Acta Universitatis Carolinae. Mathematica et Physica

Michal Staš

The regularity properties on the real line

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 51 (2010), No. Suppl, 73--82

Persistent URL: http://dml.cz/dmlcz/143756

The Regularity Properties on The Real Line

MICHAL STAŠ

Košice

We present a summarization of results on measure and category, mainly of regularity properties as the Lebesgue measurableness, the Baire Property and the Perfect Set Property. We work with the axiomatic set theory **ZF** and any using of the Axiom of Choice or any of its weak form will be emphasized.

It is well known that one cannot prove in **ZF** that there exists a Lebesgue nonmeasurable set or a set not possessing the Baire Property. For any such proof we need additional assumption e.g. the Axiom of Choice **AC**. On the other side, by J. Mycielski [5] if we assume that the Axiom of Determinacy **AD** holds true then any set of reals is Lebesgue measurable and possess the Baire Property. The common proofs of many topological results usually exploit **AC** in spite that one can prove them in **ZF** or in **ZF** with some weak form of the Axiom of Choice.

Main aim of this note is to present relationships between **LM**, **BP**, **PSP** (definitions see below) and additional corresponding assertions. We work with the Zermelo-Fraenkel axiomatic set theory **ZF** and our attention is concentrated to needed assumptions for a proof of the given statements, e.g. a necessity of the Weak Axiom of Choice[1] **wAC** or the Axiom of Dependent Choice **DC**. We shall use common set theoretical terminology and notations, say those of [3] and [4].

Institute of Mathematics, University of P. J. Šafárik, Jesenná 5, 040 01 Košice, Slovakia

The work on this research has been partially supported by the grant 1/0032/09 of Slovak Grant Agency VEGA and by the grant 51/2010/M of Faculty of Science of Pavol Jozef Šafárik University in Košice VVGS PF

2000 Mathematics Subject Classification. Primary 03E05, 42A20; Secondary 03E75, 42A28, 26A99.

Key words and phrases. Baire Property, Lebesgue measurable set, perfect set, consistency

E-mail address: michal.stas@upjs.sk

[1] The Weak Axiom of Choice wAC says that for any countable family of non-empty subsets of a given set of power c there exists a selector, for more see L. Bukovský [2].

The paper is divided into two parts. In the first part we will present the necessary definitions, elementary proofs of assertions from the set theory and some useful facts about non-measurable sets. In the second part we survey known results concerning models of **ZF** showing non-provability of some implications among investigates properties. Finally, all studied properties with their relationships will be summarized in Diagrams.

1. Definitions and Basic properties

A real line is a linearly ordered field $\mathscr{R} = \langle \mathbb{R}, =, \leq, +, \cdot, 0, 1\rangle$ satisfying the Bolzano Principle saying that every non-empty subset of $\mathbb{R}$ bounded from above has a supremum. In **ZF** one can prove that there exists a real line and, up to isomorphism, the real line is unique, for more details see [2].

We shall consider the following statements

wCH there is no set X such that $\aleph_0 < |X| < \mathfrak{c}$;
CH $\aleph_1 = \mathfrak{c}$;
WR the set of reals $\mathbb{R}$ can be well-ordered;
VS there exists a selector for a Vitali decomposition;
FU there exists a free ultrafilter on ω;
Lk a set of cardinality $\mathfrak{k}$ can be linearly ordered;[2]
InC $\aleph_1$ and $\mathfrak{c}$ are incomparable;
In1 $\mathfrak{c} < \mathfrak{k}$;
In2 $\aleph_1 < \aleph_1 + \mathfrak{c} < \aleph_1 + \mathfrak{k}$;
In3 $\mathfrak{c} \neq 2^{\aleph_1}$;
BS there exists a Bernstein set;
LM every subset of reals is Lebesgue measurable;
BP every subset of reals possesses the Baire Property;
PSP every uncountable set of reals contains a perfect subset;[3]
LDe there exists a selector for the Lebesgue decomposition.[4]

A subset $B \subseteq \mathbb{R}$ is called a **Bernstein set** if $|B| = |\mathbb{R} \setminus B| = \mathfrak{c}$ and neither B nor $\mathbb{R} \setminus B$ contains a perfect subset. There exist many different concepts how to construct a Bernstein set with some special properties, but these methods are based on a well-ordering of the real line.

