shows that pole assignment can be performed via LMIs:

Lemma 2.1. The closed-loop model (2) has the same poles of matrix J if there exist matrices $T = T^T > 0$ and M such that LMIs $-\epsilon \prec AT - BM - TJ \prec \epsilon$ hold.

Proof. If matrices $A - BK$ and J share the same eigenvalues, then a similarity transformation should exist between them, i. e., $T^{-1}(A - BK)T = J$ with nonsingular $T = T^T > 0$. It is clear that $(A - BK)T = TJ$, which implies $AT - BKT - TJ = 0$. Defining $M = KT$, it is clear that the desired LMIs arise from the fact that if $AT - BM - TJ = 0$ then an arbitrarily small $\epsilon > 0$ exists such that the desired LMIs hold, thus concluding the proof. □

Remark 2.2. In the sequel, gain matrix K will be calculated as in the previous lemma as a first step leading to the sliding subspace design; it will be therefore assumed to be available.

Reduced-order sliding subspace design [14]: The following pair of equations defines the sliding mode that will be attained with a suitable switching type control law. Combining (2) and (3) leads to (4):

$$g(t) = Cx(t) = 0, \tag{3}$$

$$\delta g = C\delta x = C(Ax + Bu) = C(A - BK)x = 0, \tag{4}$$

where $C \in \mathbb{R}^{m\times n}$ is a full-rank sliding subspace matrix.

Assuming that

A3) Matrix CB is full rank, i. e., $\operatorname{rank}(CB) = m$,

it can be shown from (4) that the equivalent feedback matrix is $K = (CB)^{-1}CA$.

Problem statement 1: Find C such that the sliding mode dynamics (2)–(4) have the pole assignment given by K under assumptions A1–A3.

Full-order (integral) sliding subspace design [14]: This approach is based on adding m integrators to the system (1), thus leading to the following equations which define the integral sliding mode scheme under an appropriate switching type control law. Note that (7) stems from (2), (5), and (6):

$$g(t) = Dx + \sigma = 0, \tag{5}$$

$$\delta\sigma = Ex + q\sigma, \tag{6}$$

$$\begin{aligned}\delta g &= D(Ax + Bu) + Ex + q\sigma = D(A - BK)x + Ex + q\sigma \\ &= \big(D(A - BK) + E\big)x + q\sigma,\end{aligned} \tag{7}$$

with sliding subspace matrices $D \in \mathbb{R}^{m\times n}$ and $E \in \mathbb{R}^{m\times n}$, q being a parameter which equals to 1 for the discrete-time case and 0 for the continuous-time case.

Assuming that

A4) Matrix DB is full rank, i. e., $\operatorname{rank}(DB) = m$,

it can be shown from (7) that the equivalent feedback matrix is $K = (DB)^{-1}\big(D(A - qI) + E\big)$.

Equations (2) and (6) can be put together in the following matrix form:

$$\begin{bmatrix} \delta x \\ \delta\sigma \end{bmatrix} = \begin{bmatrix} A - BK & 0 \\ E & qI \end{bmatrix} \begin{bmatrix} x \\ \sigma \end{bmatrix}. \tag{8}$$

Problem statement 2: Find D and E such that the sliding mode dynamics (2),(5)–(7) have the pole assignment given by K under assumptions A1, A2, and A4.

The following well-known matrix property will play an essential role in deriving the results presented in this paper:

Finsler's lemma [11]: Let $x \in \mathbb{R}^n$, $Q = Q^T \in \mathbb{R}^{n\times n}$, and $R \in \mathbb{R}^{m\times n}$ such that $\operatorname{rank}(R) < n$; the following expressions are equivalent:

$$x^T Qx < 0, \;\; \forall x \in \{x \in \mathbb{R}^n : x \neq 0, Rx = 0\} \tag{9}$$

$$\exists H \in \mathbb{R}^{n\times m} : Q + HR + R^T H^T < 0. \tag{10}$$

3. MAIN RESULTS

Consider the following quadratic Lyapunov funcion candidate:

$$V = g^T(t)\bar{P}g(t), \;\; \bar{P} = \bar{P}^T > 0. \tag{11}$$

Let $P = C^T \bar{P} C$. Then using (3), (11) can be rewritten as:

$$V = g^T(t)\bar{P}g(t) = x^T(t)C^T\bar{P}Cx(t) = x^T(t)Px(t), \;\; P = P^T > 0. \tag{12}$$

