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DEGENERATING CAHN-HILLIARD SYSTEMS COUPLED WITH MECHANICAL EFFECTS AND COMPLETE DAMAGE PROCESSES

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Abstract. This paper addresses analytical investigations of degenerating PDE systems for phase separation and damage processes considered on nonsmooth time-dependent domains with mixed boundary conditions for the displacement field. The evolution of the system is described by a degenerating Cahn-Hilliard equation for the concentration, a doubly nonlinear differential inclusion for the damage variable and a quasi-static balance equation for the displacement field.

The analysis is performed on a time-dependent domain which characterizes the nondegenerated elastic material regions. We choose a notion of weak solutions which consists of weak formulations of the Cahn-Hilliard system and the momentum balance equation, a variational inequality for the damage evolution and an energy inequality. For the introduced degenerating system, we prove global-in-time existence of weak solutions. The main results are sketched from our recent paper [WIAS preprint no. 1759 (2012)].

Keywords: Cahn-Hilliard system; phase separation; complete damage; elliptic-parabolic degenerating system; linear elasticity; energetic solution; weak solution; doubly nonlinear differential inclusion; existence result; rate-dependent system

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1. Introduction and notation

In modern materials such as multicomponent alloys, many different physical processes can influence the microstructure. In the case of solder materials, phase separation, elastic deformations and damage processes may deteriorate the pursued quality for their use as microelectronic components.

Phase separation and damage processes are, however, usually treated by two separate models in the mathematical literature. To describe phase separation processes in alloys, phase-field models of Cahn-Hilliard and Allen-Cahn type coupled with elasticity are well adapted [8], [2], [7], [5], [1]. On the other hand, damage processes for standard materials are often modeled as unilateral processes within a gradient-of-damage theory [6], see also [3], [12], [17], [16] for analytical results. In the case of solder alloys, experimental studies indicate rate-dependence of the underlying physical processes (see [15]) which should be accounted for in the damage model. A phase-field approach which describes *both* the phase separation and the damage processes in a unifying model has been recently introduced in [9] and further investigated in [10]. In these papers, the damage process is considered as *incomplete*, i.e., maximal damaged regions still exhibit elastic properties. The reason why incomplete damage models are more feasible for mathematical investigations is that a uniform convexity assumption on the elastic energy density prevents the material from a complete degeneration. Mathematical works of complete damage models covering global-in-time existence are rare and are mainly focused on purely rate-independent systems [4], [13] by using Γ-convergence techniques to recover energetic properties in the limit. Existence results for rate-dependent complete damage systems in thermoviscoelastic materials have been recently published in [16] (see also [14]).

The main goal of this proceedings article is to present recent results from [11] concerning existence of weak solutions of a system coupling damage processes and elastic Cahn-Hilliard systems as in [9], [10] but allowing for complete damage and degenerating Cahn-Hilliard mobilities. The reference domain is assumed to be bounded and Lipschitz. Let us remark that, in our case, the mobility tensor is constant with respect to the concentration but depends on the damage variable and even may degenerate with respect to the damage variable. By the authors' best knowledge, that case has not yet been studied except in the recent paper [11]. The challenge is to derive an $L^2(H^2)$-type a priori estimate for the chemical potential. By the use of the so-called conical Pioncaré inequality, local estimates of this types can be obtained.

In this paper, generally speaking, we will investigate the coupled PDE system:

Cahn-Hilliard equations:

$$c_t = \operatorname{div}(m^\delta(z)\nabla\mu) \quad \text{in } \Omega_T, \tag{1.1a}$$

$$\mu = -\Delta c + \Psi'(c) + W^\delta_{,c}(c,\varepsilon(u),z) \quad \text{in } \Omega_T, \tag{1.1b}$$

quasi-static equilibrium of the forces:

$$-\operatorname{div}(W^\delta_{,e}(c,\varepsilon(u),z)) = f \quad \text{in } \Omega_T, \tag{1.1c}$$

evolution inclusion for the damage processes:

$$z_t - \Delta_p z + W^\delta_{,z}(c,\varepsilon(u),z) + g'(z) + \xi + \eta = 0 \quad \text{in } \Omega_T, \tag{1.1d}$$

$$\xi \in \partial I_{[0,\infty)}(z) \quad \text{in } \Omega_T, \tag{1.1e}$$

$$\eta \in \partial I_{(-\infty,0]}(z_t) \quad \text{in } \Omega_T. \tag{1.1f}$$

The variables in our model are the chemical concentration $c\colon \Omega_T \to \mathbb{R}^N$, the displacement field $u\colon \Omega_T \to \mathbb{R}^n$, the degree of damage $z\colon \Omega_T \to [0,1]$, the chemical potential $\mu\colon \Omega_T \to \mathbb{R}^N$ and the subgradients ξ and η. We denote by I_A and $\mathbf{1}_A$ the indicator function and characteristic function, respectively, for the set $A \subseteq X$.

We impose mixed boundary conditions for u and natural boundary conditions for the variables. The full list of the initial-boundary conditions considered is given in the following, where Γ_{D} and Γ_{N} denote the Dirichlet and Neumann boundaries.

Mixed boundary conditions:

$$u = b \quad \text{on } (\Gamma_{\mathrm{D}})_T, \tag{1.2a}$$

$$W^\delta_{,e}(c, \varepsilon(u), z) \cdot \nu = 0 \quad \text{on } (\Gamma_{\mathrm{N}})_T, \tag{1.2b}$$

Neumann boundary conditions:

$$\nabla z \cdot \nu = \nabla c \cdot \nu = m^\delta(z) \nabla \mu \cdot \nu = 0 \quad \text{on } (\partial\Omega)_T, \tag{1.2c}$$

initial values:

$$c(0) = c^0, \ z(0) = z^0 \quad \text{in } \Omega. \tag{1.2d}$$

The involved (nonlinear) functions and operators are explained below:

▷ *phase separation mobility tensor* m^δ depending on z is assumed to be

$$m^\delta(z) = m(z) + \delta \quad \text{with } \delta \geqslant 0,\ m(z) \geqslant 0 \text{ and } m(z) = 0 \text{ iff } z = 0, \tag{1.3}$$

▷ *elastic energy density* W^δ depending on c, $\varepsilon(u)$ and z is assumed to be

$$W^\delta(c, e, z) = (h(z) + \delta)\varphi(c, e) \quad \text{with } h(z) \geqslant 0 \text{ and } h(z) = 0 \text{ iff } z = 0, \tag{1.4}$$

▷ *chemical energy density* Ψ depending on c,

▷ *linearized strain tensor* $\varepsilon(u) := \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{t}})$,

▷ *damage potential* g depending on z,

▷ *volume forces* f depending on spatial and time coordinates and

▷ *p-Laplacian* $\Delta_p \cdot := \operatorname{div}(|\nabla \cdot|^{p-2}\nabla \cdot)$,

▷ *boundary displacement data* b depending on spatial and time coordinates.

The precise assumptions are listed in the next section.

The case $\delta > 0$ is considered as the regular (incomplete damage) case whereas $\delta = 0$ is the degenerated case (allowing for complete damage). The latter one requires a special treatment. In this context, it is appropriate to refine the notion of solution by introducing a time-dependent domain $\Omega(\cdot) := \{(x,t) \in \overline{\Omega_T};\ x \in \Omega(t)\}$ which incorporates the not completely damaged material. The main idea is to consider the path-connected components of the not completely damaged material which are connected to the Dirichlet boundary Γ_{D}. To be more precise, we introduce the following notion.