Theorem 1 (F. Bernstein [1]) *If the real line can be well-ordered, then there exists a Bernstein set, i.e.* **WR** $\rightarrow$ **BS**.

A Bernstein set is a classical example of a non-measurable set.

---

[2] Definition of a cardinality $\mathfrak{k}$ is on the following page.
[3] A perfect set is a non-empty closed set without isolated points.
[4] The Lebesgue decomposition is a family $\{\{A \subseteq \omega : \mathrm{ot}(\omega, \pi^{-1}(A)) = \xi\} : \xi < \omega_1\}$, where $\pi : \omega \times \omega \rightarrow \omega$ is a pairing function.

Theorem 2 (F. Bernstein [1]) *A Bernstein set does not possess the Baire Property and is not Lebesgue measurable, i.e.* **BP** $\to$ ¬**BS** *and* **LM** $\to$ ¬**BS**.

In the next, we show that the opposite need not be true. By the definition of the Bernstein set we have **PSP** $\to$ ¬**BS**.

Let $\langle X, +, 0\rangle$ be an Abelian Topological Polish group. A set $V \subseteq X$ is called a **Vitali set** if there exists a countable dense subset D such that

$$(\forall x, y)\,((x, y \in V \wedge x \neq y) \to x - y \notin D),$$

$$(\forall x \in X)(\exists y \in V)\, x - y \in D.$$

Note that, for every $x \in X$ there exists exactly one $y \in V$ such that $x - y \in D$. It is easy to verify that the family $\{\{y \in X : x - y \in D\} : x \in X\}$ is a decomposition of the set X and we call it the **Vitali decomposition**. A selector for the Vitali decomposition is a Vitali set.

Theorem 3 (G. Vitali [12]) *If the real line can be well-ordered, then there exists a Vitali set, i.e.* **WR** $\to$ **VS**.

A Vitali set is an another example of a non-measurable set

Theorem 4 (G. Vitali [12]) *A Vitali set does not possess the Baire Property and is not Lebesgue measurable, i.e.* **BP** $\to$ ¬**VS** *and* **LM** $\to$ ¬**VS**.

Let us consider the family $\mathscr{P}(\omega)$ of all subsets of ω. $\mathscr{P}(\omega)$ is a Boolean algebra and the set

$$\mathrm{Fin} = \{A \subseteq \omega : |A| < \aleph_0\}$$

of all finite subsets of ω is an ideal of algebra $\mathscr{P}(\omega)$. So we can consider the quotient algebra $\mathscr{P}(\omega)/\mathrm{Fin}$ and we denote by $\mathfrak{k}$ its cardinality

$$\mathfrak{k} = |\mathscr{P}(\omega)/\mathrm{Fin}|.$$

We define relation $\ll$ between cardinalities of sets as

$$|A| \ll |B| \equiv (\exists f)\,(f : B \xrightarrow{\text{onto}} A).$$

The relation $\ll$ is reflexive and transitive. Evidently $|A| \leq |B|$ implies $|A| \ll |B|$, and by **AC** we have the opposite implication.

Theorem 5 *The inequalities* $\mathfrak{c} \leq \mathfrak{k}$ *and* $\mathfrak{k} \ll \mathfrak{c}$ *hold true in* **ZF**. *Moreover, if the set* $\mathscr{P}(\omega)$ *can be well-ordered, then* $\mathfrak{k} = \mathfrak{c}$, *i.e.* **In1** $\to$ ¬**WR**.

Proof. Since ${}^{<\omega}\omega$ is countable, we can construct a family $\mathscr{F} \subseteq [{}^{<\omega}\omega]^{\omega}$ of cardinality $\mathfrak{c}$ by setting

$$\mathscr{F} = \{\{s \in {}^{<\omega}\omega : s \subseteq f\} : f \in {}^{\omega}\omega\}.$$

Then $|\{s \in {}^{<\omega}\omega : s \subseteq f_1 \cap f_2\}| < \aleph_0$ for any $f_1, f_2 \in {}^{\omega}\omega$, $f_1 \neq f_2$. The second inequality follows from definitions. □

Note the following: if A, B are sets such that $|A| \leq |B|, |B| \ll |A|$ then A can be well-ordered if and only if B can be well-ordered.