Theorem 3.1. (reduced-order continuous-time case) Assume A1–A3 and $\delta x = \dot{x}$. The sliding mode dynamics (2)–(4) have the pole assignment $A - BK$ if the LMIs in the following conditions hold:

$$P(A - BK) + M + (*) < 0, \;\; -\epsilon \prec M(A - BK) \prec \epsilon \tag{13}$$

with $P = P^T = [P_1 \;\; P_2] > 0$, $P_1 \in \mathbb{R}^{n\times m}$, $P_2 \in \mathbb{R}^{n\times(n-m)}$, and $M \in \mathbb{R}^{n\times n}$ being decision variables, and sliding subspace matrix given by $C = P_1^+ M$.

Proof. Omitting arguments when convenient, the time-derivative of (12) can be written as:

$$\begin{aligned} \dot{V} &= x^T P\delta x + \delta x^T P x = x^T P(A - BK)x + x^T(A - BK)^T Px \\ &= x^T\big[P(A - BK) + (*)\big]x < 0. \end{aligned} \tag{14}$$

Applying the Finsler's lemma in (10) with $Q = P(A - BK) + (*)$ taken from (14) and $R = C$ taken from restriction (3) leads to the following equivalent condition:

$$P(A - BK) + HC + (*) < 0. \tag{15}$$

Considering $H = P_1 \in \mathbb{R}^{n\times m}$ and $M = HC = P_1C \in \mathbb{R}^{n\times n}$, the previous inequality yields

$$P(A - BK) + M + (*) < 0,$$

which is a sufficient LMI condition to guarantee $\dot{V} < 0$ under restriction $Cx = 0$ where decision variables are given by P, M, and sliding subspace matrix can be calculated from $C = P_1^+ M$.

In order to guarantee $K = (CB)^{-1}CA$, notice that:

$$CA = CBK \Leftrightarrow C(A - BK) = 0 \Leftrightarrow P_1C(A - BK) = 0 \Leftrightarrow M(A - BK) = 0.$$

The latter equality guarantees the existence of an arbitrarily small $\epsilon > 0$ such that the second LMI in (13) holds, thus concluding the proof. □

Theorem 3.2. (reduced-order discrete-time case) Assume A1 – A3 and $\delta x = x(t + 1)$. The sliding mode dynamics (2) – (4) have the pole assignment $A - BK$ if the LMIs in the following conditions hold:

$$M + (*) + (A - BK)^T P(A - BK) - P < 0, \quad -\epsilon \prec M(A - BK) \prec \epsilon \tag{16}$$

with $P = P^T = [P_1 \quad P_2] > 0$, $P_1 \in \mathbb{R}^{n\times m}$, $P_2 \in \mathbb{R}^{n\times(n-m)}$, and $M \in \mathbb{R}^{n\times n}$ being decision variables, and sliding subspace matrix given by $C = P_1^+ M$.

Proof. It follows a similar outline as for theorem 3.1 using the difference equation from (11). □

For integral sliding subspace design, consider the same quadratic Lyapunov function candidate (11). Taking into account the extended definition of the sliding surface in (5), this candidate Lyapunov function can be written as:

$$V = g^T(t)\bar{P}g(t) = \bar{x}^T[D \quad I]^T P[D \quad I]\bar{x}, \tag{17}$$

with $\bar{x} = [x^T(t) \quad \sigma^T(t)]^T$, $\bar{P} = \bar{P}^T = [D \quad I]^T P[D \quad I] > 0$, $P = \begin{bmatrix} P_{11} & (*) \\ P_{21} & P_{22} \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}$, $P_{11} \in \mathbb{R}^{n\times n}$, $P_{21} \in \mathbb{R}^{m\times n}$, $P_{22} \in \mathbb{R}^{m\times m}$.

The next theorems state BMI conditions to solve the full order sliding subspace design; they are followed by 2-steps algorithms which allow to efficiently solve these BMIs.

Theorem 3.3. (full-order continuous-time case) Assume A1, A2, A4, $\delta x = \dot{x}$, and $\delta\sigma = \dot{\sigma}$. The sliding mode dynamics (2), (5)–(7) have the pole assignment $A - BK$ if the BMIs in the following conditions hold:

$$\begin{bmatrix} P_{11}(A-BK) + P_{21}^T E + H_1 D & H_1 \\ P_{21}(A-BK) + P_{22}E + H_2 D & H_2 \end{bmatrix} + (*) < 0, \quad \begin{bmatrix} P_{11} & (*) \\ P_{21} & P_{22} \end{bmatrix} > 0, \qquad (18)$$

$$-\epsilon \prec D(A-BK) + E \prec \epsilon$$

with $P_{11} \in \mathbb{R}^{n\times n}$, D, E, $P_{21} \in \mathbb{R}^{m\times n}$, $H_1 \in \mathbb{R}^{n\times m}$, and P_{22}, $H_2 \in \mathbb{R}^{m\times m}$ being decision variables.