Definition 1.1 (Admissible subsets of $\overline{\Omega}$ with respect to Γ_{D})**.** We say that a relatively open subset $F \subseteq \overline{\Omega}$ is *admissible* with respect to the Dirichlet boundary Γ_{D} if for every path-connected component P of F the condition $\mathcal{H}^{n-1}(P \cap \Gamma_{\mathrm{D}}) > 0$ is fulfilled. Furthermore, $\mathfrak{A}_{\Gamma_{\mathrm{D}}}(F)$ denotes the maximal admissible subset of F with respect to Γ_{D}, i.e., $\mathfrak{A}_{\Gamma_{\mathrm{D}}}(F) := \bigcup\{G \subseteq F;\ G \text{ is admissible with respect to } \Gamma_{\mathrm{D}}\}$.

The PDE system (1.1)–(1.2) can be formulated on a time-dependent domain as follows. We call a 5-tuple (c,u,z,μ,η) a classical solution if

time-dependent domain:

$$\Omega(t) = \mathfrak{A}_{\Gamma_{\mathrm{D}}}(\{z(t) > 0\}) := \mathfrak{A}_{\Gamma_{\mathrm{D}}}(\{x \in \overline{\Omega};\ z(t,x) > 0\}) \tag{1.5a}$$

Cahn-Hilliard equations:

$$c_t = \operatorname{div}(m(z)\nabla\mu) \quad \text{in } \Omega(\cdot), \tag{1.5b}$$

$$\mu = -\Delta c + \Psi'(c) + W_{,c}(c,\varepsilon(u),z) \quad \text{in } \Omega(\cdot), \tag{1.5c}$$

quasi-static equilibrium of the forces:

$$-\operatorname{div}(W_{,e}(c,\varepsilon(u),z)) = f \quad \text{in } \Omega(\cdot), \tag{1.5d}$$

evolution inclusion for the damage processes:

$$z_t - \Delta_p z + W_{,z}(c,\varepsilon(u),z) + g'(z) + \eta = 0 \quad \text{in } \Omega(\cdot), \tag{1.5e}$$

$$\eta \in \partial I_{(-\infty,0]}(z_t) \quad \text{in } \Omega(\cdot). \tag{1.5f}$$

We adopt the convention $\partial\Omega(\cdot) := \{(x,t);\ x \in \partial\Omega(t)\}$ as well as $m := m^0$ and $W := W^0$. We would like to remark that $\xi = 0$ in $\Omega(\cdot)$ (ξ is the subgradient in (1.1e)) and that $\Omega(\cdot)$ is shrinking, i.e. $\Omega(t) \subseteq \Omega(s)$ whenever $t \geqslant s$, since z is monotonically decreasing in t. The boundary conditions in the degenerated case are more involved since parts of the Dirichlet or Neumann boundaries are also allowed to degenerate. We impose the following initial-boundary data:

mixed boundary conditions:

(1.6a) $$u = b \quad \text{on } \Gamma_1(\cdot) := \Omega(\cdot) \cap (\Gamma_{\mathrm{D}})_T,$$

(1.6b) $$W_{,e}(c, \varepsilon(u), z) \cdot \nu = 0 \quad \text{on } \Gamma_2(\cdot) := \Omega(\cdot) \cap (\Gamma_{\mathrm{N}})_T,$$

degenerated boundary:

(1.6c) $$z = 0 \quad \text{on } \Gamma_3(\cdot) := \partial\Omega(\cdot) \setminus (\Gamma_1(\cdot) \cup \Gamma_2(\cdot)),$$

Neumann boundary conditions:

(1.6d) $$\nabla z \cdot \nu = \nabla c \cdot \nu = m(z)\nabla\mu \cdot \nu = 0 \quad \text{on } \Gamma_1(\cdot) \cup \Gamma_2(\cdot),$$

initial values:

(1.6e) $$c(0) = c^0, \; z(0) = z^0 \quad \text{in } \Omega(0).$$

Plan of the paper. In Section 2, we will present suitable weak formulations for the regularized PDE system (1.1)–(1.2) as well as for the degenerating system (1.5)–(1.6). The main results are stated in Section 3 and are gathered from the papers [9], [10], [11]. Subsection 3.1 collects the existence theorems of these systems for different cases. While the existence proofs are briefly sketched in Subsection 3.3 and 3.4, some crucial proof techniques which are valuable in their own sense and which have already been successfully adapted to other situations (see [16]) are presented in Subsection 3.2.

2. Weak formulations

2.1. Nondegenerating case. The notion of a weak solution for system (1.1)–(1.2) was originally motivated in [9]. In order not to overburden the presentation, we assume $f = 0$.

Definition 2.1. We say that a 5-tuple (c, u, z, μ, ξ) is a weak solution in the nondegenerating case iff the functions are in the following spaces:

$$\begin{aligned}
&c \in L^\infty(0,T; H^1(\Omega;\mathbb{R}^N)) \cap H^1(0,T;(H^1(\Omega;\mathbb{R}^N))^*), \quad c(0) = c^0, \; c_1 + \ldots + c_N = 1,\\
&u \in L^\infty(0,T; H^1(\Omega;\mathbb{R}^n)), \quad u|_{(\Gamma_{\mathrm{D}})_T} = b|_{(\Gamma_{\mathrm{D}})_T},\\
&z \in L^\infty(0,T; W^{1,p}(\Omega)) \cap H^1(0,T; L^2(\Omega)), \quad z(0) = z^0, \; z \geqslant 0, \; \partial_t z \leqslant 0 \;\text{ a.e.},\\
&\mu \in L^2(0,T; H^1(\Omega;\mathbb{R}^N)),\\
&\xi \in L^2(0,T;(W^{1,p}(\Omega))^*),
\end{aligned}$$

and the following system is satisfied for a.e. $t \in (0,T)$:

Cahn-Hilliard equations:

$$c_t = \operatorname{div}(m^\delta(z)\nabla\mu) \quad \text{in } (H^1(\Omega;\mathbb{R}^N))^*, \tag{2.1a}$$

$$\mu = -\Delta c + \Psi'(c) + W^\delta_{,c}(c,\varepsilon(u),z) \quad \text{in} \quad (H^1(\Omega;\mathbb{R}^N))^*, \tag{2.1b}$$

quasi-static equilibrium of the forces:

$$-\operatorname{div}(W^\delta_{,e}(c,\varepsilon(u),z)) = 0 \quad \text{in} \quad (H^1(\Omega;\mathbb{R}^n))^*, \tag{2.1c}$$

evolution law for the damage processes (see Remark 2.2 (i)):

$$z_t - \Delta_p z + W^\delta_{,z}(c,\varepsilon(u),z) + g'(z) + \xi \leqslant 0 \quad \text{in} \quad (W^{1,p}(\Omega))^*, \tag{2.1d}$$

$$\xi \in \partial I_{W^{1,p}_+(\Omega)}(z), \tag{2.1e}$$

$$\mathcal{E}(t) + \mathcal{D}(0,t) \leqslant \mathcal{E}(0) + \mathcal{W}_{\rm ext}(0,t). \tag{2.1f}$$