Corollary 6 *A set of cardinality* $\mathfrak{k}$ *can be well-ordered if and only if the set of reals* $\mathbb{R}$ *can be well-ordered.*

Corollary 7 *If a set of cardinality* $\mathfrak{k}$ *cannot be linearly ordered, then* $\aleph_1 < \aleph_1 + \mathfrak{c} < < \aleph_1 + \mathfrak{k}$,[5] *i.e.* $\neg$**Lk** $\to$ **In2**.

Proof. Assume that a set of cardinality $\mathfrak{k}$ cannot be linearly ordered. Since a well-ordered set is linearly ordered, by Corollary 6 we have $\aleph_1 < \aleph_1 + \mathfrak{c}$ and the second inequality follows from a linear ordering of the real line. □

The implication **In2** $\to$ **In1** is trivial by a linear ordering of the real line.

Remark 8 *The following holds true*

(1) *Let* $D = \{x \in {}^{\omega}2 : \{n : x(n) = 1\} \in [\omega]^{<\omega}\}$ *be countable dense set. Then a selector for the Vitali decomposition* ${}^{\omega}2/D = \{\{y \in {}^{\omega}2 : \{n : x(n) \neq y(n)\} \in \in [\omega]^{<\omega}\} : x \in {}^{\omega}2\}$ *is a set of cardinality* $\mathfrak{k}$.
(2) *Let* $\mathbb{D}$ *be the set of all dyadic numbers. Then a selector for the Vitali decomposition* $\mathbb{T}/\mathbb{D} = \{\{y \in \mathbb{T} : x - y \in \mathbb{D}\} : x \in \mathbb{T}\}$,[6] *is a set of cardinality* $\mathfrak{k}$.
(3) *Let* $\mathbb{Q}$ *be the set of all rational numbers. Then a selector for the Vitali decomposition* $\mathbb{T}/\mathbb{Q} = \{\{y \in \mathbb{T} : x - y \in \mathbb{Q}\} : x \in \mathbb{T}\}$ *is the set of cardinality* $\mathfrak{k}$.

Proof.

(1) We can identify the sets $\mathscr{P}(\omega)$ and ${}^{\omega}2$ in a natural way, i.e. a sequence $\{a_n\}_{n=0}^{\infty} \in {}^{\omega}2$ is identified with the set $A = \{n \in \omega : a_n = 1\}$. So there exists a bijection f

$$f : \mathscr{P}(\omega)/\mathrm{Fin} \xrightarrow[\text{onto}]{1-1} {}^{\omega}2/D.$$

(2) There exists a bijection ${}^{\omega}2/D$ onto $\mathbb{T}/\mathbb{D}$.
(3) Since $\mathbb{Q}$, $\mathbb{D}$ are subgroups of $\mathbb{T}$ and $\mathbb{D}$ is subgroup of $\mathbb{Q}$, therefore by the second factor's isomorphism theorem we obtain $\mathbb{T}/\mathbb{Q} \cong (\mathbb{T}/\mathbb{D})/(\mathbb{Q}/\mathbb{D})$. Thus, $\mathfrak{k} = = \aleph_0.|\mathbb{T}/\mathbb{Q}|$. By assumption we have $\mathfrak{k} \leq \aleph_0.|\mathbb{T}/\mathbb{Q}| \leq \mathfrak{c} \leq \mathfrak{k}$.

□

Another essential notion for a construction of a non-measurable set is a tail-set. A set $A \subseteq \mathbb{T}$[7] is called a **tail-set** if the set $\{r \in \mathbb{T} : A + r = A\}$ contains a countable subset dense in $\mathbb{T}$. By the Zero-One Law Theorems saying that

[5] If a set of cardinality $\mathfrak{k}$ cannot be linearly ordered then $\aleph_1$ and $\mathfrak{c}$ can be incomparable (Solovay model [10]) or comparable (Shelah model [8]). Thus, we cannot replace $\aleph_1 < \aleph_1 + \mathfrak{c}$ by $\aleph_1 < \mathfrak{c}$.

[6] The quotient group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is identified with the unit interval $[0, 1]$, in which we have identified the points 0 and 1. The topology of $\mathbb{T}$ is induced by metric $\rho(x, y) = \|x - y\|$, where $\|a\|$ is the distance of the real a to the nearest integer.

[7] In the definition of a tail-set, the torus $\mathbb{T}$ can be replaced by the Cantor space ${}^{\omega}2$ for which the Zero-One Theorems hold true in similar sense.