Proof. Positiveness of the Lyapunov function candidate in (17) is guaranteed by the second LMI in (18).

The time-derivative of (17) can be written as:

$$\dot{V} = 2\bar{x}^T P \dot{\bar{x}} = \begin{bmatrix} x \\ \sigma \end{bmatrix}^T \left(\begin{bmatrix} P_{11} & (*) \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} A-BK & 0 \\ E & 0 \end{bmatrix} + (*) \right) \begin{bmatrix} x \\ \sigma \end{bmatrix} < 0. \qquad (19)$$

Applying the Finsler's lemma in (10) with $Q = \begin{bmatrix} P_{11} & (*) \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} A-BK & 0 \\ E & 0 \end{bmatrix} + (*)$ taken from (19) and $R = [D \quad I]$ taken from restriction (5) leads to the following equivalent condition:

$$\begin{aligned} &\begin{bmatrix} P_{11} & (*) \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} A-BK & 0 \\ E & 0 \end{bmatrix} + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \begin{bmatrix} D & I \end{bmatrix} + (*) \\ &= \begin{bmatrix} P_{11}(A-BK) + P_{21}^T E & 0 \\ P_{21}(A-BK) + P_{22}E & 0 \end{bmatrix} + \begin{bmatrix} H_1 D & H_1 \\ H_2 D & H_2 \end{bmatrix} + (*) \\ &= \begin{bmatrix} P_{11}(A-BK) + P_{21}^T E + H_1 D & H_1 \\ P_{21}(A-BK) + P_{22}E + H_2 D & H_2 \end{bmatrix} + (*) < 0, \end{aligned} \qquad (20)$$

where $H_1 \in \mathbb{R}^{n\times m}$, $H_2 \in \mathbb{R}^{m\times m}$ are new free decision variables: this corresponds to the first LMI in (18).

Finally, note that the third LMI in (18) allows to guarantee $K = (DB)^{-1}(DA+E)$, since it implies the existence of an arbitrarily small $\epsilon > 0$ guaranteeing $D(A–BK)+E = 0$.
□

Remark 3.4. Conditions in theorem 3.3 are BMIs; therefore it is suggested to solve them in two steps, as follows:

Step 1: Solve the first two LMIs in (18) for P_{11}, P_{21}, P_{22}, H_1, H_2, and $N_1 = P_{21}^T E$, $N_2 = P_{22}E$, $M_1 = H_1 D$, and $M_2 = H_2 D$. Once solved, take $D = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}^+ \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$, $E = \begin{bmatrix} P_{21}^T \\ P_{22} \end{bmatrix}^+ \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$.

Step 2: Take D (preserving N_1 and N_2) or E (preserving M_1 and M_2) from the previous step in order to solve all the LMIs in (18). If D is chosen, the third LMI in (18) turns into:

$$-\epsilon \prec \begin{bmatrix} P_{21}^T \\ P_{22} \end{bmatrix} D(A-BK) + \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \prec \epsilon;$$

otherwise, if E is chosen, the third LMI in (18) is rewritten as:

$$-\epsilon \prec \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} (A-BK) + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} E \prec \epsilon.$$

Theorem 3.5. (full-order discrete-time case) Assume A1, A2, A4, $\delta x = x(t+1)$, and $\delta\sigma = \sigma(t+1)$. The sliding mode dynamics (2), (5) – (7) have the pole assignment $A-BK$ if the BMIs in the following conditions hold:

$$\begin{bmatrix} P - \begin{bmatrix} H_1 D & H_1 \\ H_2 D & H_2 \end{bmatrix} + (*) & (*) \\ \begin{bmatrix} P_{11}(A-BK) + P_{21}^T E & P_{21}^T \\ P_{21}(A-BK) + P_{22}E & P_{22} \end{bmatrix} & P \end{bmatrix} > 0, \quad P = \begin{bmatrix} P_{11} & (*) \\ P_{21} & P_{22} \end{bmatrix} > 0, \tag{21}$$
$$-\epsilon \prec D(A-BK-I) + E \prec \epsilon$$

with $P_{11} \in \mathbb{R}^{n\times n}$, D, E, $P_{21} \in \mathbb{R}^{m\times n}$, $H_1 \in \mathbb{R}^{n\times m}$, and H_2, $P_{22} \in \mathbb{R}^{m\times m}$, being decision variables.