The last property is the so-called *energy inequality*, where the terms are given by:

energy:

$$\begin{aligned}\mathcal{E}(t) := &\int_\Omega \Big(\frac{1}{p}|\nabla z(t)|^p + \frac{1}{2}|\nabla c(t)|^2\Big)\,\mathrm{d}x\\ &+ \int_\Omega (W^\delta(c(t),\varepsilon(u(t)),z(t))) + g(z(t)) + \Psi(c(t)))\,\mathrm{d}x,\end{aligned}$$

dissipated energy:

$$\mathcal{D}(0,t) := \int_0^t\int_\Omega (m^\delta(z)|\nabla\mu|^2 + |\partial_t z|^2)\,\mathrm{d}x\,\mathrm{d}t,$$

external work:

$$\mathcal{W}_{\rm ext}(0,t) := \int_0^t\int_\Omega (W^\delta_{,e}(c,\varepsilon(u),z)\colon\ \varepsilon(\partial_t b))\,\mathrm{d}x\,\mathrm{d}t.$$

R e m a r k 2.2. (i) The inequality (2.1d) should be read as

$$\int_\Omega (\partial_t z\,\zeta + |\nabla z|^{p-2}\nabla z\cdot\nabla\zeta + W^\delta_{,z}(c,\varepsilon(u),z)\zeta + g'(z)\zeta + \xi\zeta)\,\mathrm{d}x \leqslant 0$$

for all $\zeta \in W^{1,p}_+(\Omega) := \{f \in W^{1,p}(\Omega);\ f \geqslant 0\}$ or, equivalently,

$$\int_\Omega (\partial_t z\,\zeta + |\nabla z|^{p-2}\nabla z\cdot\nabla\zeta + W^\delta_{,z}(c,\varepsilon(u),z)\zeta + g'(z)\zeta + \xi\zeta)\,\mathrm{d}x \geqslant 0$$

for all $\zeta \in W^{1,p}_-(\Omega) := \{f \in W^{1,p}(\Omega);\ f \leqslant 0\}$ as presented in [9].

(ii) Let us remark that if a weak solution possesses more regularity, e.g.,

$$c \in H^1(0,T;H^1(\Omega;\mathbb{R}^N)), \quad u \in H^1(0,T;H^1(\Omega;\mathbb{R}^n)), \quad z \in H^1(0,T;W^{1,p}(\Omega)),$$

we find an $\eta \in L^2(0,T;(W^{1,p}(\Omega))^*)$ such that the evolution inclusion

$$\begin{aligned}
&z_t - \Delta_p z + W^{\delta}_{,z}(c,\varepsilon(u),z) + g'(z) + \xi + \eta = 0 \quad \text{in } (W^{1,p}(\Omega))^*,\\
&\xi \in \partial I_{W^{1,p}_+(\Omega)}(z) \quad \text{in } (W^{1,p}(\Omega))^*,\\
&\eta \in \partial I_{W^{1,p}_-(\Omega)}(\partial_t z) \quad \text{in } (W^{1,p}(\Omega))^*
\end{aligned}$$

holds for a.e. $t \in (0,T)$.

2.2. Degenerating case. To proceed, we need the following definition. Let $\Omega(\cdot) := \{\Omega(t)\}_{t\in[0,T]}$ be a time-dependent domain such that, for all $t \in [0,T]$, $\Omega(t)$ is relatively open in $\overline{\Omega}$ and $\Omega(\cdot)$ is assumed to be shrinking, i.e. $\Omega(t) \subseteq \Omega(s)$ whenever $t \geqslant s$. We introduce the space of local Sobolev functions $L^2_t H^1_{x,\,\mathrm{loc}}(\Omega(\cdot);\mathbb{R}^N)$ as

$$\begin{aligned}
L^2_t H^q_{x,\mathrm{loc}}(\Omega(\cdot);\mathbb{R}^N) := \{v\colon\ \Omega(\cdot) \to \mathbb{R}^N\colon\ &\forall\, t \in (0,T],\ \forall\, U \subset\subset \Omega(t) \text{ open:}\\
&v|_{U\times(0,t)} \in L^2(0,t;H^q(U;\mathbb{R}^N))\}.
\end{aligned}$$

Definition 2.3. We say that a 4-tuple (c,u,z,μ) is a weak solution of system (1.5)–(1.6) in the degenerating case iff the functions are in the following spaces:

$$\begin{aligned}
&c \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^N)) \cap H^1(0,T;(H^1(\Omega;\mathbb{R}^N))^*), \quad c(0) = c^0, c_1 + \ldots + c_N = 1,\\
&u \in L^2_t H^1_{x,\mathrm{loc}}(\Omega(\cdot);\mathbb{R}^n), \quad u|_{(\Gamma_{\mathrm{D}})_T\cap\Omega(\cdot)} = b|_{(\Gamma_{\mathrm{D}})_T\cap\Omega(\cdot)},\\
&z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), \quad z(0) = z^0,\ z \geqslant 0,\ \partial_t z \leqslant 0 \text{ a.e.},\\
&\mu \in L^2_t H^1_{x,\mathrm{loc}}(\Omega(\cdot);\mathbb{R}^N),
\end{aligned}$$

and the following system is satisfied for a.e. $t \in (0,T)$:

time-dependent domain:

$$\Omega(t) = \mathfrak{A}_{\Gamma_{\mathrm{D}}}(\{z(t) > 0\}) \tag{2.2a}$$

Cahn-Hilliard equations:

$$c_t = \operatorname{div}(m(z)\nabla\mu) \quad \text{in } (H^1_{\mathrm{c}}(\Omega(t);\mathbb{R}^N))^*, \tag{2.2b}$$

$$\mu = -\Delta c + \Psi'(c) + W_{,c}(c,e,z) \quad \text{in } (H^1_{\mathrm{c}}(\Omega(t);\mathbb{R}^N))^*, \tag{2.2c}$$

quasi-static equilibrium of the forces:

$$-\operatorname{div}(W_{,e}(c,e,z)) = 0 \quad \text{in } (H^1_{\mathrm{c}}(\Omega(t);\mathbb{R}^n))^*, \tag{2.2d}$$

evolution law for the damage processes:

$$z_t - \Delta_p z + W_{,z}(c,e,z) + g'(z) \leqslant 0 \quad \text{in } (W^{1,p}_{\mathrm{c}}(\Omega(t)))^*, \tag{2.2e}$$

$$\mathcal{E}(t) + \mathcal{D}(0,t) + \mathcal{J}(0,t) \leqslant \mathcal{E}^+(0) + \mathcal{W}_{\mathrm{ext}}(0,t), \tag{2.2f}$$

where the terms in the energy inequality are given by:

energy:

$$\begin{aligned}\mathcal{E}(t) := \mathcal{E}(t,c(t),u(t),z(t)) &:= \int_{\Omega(t)} \Big(\frac{1}{p}|\nabla z(t)|^p + \frac{1}{2}|\nabla c(t)|^2\Big)\,\mathrm{d}x \\ &+ \int_{\Omega(t)} (W(c(t),\varepsilon(u(t)),z(t))) + g(z(t)) + \Psi(c(t)))\,\mathrm{d}x,\end{aligned}$$

energy jump term:

$$\begin{aligned}\mathcal{J}(0,t) &:= \sum_{s\in J_{\Omega(\cdot)}\cap(0,t]} (\mathcal{E}^-(s) - \mathcal{E}^+(s)),\\ \mathcal{E}^-(t) &:= \lim_{s\uparrow t}\big(\operatorname*{ess\,inf}_{\tau\in(s,t)} \mathcal{E}(\tau)\big) \text{ and } \mathcal{E}^+(t)\in\mathbb{R}_+ \text{ satisfies:}\\ &\quad \mathcal{E}^+(t) \leqslant \inf_{\zeta\in H^1_{\Gamma_{\mathrm{D}}\cap\Omega(t)}(\Omega(t);\mathbb{R}^n)} \mathcal{E}(t,c(t),b(t)+\zeta,z(t)),\end{aligned}$$

dissipated energy:

$$\mathcal{D}(0,t) := \int_0^t\int_{\Omega(s)} (m(z)|\nabla\mu|^2 + |\partial_t z|^2)\,\mathrm{d}x\,\mathrm{d}t,$$

external work:

$$\mathcal{W}_{\mathrm{ext}}(0,t) := \int_0^t\int_{\Omega(s)} (W_{,e}(c,\varepsilon(u),z)\colon\, \varepsilon(\partial_t b))\,\mathrm{d}x\,\mathrm{d}t.$$

Here, $J_{\Omega(\cdot)}$ denotes the jump set of $\mathbf{1}_{\Omega(\cdot)}\colon\,[0,T]\to L^2(\Omega)$, i.e.,

$$J_{\Omega(\cdot)} := \{t\in(0,T);\ \lim_{s\uparrow t}\mathbf{1}_{\Omega(\cdot)}(s) \neq \lim_{s\downarrow t}\mathbf{1}_{\Omega(\cdot)}(s)\}.$$

Remark 2.4. (i) Note that $H^1_{\mathrm{c}}(\Omega(t);\mathbb{R}^k) := \{f\in H^1(\Omega(t);\mathbb{R}^k);\ \operatorname{supp}(f)\subseteq\Omega(t)\}$ and (2.2e) is a short form for (the following holds for a.e. $t\in(0,T)$)

$$\int_{\Omega(t)} (z_t\,\zeta + |\nabla z|^{p-2}\nabla z\cdot\nabla\zeta + W_{,z}(c,\varepsilon(u),z)\zeta + g'(z)\zeta)\,\mathrm{d}x \leqslant 0$$

for all $\zeta\in W^{1,p}_+(\Omega(t);\mathbb{R}^N)$ with $\operatorname{supp}(\zeta)\subseteq\Omega(t)$.

(ii) If the damage function z is truncated on the time-dependent domain $\Omega(\cdot)$, i.e., $\tilde{z} = z\mathbf{1}_{\Omega(\cdot)}$, we obtain an equivalent weak formulation in an SBV-framework as in [11], i.e., $\tilde{z} \in SBV(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{1,p}(\Omega))$.

(iii) Assuming better regularity (cf. Subsection 2.1), we obtain

$$\mathcal{E}^+(t) = \lim_{s\downarrow t} \mathcal{E}(s), \quad \mathcal{E}^-(t) = \lim_{s\uparrow t} \mathcal{E}(s)$$

and the corresponding evolution inclusion holds.

In the degenerate limit, we might be confronted with infinitely many material exclusions in $\Omega(\cdot)$ which occur in arbitrary short time intervals. To handle this case, we introduce a notion of weak solutions with a given fineness $\eta > 0$. Roughly speaking, we neglect arbitrarily small material exclusions in the energy inequality.

Definition 2.5. A weak solution with fineness η is a 5-tuple (c, e, u, z, μ) with

$$\begin{aligned}
&c \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^N)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^N)),\ c(0) = c^0,\ c_1 + \ldots + c_N = 1,\\
&e \in L^2(\Omega_T;\mathbb{R}^{n\times n}),\\
&u \in L^2_t H^2_{x,\mathrm{loc}}(\Omega(\cdot);\mathbb{R}^n)),\ u|_{(\Gamma_{\mathrm{D}})_T\cap\Omega(\cdot)} = b|_{(\Gamma_{\mathrm{D}})_T\cap\Omega(\cdot)},\ \varepsilon(u) = e \text{ in } \Omega(\cdot),\\
&z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)),\ z(0) = z^0,\ z \geqslant 0,\ \partial_t z \leqslant 0 \ \text{ a.e.},\\
&\mu \in L^2_t H^2_{x,\mathrm{loc}}(\Omega(\cdot);\mathbb{R}^N)
\end{aligned}$$

together with another time-dependent shrinking domain $\Xi(\cdot)$ such that (2.2) is fulfilled when we substitute $\Omega(\cdot)$ by $\Xi(\cdot)$ in the definition of $\mathcal{E}$, $\mathcal{J}$, $\mathcal{D}$ and $\mathcal{W}_{\mathrm{ext}}$ (which occur in the energy inequality (2.2f)). The domain should satisfy

$$\begin{aligned}
&\forall\, t \in [0,T]\colon\ \Omega(\cdot) \subseteq \Xi(\cdot) \text{ and } \mathcal{L}^n(\Xi(t) \setminus \Omega(t)) < \eta,\\
&\forall\, t \in [0,T] \setminus \bigcup_{t \in C_{\Omega(\cdot)}} [t, t+\eta)\colon\ \Omega(t) = \Xi(t).
\end{aligned}$$

Here, $C_{\Omega(\cdot)}$ denotes the set of cluster points from the right hand side of the jump set $J_{\Omega(\cdot)}$.

3. Existence results

3.1. Overview. For all the following theorems, we assume that Ω is a bounded Lipschitz domain, that the coefficient functions g, h and m are continuously differentiable with the properties (1.3) and (1.4) and, as already mentioned, $f = 0$. The initial-boundary values are given by $c^0 \in H^1(\Omega;\mathbb{R}^N)$, $z^0 \in W^{1,p}(\Omega)$ and $b \in W^{1,1}(0,T;W^{1,\infty}(\Omega;\mathbb{R}^n))$ with $c^0_1 + \ldots + c^0_N = 1$ and $0 \leqslant z^0 \leqslant 1$.

Theorem 3.1 (Nondegenerating case-polynomial growth conditions)**.** *Let $p = 2$ or $p > n$, and $\Gamma_{\rm D} \subseteq \partial\Omega$ with $\mathcal{H}^{n-1}(\Gamma_{\rm D}) > 0$. We assume the following growth conditions (C, $\eta > 0$ are constants):*

$$\begin{aligned}
&\eta|e_1 - e_2|^2 \leqslant (\partial_e\varphi(c, e_1) - \partial_e\varphi(c, e_2)) : (e_1 - e_2),\\
&\varphi(e, z) \leqslant C(|c|^2 + |e|^2 + 1), \quad |\partial_c\varphi(c, e)| \leqslant C(|c|^2 + |e|^2 + 1),\\
&|\partial_e\varphi(e_1 + e_2, c)| \leqslant C(\varphi(c, e_1) + |e_2| + 1), \quad |\Psi'(c)| \leqslant C(|c|^{2^\star/2} + 1).
\end{aligned}$$

Then there exists a weak solution (c, u, z, μ, ξ) on the time interval $[0, T]$ in the sense of Definition 2.1*.*

Theorem 3.2 (Nondegenerating case-logarithmic growth conditions)**.** *Let $p > n$ or $p = 2$ and $\Gamma_{\rm D} = \partial\Omega$. Furthermore, suppose that φ satisfies the assumptions from Theorem* 3.1 *and, additionally, suppose $c_k^0 > 0$ a.e. in Ω for all $k = 1, \ldots, N$. The chemical energy density Ψ is assumed to be of the logarithmic form*

$$\Psi(c) = \theta\sum_{k=1}^{N} c_k \log(c_k) + \frac{1}{2}c \cdot Ac, \quad \theta > 0,\ A \in \mathbb{R}^{n\times n}_{\rm sym}.$$