Proof. Positiveness of the Lyapunov function candidate in (17) is guaranteed by the second LMI in (21); its one-step variation is developed as follows:

$$\begin{aligned} \Delta V &= V(t+1) - V(t) = \bar{x}^T(t+1)P\bar{x}(t+1) - \bar{x}^T(t)P\bar{x}(t) \\ &= \bar{x}^T(t)\left[\begin{bmatrix} A-BK & 0 \\ E & I \end{bmatrix}^T P \begin{bmatrix} A-BK & 0 \\ E & I \end{bmatrix} - P \right]\bar{x}(t) < 0. \end{aligned} \tag{22}$$

Applying the Finsler's lemma in (10) with $Q = \begin{bmatrix} A-BK & 0 \\ E & I \end{bmatrix}^T P \begin{bmatrix} A-BK & 0 \\ E & I \end{bmatrix} - P$ taken from (22) and $R = [D \quad I]$ taken from restriction (5) leads to the following equivalent condition:

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \begin{bmatrix} D & I \end{bmatrix} + (*) + \begin{bmatrix} A-BK & 0 \\ E & I \end{bmatrix}^T P \begin{bmatrix} A-BK & 0 \\ E & I \end{bmatrix} - P < 0,$$

where $H_1 \in \mathbb{R}^{n\times m}$ and $H_2 \in \mathbb{R}^{m\times m}$ are new free decision variables. By Schur complement, this inequality renders:

$$\begin{bmatrix} P - \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \begin{bmatrix} D & I \end{bmatrix} + (*) & (*) \\ \begin{bmatrix} (A-BK) & 0 \\ E & I \end{bmatrix} & P^{-1} \end{bmatrix} > 0.$$

Pre- and post-multiplying the previous expression by $\begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix}$ yields

$$\begin{bmatrix} P - \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \begin{bmatrix} D & I \end{bmatrix} + (*) & (*) \\ P \begin{bmatrix} (A-BK) & 0 \\ E & I \end{bmatrix} & P \end{bmatrix} = \begin{bmatrix} P - \left[\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} D \ \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \right] + (*) & (*) \\ \begin{bmatrix} P_{11}(A-BK) + P_{21}^T E & P_{21}^T \\ P_{21}(A-BK) + P_{22}E & P_{22} \end{bmatrix} & P \end{bmatrix} > 0,$$

which is equivalent to the first LMI in (21). As in theorem 3.3, the third LMI in (21) guarantees $K = (DB)^{-1}(DA + E - D)$, a requisite coming from (7) with $q = 1$. □

Remark 3.6. Conditions in theorem 3.5 are BMIs. The two-step procedure in remark 3.4 can be adapted to solve them systematically.

Remark 3.7. Results in theorems 3.1 and 3.2 include those in [14] for reduced-order design as particular cases since they can be recovered from the second LMI in (13) and (16) (for the continuous- and discrete-time case, respectively) if $CB = I$. Similarly, integral full-order design in [14] can be obtained from theorems 3.3 and 3.5 if $DB = I$ is assumed.

Remark 3.8. LQR design as in [14] can be used for controller design of matrix K without affecting the validity of the LMI results above; moreover, due to this LMI structure, additional constraints can be easily included (for instance, performance requirements such as those described in [4]).

Some examples are provided in the next section to illustrate the effectiveness of the proposed approach.

4. EXAMPLES

In the sequel, the proposed methodology is applied to three plants already considered in the literature for sliding subspace design: a continuous and a discretized model of an aircraft in examples 4.1 and 4.2, respectively; an experimental furnace in 4.3; finally, a DC motor setup in 4.4.