Then, there exists a weak solution (c, u, z, μ, ξ) on the time interval $[0, T]$ in the sense of Definition 2.1 *and $c_k > 0$ a.e. in Ω_T for all $k = 1, \ldots, N$.*

Theorem 3.3 (Degenerating case-polynomial growth conditions)**.** *Let $p > n$, $\eta > 0$ and $\Gamma_{\rm D} \subseteq \partial\Omega$ with $\mathcal{H}^{n-1}(\Gamma_{\rm D}) > 0$. We assume $\varphi(e, z) = \varphi^1 e : e + \varphi^2(c) : e + \varphi^3(c)$ with a positive definite matrix φ^1 and $\varphi^2 \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^{n\times n}_{\rm sym})$, $\varphi^3 \in \mathcal{C}^1(\mathbb{R})$ which satisfy (C, $\eta > 0$ are constants)*

$$\begin{gathered}
|\varphi^2(c)|, |\varphi^2_{,c}(c)| \leqslant C(1 + |c|), \quad |\varphi^3(c)|, |\varphi^3_{,c}(c)| \leqslant C(1 + |c|^2),\\
|\Psi_{,c}(c)| \leqslant C(1 + |c|^{2^\star/2}), \quad \eta \leqslant h'(z).
\end{gathered}$$

Furthermore, we suppose that the set $\{z^0 > 0\}$ is admissible with respect to $\Gamma_{\rm D}$. Then there exists a weak solution (c, e, u, z, μ) with fineness η on the time interval $[0, T]$ in the sense of Definition 2.5*.*

R e m a r k 3.4. We also obtain maximal local-in-time solutions in the sense of Definition 2.3. Furthermore, the condition $p > n$ in Theorem 3.3 is needed since the existence proof strongly relies on the compact embedding $L^\infty(0, T; W^{1,p}(\Omega)) \cap H^1(0, T; L^2(\Omega)) \hookrightarrow \mathcal{C}(\overline{\Omega_T})$.

3.2. Auxiliary results. The following results are crucial for the proof of the existence theorems in the next subsections. Let us mention that whenever we consider a family $\{f_\tau\}_{\tau>0}$ of functions in this work, we actually mean a sequence $\{f_{\tau_k}\}_{k\in\mathbb{N}}$ with $\tau_k \downarrow 0$. In this subsection, we always assume $p > n$.

Lemma 3.5 (see [10]). *Let $f \in L^p(\Omega;\mathbb{R}^n)$, $g \in L^1(\Omega)$ and $z \in W^{1,p}_+(\Omega)$ with $f\cdot\nabla z \geqslant 0$ a.e. in Ω and $\{f=0\} \supseteq \{z=0\}$ in an a.e. sense. Furthermore, we assume that*

$$\int_\Omega (f\cdot\nabla\zeta + g\zeta)\,\mathrm{d}x \geqslant 0 \quad \text{for all } \zeta \in W^{1,p}_-(\Omega) \text{ with } \{\zeta=0\} \supseteq \{z=0\}.$$

Then

$$\int_\Omega (f\cdot\nabla\zeta + g\zeta)\,\mathrm{d}x \geqslant \int_{\{z=0\}} [g]^+\zeta\,\mathrm{d}x \quad \text{for all } \zeta \in W^{1,p}_-(\Omega)$$

with $[g]^+ := \max(0,g)$.

Remark 3.6. In the work [10], g is assumed to be in $L^p(\Omega)$. But the proof extends to $g \in L^1(\Omega)$ without any modifications.

In the next lemma, the notation $\{\zeta=0\} \supseteq \{f=0\}$ for functions in $L^\infty(0,T;W^{1,p}(\Omega))$ shall be read as $\{x\in\overline{\Omega};\ \zeta(x,t)=0\} \supseteq \{x\in\overline{\Omega};\ f(x,t)=0\}$ for a.e. $t\in(0,T)$.

Lemma 3.7 (see [10]). *Let*

- ▷ *$f_\tau, f \in L^\infty(0,T;W^{1,p}_+(\Omega))$ with $f_\tau(t)\to f(t)$ weakly in $W^{1,p}(\Omega)$ as $\tau\downarrow 0$ for a.e. $t\in(0,T)$,*
- ▷ *$\zeta \in L^\infty(0,T;W^{1,p}_+(\Omega))$ with $\{\zeta=0\}\supseteq\{f=0\}$.*

Then there exist a sequence $\zeta_\tau \in L^\infty(0,T;W^{1,p}_+(\Omega))$ and constants $\nu_{\tau,t}>0$ such that

- ▷ *$\zeta_\tau\to\zeta$ strongly in $L^q(0,T;W^{1,p}(\Omega))$ as $\tau\downarrow 0$ for all $q\geqslant 1$,*
- ▷ *$\zeta_\tau\to\zeta$ weakly-star in $L^\infty(0,T;W^{1,p}(\Omega))$ as $\tau\downarrow 0$,*
- ▷ *$\zeta_\tau\leqslant\zeta$ a.e. in Ω_T for all $M\in\mathbb{N}$ (in particular $\{\zeta_\tau=0\}\supseteq\{\zeta=0\}$),*
- ▷ *$\nu_{\tau,t}\zeta_\tau(t)\leqslant f_\tau(t)$ in Ω for a.e. $t\in(0,T)$ and for all $\tau>0$.*

If, in addition, $\zeta\leqslant f$ a.e. in Ω_T then the last condition can be refined to $\zeta_\tau\leqslant f_\tau$ a.e. in Ω_T for all $\tau>0$.

3.3. Sketch of the existence proof of the nondegenerated case. Here we restrict ourselves to a polynomial growth condition for Ψ as in Theorem 3.1. The logarithmic case can be proved with a higher integrability argument for the strain tensor and with an appropriate regularization technique for Ψ, see [10]. The proof consists of the following steps which are sketched below.

1. Regularization. The proof is based on the following ε-regularization ($q > n$):

$$c_t = \operatorname{div}(m^\delta(z)\nabla\mu) \quad \text{in } (H^1(\Omega;\mathbb{R}^N))^*, \tag{3.1a}$$

$$\mu = -\Delta c + \Psi'(c) + W^\delta_{,c}(c,\varepsilon(u),z) + \varepsilon c_t \quad \text{in } (H^1(\Omega;\mathbb{R}^N))^*, \tag{3.1b}$$

$$-\operatorname{div}(W^\delta_{,e}(c,\varepsilon(u),z)) - \varepsilon\operatorname{div}(|\nabla u|^2\nabla u) = 0 \quad \text{in } (W^{1,4}(\Omega;\mathbb{R}^n))^*, \tag{3.1c}$$

$$z_t - \Delta_p z - \varepsilon\Delta_q z + W^\delta_{,z}(c,\varepsilon(u),z) + g'(z) + \xi \leqslant 0 \quad \text{in } (W^{1,q}(\Omega))^*, \tag{3.1d}$$

$$\xi \in \partial I_{W^{1,q}_+(\Omega)}(z), \tag{3.1e}$$

$$\mathcal{E}_\varepsilon(t) + \mathcal{D}_\varepsilon(0,t) \leqslant \mathcal{E}_\varepsilon(0) + \mathcal{W}_{\varepsilon,\mathrm{ext}}(0,t) \tag{3.1f}$$

with

$$\begin{aligned}
\mathcal{E}_\varepsilon(t) &:= \int_\Omega \Big(\frac{1}{p}|\nabla z(t)|^p + \frac{\varepsilon}{q}|\nabla z(t)|^q + \frac{1}{2}|\nabla c(t)|^2 + \frac{\varepsilon}{4}|\nabla u(t)|^4\Big)\,\mathrm{d}x \\
&\quad + \int_\Omega (W^\delta(c(t),\varepsilon(u(t)),z(t))) + g(z(t)) + \Psi(c(t)))\,\mathrm{d}x, \\
\mathcal{D}_\varepsilon(0,t) &:= \int_0^t\int_\Omega (m^\delta(z)|\nabla\mu|^2 + |\partial_t z|^2 + \varepsilon|\partial_t c|^2)\,\mathrm{d}x\,\mathrm{d}t, \\
\mathcal{W}_{\varepsilon,\mathrm{ext}}(0,t) &:= \int_0^t\int_\Omega (W^\delta_{,e}(c,\varepsilon(u),z)\colon\ \varepsilon(\partial_t b) + \varepsilon|\nabla u|^2\nabla u\colon\ \nabla(\partial_t b))\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The q-Laplacian in the regularized system with $q > n$ is not needed in the case $p > n$.

2. Time-discretization via semi-implicit Euler scheme. Existence of weak solutions for the regularized system is proved by a time-discretization method via a semi-implicit Euler scheme. Let $\tau > 0$ denote the discretization fineness and let $M_\tau := \lfloor T/\tau\rfloor$ be the number of discrete time steps associated with the fineness τ. A weak solution for the time-discrete system at the discrete time step $k \in \{0,\dots,M_\tau\}$ is given by

$$\begin{aligned}
&\tau^{-1}(c^k_\tau - c^{k-1}_\tau) = \operatorname{div}(m^\delta(z^{k-1}_\tau)\nabla\mu^k_\tau), \\
&\mu^k_\tau = -\Delta c^k_\tau + \Psi'(c^k_\tau) + W^\delta_{,c}(c^k_\tau,\varepsilon(u^k_\tau),z^k_\tau) + \varepsilon\tau^{-1}(c^k_\tau - c^{k-1}_\tau), \\
&-\operatorname{div}(W^\delta_{,e}(c^k_\tau,\varepsilon(u^k_\tau),z^k_\tau)) - \operatorname{div}(|\nabla u^k_\tau|^2\nabla u^k_\tau) = 0, \\
&\tau^{-1}(z^k_\tau - z^{k-1}_\tau) - \Delta_p z^k_\tau - \varepsilon\Delta_q z^k_\tau + W^\delta_{,z}(c^k_\tau,\varepsilon(u^k_\tau),z^k_\tau) + g'(z^k_\tau) + \xi^k_\tau = 0, \\
&\xi^k_\tau \in \partial I_{A^k_\tau}(z)
\end{aligned}$$

with the constraint set $A^k_\tau := \{\zeta \in W^{1,q}(\Omega);\ 0 \leqslant \zeta \leqslant z^{k-1}_\tau\}$.

The differential inclusion is equivalent to the following variational inequality:

$$\begin{aligned}
-\int_\Omega &(\tau^{-1}(z^k_\tau - z^{k-1}_\tau)(\zeta - z^k_\tau) + W^\delta_{,z}(c^k_\tau,\varepsilon(u^k_\tau),z^k_\tau)(\zeta - z^k_\tau) + g'(z^k_\tau)(\zeta - z^k_\tau) \\
&+ |\nabla z^k_\tau|^{p-2}\nabla z^k_\tau\cdot\nabla(\zeta - z^k_\tau) + |\nabla z^k_\tau|^{q-2}\nabla z^k_\tau\cdot\nabla(\zeta - z^k_\tau))\,\mathrm{d}x \leqslant 0
\end{aligned} \tag{3.2}$$

holding for all $\zeta \in A^k_\tau$.

Now, given $(c_\tau^{k-1}, u_\tau^{k-1}, z_\tau^{k-1}, \mu_\tau^{k-1})$, existence of weak solutions $(c_\tau^k, u_\tau^k, z_\tau^k, \mu_\tau^k)$ can be proved by minimizing an appropriate functional via the direct method.

3. A priori estimates and compactness properties. For a sequence of functions $\{v_\tau^k\}$, we denote the piecewise constant interpolants by $v_\tau(t) := v_\tau^k$ with $k = \lceil t/\tau \rceil$ and $v_\tau^-(t) := v_\tau^k$ with $k = \lfloor t/\tau \rfloor$. The discrete energy estimate yields the following a priori estimates (for all $\tau > 0$):

$$\begin{aligned}
&\{c_\tau\} \text{ in } L^\infty(0,T;H^1(\Omega;\mathbb{R}^N)), \quad \{\tau^{-1}(c_\tau - c_\tau^-)\} \text{ in } L^2(0,T;L^2(\Omega;\mathbb{R}^N)),\\
&\{z_\tau\} \text{ in } L^\infty(0,T;W^{1,q}(\Omega)), \quad \{\tau^{-1}(z_\tau - z_\tau^-)\} \text{ in } L^2(0,T;L^2(\Omega)),\\
&\{u_\tau\} \text{ in } L^\infty(0,T;W^{1,4}(\Omega;\mathbb{R}^n)), \quad \{\mu_\tau\} \text{ in } L^2(0,T;H^1(\Omega;\mathbb{R}^N)).
\end{aligned}$$

The corresponding convergence properties (with respect to a subsequence) can be obtained by standard compactness theorems and by a compactness theorem from Aubin/Lions.

4. Strong convergence properties. By using uniform monotonicity estimates and by exploiting the equations for the discrete system, it is possible to prove the following strong convergence properties:

$$\begin{aligned}
&c_\tau \to c \quad \text{strongly in } L^2(0,T;H^1(\Omega;\mathbb{R}^N)),\\
&u_\tau \to u \quad \text{strongly in } L^4(0,T;W^{1,4}(\Omega;\mathbb{R}^n)),\\
&z_\tau \to z \quad \text{strongly in } L^q(0,T;W^{1,q}(\Omega)).
\end{aligned}$$

We sketch the proof for $\{z_\tau\}$ since the other convergence properties follow with much less effort. Applying Lemma 3.7, we obtain a sequence of approximations $\{\zeta_M\} \subseteq L^\infty(0,T;W_+^{1,q}(\Omega))$ with the properties

$$\zeta_\tau \to z \quad \text{in } L^q(0,T;W^{1,q}(\Omega)), \tag{3.3a}$$

$$0 \leqslant \zeta_\tau \leqslant z_\tau^- \quad \text{for all } \tau > 0. \tag{3.3b}$$