Example 4.1. Reduced-Order Continuous-Time Case: Consider the following linearization of a continuous-time model of an aircraft given in [14, 20]:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 0 & 1.132 & 0 & -1 \\ 0 & -0.0638 & -0.1712 & 0 & 0.0705 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0.0468 & 0 & -0.8556 & -1.013 \\ 0 & -0.2908 & 0 & 1.0532 & -0.6059 \end{bmatrix} x(t) \\ &+ \begin{bmatrix} 0 & 0 & 0 \\ -0.12 & 1 & 0 \\ 0 & 0 & 0 \\ 4.419 & 0 & -1.656 \\ 1.575 & 0 & -0.0732 \end{bmatrix} u(t). \end{aligned} \tag{23}$$

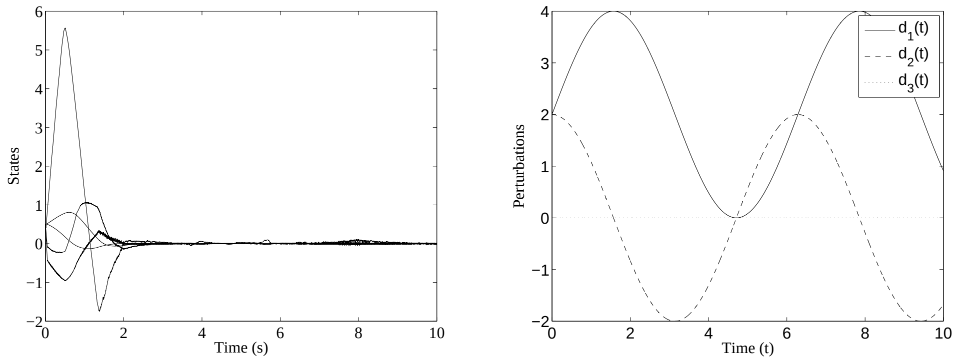

Fig. 1. Simulation results for perturbed model (25) in example 4.1.

Applying theorem 3.1 for sliding subspace design of (23), the matrix C is designed using LMI conditions (13). As a first step, gain K is obtained using LMI conditions in lemma 2.1 such that the system $(A - BK)$ has eigenvalues in $(-2.5894 + 1.3723i, -2.5894 - 1.3723i, 0, 0, 0)$:

$$K = \begin{bmatrix} 0 & -0.2123 & -1.5405 & 6.2176 & 0.9541 \\ 0 & -0.0893 & -2.0488 & -19.1819 & 1.6804 \\ 0 & -0.5947 & -4.2985 & 15.1074 & 3.3237 \end{bmatrix}. \tag{24}$$

Theorem 3.1 produces the following sliding surface matrix:

$$C = \begin{bmatrix} -2.4818 & -0.0649 & 0.6507 & -2.5232 & 1.0110 \\ 13.6457 & -4.6916 & 0.3449 & -14.878 & -5.7447 \\ 2.5973 & -0.5519 & -3.2672 & -1.5806 & -1.1827 \end{bmatrix}.$$

Consider a perturbed version of LTI model (23)

$$\delta x = Ax(t) + B\big(u(t) + d(t)\big), \tag{25}$$

where $d(t) = [2 + 2\sin(t) \quad 2\cos(t) \quad 0]^T$ is a matched disturbance on the input. Applying the switching control law $u = -10\text{sign}(Cx)$ to (25) under the initial conditions $x(0) = [0.5 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5]^T$, results in Figure 1 arise. They show an important feature of the designed control law: it is able to remain insensitive to matched perturbations, thus accomplishing the task of remaining in the sliding subspace. It is important to stress that these results cannot be obtained with the techniques in [16] since the assigned poles are not all the same; they cannot be obtained with those in [14] either since $CB \neq I$.

Example 4.2. Full-Order Discrete-Time Case: The following model corresponds to the discretization of (23) with sampling period $T = 0.1$:

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 1 & 0.0014 & 0.1132 & 0.0005 & -0.0967 \\ 0 & 0.9945 & -0.0171 & -0.0005 & 0.0068 \\ 0 & 0.0003 & 1 & 0.0957 & -0.0048 \\ 0 & 0.0060 & 0 & 0.9131 & -0.0936 \\ 0 & -0.0277 & 0.0002 & 0.0973 & 0.9287 \end{bmatrix} x(t) \\ &+ \begin{bmatrix} -0.0076 & 0 & 0.0003 \\ -0.0115 & 0.0997 & 0 \\ 0.0212 & 0 & -0.0081 \\ 0.4152 & 0.0003 & -0.1598 \\ 0.1742 & -0.0014 & -0.0154 \end{bmatrix} u(t). \end{aligned} \tag{26}$$

Using pole placement for a full-order case at (0.5138, 0.5138, 0.95, 0.98, 0.98), gain matrix K is obtained as follows:

$$K = \begin{bmatrix} -0.6367 & -0.2062 & -3.7755 & -0.0654 & 2.8486 \\ 8.6056 & 0.4029 & 46.9047 & 13.2960 & -41.4777 \\ -1.6391 & -0.5713 & -10.3666 & -2.7456 & 7.9230 \end{bmatrix}. \tag{27}$$

Applying theorem 3.5 for sliding subspace design for (26), matrices D and E can be found using LMI conditions in (21):

$$D = \begin{bmatrix} 0.0046 & 0.0053 & 0.0365 & -0.0838 & -0.0213 \\ -0.0027 & 0.0431 & -0.0091 & 0.0174 & -0.0052 \\ 0.0043 & -0.0001 & 0.0284 & -0.0557 & -0.0297 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.0066 & 0.0003 & 0.0281 & -0.0360 & -0.0322 \\ 0.0378 & 0.0021 & 0.2090 & 0.0670 & -0.1827 \\ 0.0028 & 0 & 0.0103 & -0.0277 & -0.0141 \end{bmatrix}.$$

Note that this solution is no longer subject to the restriction $D = B^+$ as in [14]; moreover, since it concerns integral sliding design, the methodology in [16] no longer applies.

Figure 2 results from applying the equivalent control law $u = -Kx$ to (26) under the initial conditions $x(0) = [0.5 \quad -0.5 \;\; 0.5 \quad -0.3 \;\; 0.1]^T$ and $\sigma(0) = -Dx(0)$. Note that the system already starts on the sliding surface, an advantage that comes with full-order sliding surface design.

Example 4.3. Reduced Order Continuous-Time Sliding Surface Design of an Experimental Furnace: Consider the model of a two-input two-output experimental furnace [16] with system matrices

$$A = \begin{bmatrix} -0.0186 & -0.0065 & 0.0064 & -0.0012 \\ 0.0026 & -0.1354 & 0.0020 & -0.0028 \\ -0.1311 & 0.0349 & -0.4684 & -0.0095 \\ 1.0120 & -0.7736 & -0.2741 & -0.1523 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{28}$$

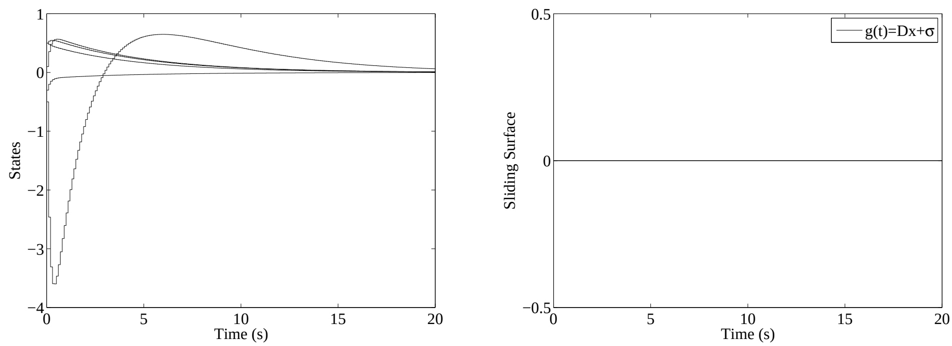

Fig. 2. Simulation results for model (26) in example 4.2.

The two-dimensional output (given by the temperature and excess oxygen concentration) is chosen as the sliding surface.

Applying theorem 3.1 for sliding subspace design of (28), the matrix C is designed using LMI conditions (13). As a first step, gain K is obtained using LMI conditions in lemma 2.1 such that the system $(A - BK)$ has eigenvalues in $(-2.5,\ -1.2,\ 0,\ 0)$:

$$K = \begin{bmatrix} -7.2131 & -42.3726 & 2.8164 & -1.3358 \\ -4.6999 & 26.2018 & 1.0711 & 0.1089 \end{bmatrix}. \tag{29}$$

Theorem 3.1 produces the following sliding surface matrix:

$$C = \begin{bmatrix} 1.3452 & -28.0601 & -0.0439 & 0.0716 \\ 0.4359 & -7.6257 & -0.0118 & 0.0196 \end{bmatrix}.$$

Once a switching control law with a suitable gain is applied based on the previously designed sliding surface, the states remain in the sliding subspace thus designed.