A uniform p-monotonicity estimate yields

$$\begin{aligned}
&C_{\mathrm{uc}} \int_{\Omega_T} \varepsilon |\nabla z_\tau - \nabla z|^q \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} |\nabla z_\tau - \nabla z|^p \,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant \int_{\Omega_T} ((\varepsilon|\nabla z_\tau|^{q-2} + |\nabla z_\tau|^{p-2})\nabla z_\tau - (\varepsilon|\nabla z|^{q-2} + |\nabla z|^{p-2})\nabla z) \cdot \nabla(z_\tau - z) \,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{\Omega_T} (\varepsilon|\nabla z_\tau|^{q-2} + |\nabla z_\tau|^{p-2})\nabla z_\tau \cdot \nabla(z_\tau - \zeta_\tau) \,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_{\Omega_T} (\varepsilon|\nabla z|^{q-2} + |\nabla z|^{p-2})\nabla z \cdot \nabla(z_\tau - z) \,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_{\Omega_T} (\varepsilon|\nabla z_\tau|^{q-2} + |\nabla z_\tau|^{p-2})\nabla z_\tau \cdot \nabla(\zeta_\tau - z) \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The first term on the right hand side can be estimated by using the time-integrated version of the variational inequality (3.2) tested with $\zeta = \zeta_\tau$ (thanks to (3.3b)), i.e.,

$$\int_{\Omega_T} (\varepsilon|\nabla z_\tau|^{q-2} + |\nabla z_\tau|^{p-2})\nabla z_\tau \cdot \nabla(z_\tau - \zeta_\tau)\,\mathrm{d}x\,\mathrm{d}t \\ \leqslant \int_{\Omega_T} (W^\delta_{,z}(c_\tau, \varepsilon(u_\tau), z_\tau) + g'(z_\tau) + \tau^{-1}(z_\tau - z_\tau^-))(\zeta_\tau - z_\tau)\,\mathrm{d}x\,\mathrm{d}t.$$

We can now apply (3.3a) and the convergence properties arising from the a priori estimates to show $\|\nabla z_\tau - \nabla z\|^q_{L^q} + \|\nabla z_\tau - \nabla z\|^p_{L^p} \to 0$ as $\tau \downarrow 0$.

5. Establishing the continuous limit (in)equalities. By exploiting the strong convergence properties, we can now pass to the limit in the time-discrete system. The challenging part is to establish the variational inequality (3.1d)–(3.1e). We give the main ideas below.

First of all, we consider only test-functions $\zeta \in L^\infty(0,T;W^{1,p}_-(\Omega))$ with the constraint $\{\zeta = 0\} \supseteq \{z = 0\}$. Lemma 3.7 gives a sequence $\{\zeta_\tau\} \subseteq L^\infty(0,T;W^{1,p}_-(\Omega))$ with

$$\zeta_\tau \to z \quad \text{in } L^q(0,T;W^{1,q}(\Omega)), \\ 0 \leqslant -\nu_{\tau,t}\zeta_\tau \leqslant z_\tau \quad \text{for all } \tau > 0.$$

In particular, (3.2) holds for $\zeta = \nu_{\tau,t}\zeta_\tau(t) + z_\tau(t)$. Dividing the resulting inequality by $\nu_{\tau,t} > 0$, integrating from 0 to T, passing to $\tau \downarrow 0$ and switching to an "a.e. t" formulation we obtain

$$\int_\Omega ((\varepsilon|\nabla z|^{q-2} + |\nabla z|^{p-2})\nabla z \cdot \nabla\zeta + (W^\delta_{,z}(c, \varepsilon(u), z) + g'(z) + \partial_t z)\zeta)\,\mathrm{d}x \geqslant 0$$

holding for all $\zeta \in W^{1,p}_-(\Omega)$ with $\{\zeta = 0\} \supseteq \{z(t) = 0\}$.

Lemma 3.5 shows

$$\int_\Omega (\varepsilon|\nabla z|^{q-2} + |\nabla z|^{p-2})\nabla z \cdot \nabla\zeta + (W^\delta_{,z}(c, \varepsilon(u), z) + g'(z) + \partial_t z)\zeta\,\mathrm{d}x \\ \geqslant \int_{\{z(t)=0\}} [W^\delta_{,z}(c, \varepsilon(u), z) + g'(z) + \partial_t z]^+\zeta\,\mathrm{d}x$$

for every $\zeta \in W^{1,p}_-(\Omega)$. Now, the variational inequality (3.1d) follows by setting

$$\xi := -\mathbf{1}_{\{z=0\}}[W^\delta_{,z}(c, \varepsilon(u), z) + g'(z) + \partial_t z]^+.$$

6. Passing to the limit in the regularization. Finally, we pass $\varepsilon \downarrow 0$ in (3.1) by performing a limit analysis and end up with system (2.1). For details, we refer to [10].

3.4. Sketch of the existence proof of the degenerating case. For every $\delta > 0$, we take a weak solution $(c_\delta, u_\delta, z_\delta, \mu_\delta, \xi_\delta)$ according to Theorem 3.1 and perform the following steps.

1. A priori estimates. By Gronwall's lemma, the left hand side of the energy inequality (2.1f) stays bounded in the transition $\delta \downarrow 0$. We obtain the following a priori estimates:

$$
\begin{aligned}
&\{c_\delta\} \text{ in } L^\infty(0,T;H^1(\Omega;\mathbb{R}^N)), \quad \{\partial_t c_\delta\} \text{ in } L^2(0,T;(H^1(\Omega;\mathbb{R}^N))^*),\\
&\{z_\delta\} \text{ in } L^\infty(0,T;W^{1,p}(\Omega)), \quad \{\partial_t z_\delta\} \text{ in } L^2(0,T;L^2(\Omega)),\\
&\{\hat{e}_\delta\} \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^{n\times n})), \quad \{W^\delta(c_\delta,e_\delta,z_\delta)\} \text{ in } L^\infty(0,T;L^1(\Omega)),\\
&\{m^\delta(z_\delta)^{1/2}\nabla\mu_\delta\} \text{ in } L^2(0,T;L^2(\Omega;\mathbb{R}^{N\times n}))
\end{aligned}
$$

with $e_\delta := \varepsilon(u_\delta)$ and $\hat{e}_\delta := e_\delta \mathbf{1}_{\{z_\delta>0\}}$.

2. Convergence to a limit system. The a priori estimates lead to the following convergence properties (with respect to a subsequence):

$$
\begin{aligned}
&c_\delta \to c \quad \text{weakly-star in } L^\infty(0,T;H^1(\Omega;\mathbb{R}^N)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^N))\\
&\qquad\quad \text{and strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^N)),\\
&z_\delta \to z \quad \text{weakly-star in } L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))\\
&\qquad\quad \text{and strongly in } \mathcal{C}(\overline{\Omega_T}),\\
&\hat{e}_\delta \to \hat{e} \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n})),\\
&\mu_\delta \to \mu \quad \text{weakly in } L^2_t H^1_{x,\,\mathrm{loc}}(\Omega(\cdot);\mathbb{R}^N),\\
&m^\delta(z_\delta)\nabla\mu_\delta \to m(z)\nabla\mu \quad \text{weakly in } L^2(\Omega(\cdot);\mathbb{R}^N),\\
&W^\delta_{,e}(c_\delta,e_\delta,z_\delta) \to W_{,e}(c,\hat{e},z) \quad \text{weakly in } L^2(\Omega(\cdot);\mathbb{R}^{n\times n}),\\
&W^\delta_{,c}(c_\delta,e_\delta,z_\delta) \to W_{,c}(c,\hat{e},z) \quad \text{weakly in } L^2(\Omega(\cdot);\mathbb{R}^N).
\end{aligned}
$$

By proving a suitable representation result for $\Omega(\cdot) = \mathfrak{A}_{\Gamma_{\mathrm{D}}}(\{z(\cdot) > 0\})$ with Lipschitz domains, it is possible to prove $\hat{e} = \varepsilon(u)$ in $\Omega(\cdot)$ for some $u \in L^2_t H^1_{x,\,\mathrm{loc}}(\Omega(\cdot);\mathbb{R}^n)$ with $u = b$ on $(\Gamma_{\mathrm{D}})_T \cap \Omega(\cdot)$.