Example 4.4. Full-Order Continuous-Time Sliding Surface Design of a DC Motor: Consider the following simple model of a DC motor taken from [15]:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 88.7574 \\ 0 & -0.6000 & -24 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t). \tag{30}$$

Considering the pole placement for the sliding motion at $(-2.22 + 1.5i,\ -2.22 - 1.5i,\ -0.72)$ and via lemma 2.1, the following matrix K is found:

$$K = \begin{bmatrix} 0.0586 & -0.4830 & -18.8326 \end{bmatrix}. \tag{31}$$

Using LMI conditions (18) in theorem 3.3 for the system given in (30), the following matrices D and E are found under the controller gain in (31):

$$D = \begin{bmatrix} -0.0117 & 17.7281 & -2.6670 \end{bmatrix},\ E = \begin{bmatrix} -0.1563 & -0.3000 & -1587.3 \end{bmatrix}.$$

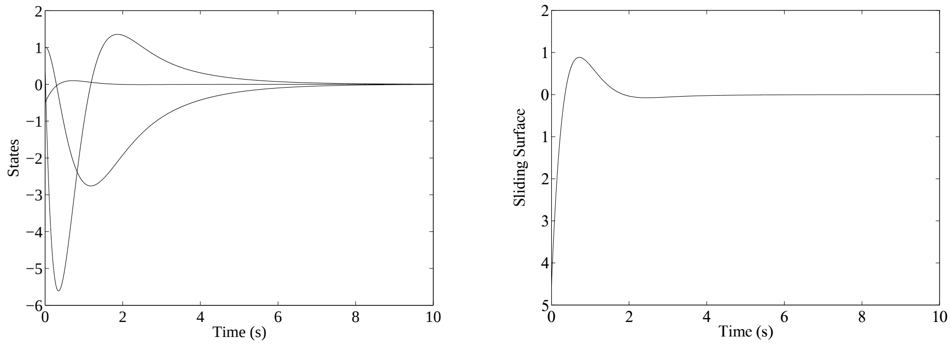

Fig. 3. Simulation results for example 4.4.

Applying the equivalent control law $u_{eq} = -(DB)^{-1}(DA+E) = -Kx$ to the LTI system (30) where (6) holds, the simulation results shown in Figure 3 are obtained. These simulations are run from the initial conditions $x(0) = \begin{bmatrix} 1 & 0.5 & -0.5 \end{bmatrix}^T$ and $\sigma(0) = -20 \neq -Dx(0)$ [14].

5. CONCLUSION AND PERSPECTIVES

A novel technique for sliding surface design based on linear and bilinear matrix inequalities has been presented. The proposed approach has been developed for reduced- and integral sliding mode, both for continuous- and discrete-time case in a single unified framework. The results thus offered prove to be more general and flexible than others recently appeared since the sliding motion is systematically designed via convex optimization techniques which are efficiently implemented by commercially available software. Examples are provided to illustrate the effectiveness of the proposed approach.

ACKNOWLEDGMENT

This work has been supported by the Mexican Council of Science and Technology (CONACYT) through the SEP-CONACYT Project CB-2011-168406 and by the Mexican Agency PROMEP via project 103.5/13/7283. The authors would like to thank Prof. Leonid Fridman and Prof. Branislava Draženović for their valuable comments during the preparation of the current manuscript.

(Received April 2, 2013)

REFERENCES

[1] K. Abidi, J.-X. Xu, and Y. Xinghuo: On the discrete-time integral sliding-mode control. IEEE Trans. Automat. Control *52* (2007), 4, 709–715.

[2] J. Ackermann and V. Utkin: Sliding mode control design based on Ackermann's formula. IEEE Trans. Automat. Control *43* (1998), 2, 234–237.

[3] D. Arezelier, M. Angulo, and J. Bernussou: Sliding surface design by quadratic stabilization and pole placement. In: Proc. 4th European Control Conference, 1997.

[4] S. Boyd, L. El Ghaoui, E. Féron, and V. Balakrishnan: Linear matrix inequalities in system and control theory. Stud. Appl. Math. *15* (1994).

[5] F. Castaños and L. Fridman: Analysis and design of integral sliding manifolds for systems with unmatched perturbations. IEEE Trans. Automat. Control *51* (2006), 5, 853–858.

[6] J.-L. Chang: Discrete sliding mode control of MIMO linear systems. Asian J. Control *4* (2002), 2, 217–222.

[7] Y.-P. Chen and J.-L. Chang: A new method for constructing sliding surfaces of linear time-invariant systems. Internat. J. System Sci. *31* (2000), 4, 417–420.