Besides the energy inequality (2.1f), we are able to pass to the limit in (2.1) for test-functions supported in $\Omega(\cdot)$ via lower semi-continuity arguments (and setting $e := \hat{e}\,\mathbf{1}_{\{z>0\}}$). Note that $\xi = 0$ in $\Omega(\cdot)$.

3. Establishing energy inequality via Γ*-convergence.* The initial value z^0 may contain complete damaged regions (this possibility is important in the next step of the proof). If this is the case, the initial displacements u^0_δ need not converge strongly to a function u^0 in $H^1(\Omega;\mathbb{R}^n)$. To obtain an energy inequality regardless of this difficulty (see next step), we use a Γ-convergence technique for the convergence of the right hand side of the energy inequality. We define the reduced energy to be

$$\mathfrak{E}_\delta(c,\xi,z) := \begin{cases} \min\limits_{\zeta\in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^n)} \mathcal{E}(c,\xi+\zeta,z) & \text{if } 0\leqslant z\leqslant 1, \\ \infty & \text{else.} \end{cases}$$

The Γ-limit of $\{\mathfrak{E}_\delta\}$ for $\delta\downarrow 0$ in the space $H^1_{\mathrm{weak}}(\Omega)\times W^{1,\infty}(\Omega;\mathbb{R}^n)\times W^{1,p}_{\mathrm{weak}}(\Omega)$ is denoted by $\mathfrak{E}$.

Considering a recovery sequence $(c^0_\delta,b^0_\delta,z^0_\delta)$ of (c^0,b^0,z^0) with $b^0:=b(0)$ as initial value, we are able to prove the energy inequality

$$\mathcal{E}(t)+\mathcal{D}(0,t)\leqslant\mathfrak{E}(0)+\mathcal{W}_{\mathrm{ext}}(0,t)\quad\text{with } \mathfrak{E}(0):=\mathfrak{E}(c^0,b^0,z^0).$$

4. Concatenation of weak solutions. To obtain the full energy inequality with the jump term, we use a concatenation property of weak solutions constructed above. We only sketch the argument. The full version requires the application of Zorn's lemma.

Let (c,e,u,z,μ) be the weak solution to (c^0,z^0) and b from the previous step. Furthermore, let t_1 denote the time when a material exclusion occurs, i.e. $t_1\in J_{\Omega(\cdot)}$. Then we set $c^1:=c(t_1)$ and $z^1:=z(t_1)\mathbf{1}_{\Omega(t_1)}$ and use these functions as new initial values. By the argumentation above, we get a further weak solution $(\tilde c,\tilde e,\tilde u,\tilde z,\widetilde\mu)$ starting from time t_1. In conclusion, we have

$$\mathcal{E}(t)+\mathcal{D}(0,t)\leqslant\mathfrak{E}(0)+\mathcal{W}_{\mathrm{ext}}(0,t)\quad\text{for a.e. } t\in(0,t_1), \tag{3.4a}$$

$$\mathcal{E}(t)+\mathcal{D}(t_1,t)\leqslant\mathfrak{E}(t_1)+\mathcal{W}_{\mathrm{ext}}(t_1,t)\quad\text{for a.e. } t\in(t_1,T). \tag{3.4b}$$

The energy inequality (3.4a) implies

$$\lim_{t\uparrow t_1}(\operatorname*{ess\,inf}_{s\in(t,t_1)}\mathcal{E}(s))+\mathcal{D}(0,t_1)\leqslant\mathfrak{E}(0)+\mathcal{W}_{\mathrm{ext}}(0,t_1).$$

Adding this to the inequality (3.4b) shows for a.e. $t\in(t_1,T)$

$$\mathcal{E}(t)+\lim_{t\uparrow t_1}(\operatorname*{ess\,inf}_{s\in(t,t_1)}\mathcal{E}(s))-\mathfrak{E}(t_1)+\mathcal{D}(0,t)\leqslant\mathfrak{E}(0)+\mathcal{W}_{\mathrm{ext}}(0,t).$$

Due to $\mathcal{E}^-(t_1)=\lim\limits_{t\uparrow t_1}(\operatorname{ess\,inf}_{s\in(t,t_1)}\mathcal{E}(s))$, $\mathcal{E}^+(t_1)\geqslant\mathfrak{E}(t_1)$ and $\mathcal{E}^+(0)\geqslant\mathfrak{E}(0)$, we have proved an energy inequality which accounts for the energy jump at t_1. By extending this argument via Zorn's lemma, we obtain the energy inequality in the sense of Definition 2.5.

References

[1] *L. Bartkowiak, I. Pawłow*: The Cahn-Hilliard-Gurtin system coupled with elasticity. Control Cybern. *34* (2005), 1005–1043.

[2] *E. Bonetti, P. Colli, W. Dreyer, G. Gilardi, G. Schimperna, J. Sprekels*: On a model for phase separation in binary alloys driven by mechanical effects. Physica D *165* (2002), 48–65.

[3] *E. Bonetti, G. Schimperna, A. Segatti*: On a doubly nonlinear model for the evolution of damaging in viscoelastic materials. J. Differ. Equations *218* (2005), 91–116.

[4] *G. Bouchitté, A. Mielke, T. Roubíček*: A complete-damage problem at small strains. Z. Angew. Math. Phys. *60* (2009), 205–236.

[5] *M. Carrive, A. Miranville, A. Piétrus*: The Cahn-Hilliard equation for deformable elastic continua. Adv. Math. Sci. Appl. *10* (2000), 539–569.

[6] *M. Frémond, B. Nedjar*: Damage, gradient of damage and principle of virtual power. Int. J. Solids Struct. *33* (1996), 1083–1103.

[7] *H. Garcke*: On Mathematical Models for Phase Separation in Elastically Stressed Solids. Habilitation thesis. University Bonn, 2000.

[8] *M. E. Gurtin*: Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance. Physica D *92* (1996), 178–192.

[9] *C. Heinemann, C. Kraus*: Existence of weak solutions for Cahn-Hilliard systems coupled with elasticity and damage. Adv. Math. Sci. Appl. *21* (2011), 321–359.

[10] *C. Heinemann, C. Kraus*: Existence results for diffuse interface models describing phase separation and damage. Eur. J. Appl. Math. *24* (2013), 179–211.

[11] *C. Heinemann, C. Kraus*: A degenerating Cahn-Hilliard system coupled with complete damage processes. WIAS preprint no. 1759, (2012), 23 pages.

[12] *D. Knees, R. Rossi, C. Zanini*: A vanishing viscosity approach to a rate-independent damage model. Math. Models Methods Appl. Sci. *23* (2013), 565–616.

[13] *A. Mielke*: Complete-damage evolution based on energies and stresses. Discrete Contin. Dyn. Syst., Ser. S *4* (2011), 423–439.

[14] *A. Mielke, T. Roubíček, J. Zeman*: Complete damage in elastic and viscoelastic media and its energetics. Comput. Methods Appl. Mech. Eng. *199* (2010), 1242–1253.

[15] *F. M. Nor, L. W. Keat, N. Kamsah, M. N. Tamin*: Damage mechanics model for interface fracture process in solder interconnects. 10th Electronics Packaging Technology Conference. 2008, pp. 821–827.

[16] *E. Rocca, R. Rossi*: A degenerating PDE system for phase transitions and damage. Math. Models Methods Appl. Sci., arXiv:1205.3578v1 (2012), 53 pages.

[17] *M. Thomas, A. Mielke*: Damage of nonlinearly elastic materials at small strain-existence and regularity results. ZAMM, Z. Angew. Math. Mech. *90* (2010), 88–112.

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