[8] H. Choi: On the existence of linear sliding surface for a class of uncertain dynamic systems with mismatched uncertainties. Automatica *37* (1999), 1707–1715.

[9] H. Choi: LMI-Based Sliding Surface Design for Integral Sliding Mode Control of Mismatched Uncertain Systems. IEEE Trans. Automat. Control *52* (2007), 4, 736–742.

[10] E. Cruz-Zavala, J. Moreno, and L. Fridman: Uniform sliding mode controllers and uniform sliding surfaces. IMA J. Math. Control Inform. *29* (2012), 4, 491–505.

[11] M. C. De Oliveira and R. E. Skelton: Stability Tests for Constrained Linear Systems. In Perspectives in Robust Control. Springer, Berlin 1994.

[12] C. M. Dorling and A. S. I. Zinober: Two approaches to hyperplane desing in multivariable variable structure control systems. Internat. J. Control *44* (1986), 1, 65–82.

[13] C. M. Dorling and A. S. I. Zinober: Robust hyperplane desing in multivariable variable structure control systems. Internat. J. Control *48* (1988), 5, 2043–2054.

[14] B. Draženović, C. Milosavljević, B. Veselić, and V. Gligić: Comprehensive approach to sliding subspace design in linear time invariant systems. In: IEEE International Workshop on Variable Structure Systems 2012, pp. 473–478.

[15] C. Edwards: Sliding Mode Control: Theory and Applications. Taylor and Francis, London 1998.

[16] C. Edwards: A practical method for the design of sliding mode controllers using linear matrix inequalities. Automatica *40* (2004), 10, 1761–1769.

[17] L. Fridman, J. Moreno, and R. Iriarte: Sliding Modes After the First Decade of the 21st Century. Springer, Berlin 2011.

[18] C. Hermann, S. K. Spurgeon, and C. Edwards: A robust sliding mode output tracking control for a class of relative degree zero and non-minimum phase plants: A chemical process application. Internat. J. Control *72* (2001), 1194–1209.

[19] J. Y. Huang and K. S. Yeung: Arbitrary eigenvalue assignment via switching hyperplanes design scheme and extension of Ackermann's formula. In: IEEE Conference on Computer, Communication, Control and Power Engineering *4* (1993), 17–20.

[20] Y. S. Hung and A. G. J. Macfarlan: Multivariable Feedback: A Quasi-Classical Approach. Volume 40. Springer-Verlag, Berlin 1982.

[21] J. Kautsky, N. K. Nichols, and P. Van Dooren: Robust pole assignment in linear state feedbacks. Internat. J. Control *41* (1985), 2, 1129–1155.

[22] M. Kočvara and M. Sting: Penbmi, version 2.1. www.penopt.com, 2008.

[23] A. J. Mehta, B. Bandyopadhyay, and A. Inoue: Reduced-order observer design for servo system using duality to discrete-time sliding-surface design. IEEE Trans. Industr. Electronics *57* (2010), 11, 3793–3800.

[24] Y. Pan, K. D. Kumar, and G. Liu: Reduced-order design of high-order sliding mode control system. Internat. J. Robust and Nonlinear Control *21* (2011), 18, 2064–2078.

[25] K. Tanaka and H. O. Wang: Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach. John Wiley and Sons, New York 2001.

[26] U. Utkin: Sliding Modes in Control and Optimization. Springer, Berlin 1992.

[27] V. Utkin and J. Shi: Integral sliding mode in systems operating under uncertainty conditions. In: Conference on Decision and Control 1996.

Alán Tapia, Sonora Institute of Technology, 5 de Febrero 818 Sur, Col. Centro, C.P.85000, Cd. Obregon, Sonora. Mexico.
e-mail: tapia_9017@hotmail.com

Raymundo Márquez, LAMIH UMR CNRS 8201, University of Valenciennes and Hainaut-Cambrésis, ZC 59000, Valenciennes. France.
e-mail: raymundo.marquezbordon@etu.univ-valenciennes.fr

Miguel Bernal, Sonora Institute of Technology, 5 de Febrero 818 Sur, Col. Centro, C.P.85000, Cd. Obregon, Sonora. Mexico.
e-mail: miguel.bernal@itson.edu.mx

Joaquín Cortez, Sonora Institute of Technology, 5 de Febrero 818 Sur, Col. Centro, C.P.85000, Cd. Obregon, Sonora. Mexico.
e-mail: joaquin.cortez@itson.7edu.mx