Theorem 9 *If the set $A \subseteq \mathbb{T}$ is a tail-set, then the outer Lebesgue measure $\lambda^*(A)$ is either* 0 *or* 1.

Theorem 10 *If a tail-set A possesses the Baire Property, then A is either meager or comeager.*

we obtain

Theorem 11 (J. Mycielski [5]) *If* $\mathbf{AC}_2$ *holds true,*[8] *then there exist a Lebesgue non-measurable set of reals and a set which does not possess the Baire Property, i.e.* $\mathbf{LM} \to \neg\mathbf{AC}_2$ *and* $\mathbf{BP} \to \neg\mathbf{AC}_2$.

Proof. Let $p : \mathscr{P}(\omega) \overset{onto}{\to} \mathscr{P}(\omega)/\mathrm{Fin}$ be the quotient mapping. Thus $p(x) = \{y \in \mathscr{P}(\omega) : (x \setminus y) \cup (y \setminus x) \in \mathrm{Fin}\}$. For a set $x \subseteq \omega$ we denote $m(x) = \{p(x), p(\omega \setminus x)\}$. By $\mathbf{AC}_2$ there exists a selector $\mathscr{F}$ for the family

$$\mathscr{M} = \{m(x) : x \subseteq \omega\} \subseteq [\mathscr{P}(\omega) \setminus \mathrm{Fin}]^2.$$

Then the sets

$$\mathscr{A} = \{x \subseteq \omega : p(x) \in \mathscr{F}\}, \mathscr{B} = \{x \subseteq \omega : p(x) \notin \mathscr{F}\}$$

are tail-sets and $\mathscr{A} \cap \mathscr{B} = \emptyset, \mathscr{A} \cup \mathscr{B} = \mathscr{P}(\omega) \approx {}^{\omega}2$. By the Zero-One Law Theorems the sets $\mathscr{A}, \mathscr{B}$ are non-measurable and does not possess the Baire Property. □

Similarly by the same argument we have

Theorem 12 (J. Mycielski [5]) *If a set of cardinality $\mathfrak{k}$ is linearly ordered, then there exist a Lebesgue non-measurable set of reals and a set which does not possess the Baire Property, i.e.* $\mathbf{LM} \to \neg\mathbf{Lk}$ *and* $\mathbf{BP} \to \neg\mathbf{Lk}$.

Proof. If the set $\mathscr{P}(\omega)/\mathrm{Fin}$ can be linearly ordered, then one could define a selector for the family $\mathscr{M}$ of the proof of Theorem 11. □

Notion of a tail-set, having special properties by the Zero-One Law Theorems, can be used to prove assertion for a **free ultrafilter on** ω, i.e. a filter $\mathscr{J} \subseteq \mathscr{P}(\omega)$ does not containing any finite set and for every $A \subseteq \omega$, either $A \in \mathscr{J}$ or $\omega \setminus A \in \mathscr{J}$.

Theorem 13 (W. Sierpiński [9]) *A free ultrafilter on ω is a Lebesgue non-measurable set and does not possess the Baire Property, i.e.* $\mathbf{LM} \to \neg\mathbf{FU}$ *and* $\mathbf{BP} \to \neg\mathbf{FU}$.

Proof. Since we can identify the sets ${}^{\omega}2$ and $\mathscr{P}(\omega)$, we consider $\mathscr{P}(\omega)$ with the topology induced from the Cantor space. Moreover, $x + \mathscr{I} = \mathscr{I}$ for any finite $x \subseteq \omega$ and free ultrafilter $\mathscr{I}$. Therefore a free ultrafilter on ω considered as a subset of ${}^{\omega}2$ is a tail-set and also its complement. They have equal outer measure and are homeomorphic, so the statement follows from the Zero-One Law Theorems. □

[8] The Axiom of Choice $\mathbf{AC}_2$ says that for every family of two elements sets there exists a selector.

Let us remark that by transfinite induction we can construct a non-measurable tail-set which does not possess a Baire Property and is not a free ultrafilter on ω.

Properties of measure and topological properties that are connected with the Baire Property and the first Baire category offer us that there exists some kind of duality between measure and category. A great deal of dual results holds true but yet it is not in general. J. Raisonnier [7] proved in the theory **ZF** + **wAC** that

Theorem 14 (J. Raisonnier) *If* $\aleph_1 \leq \mathfrak{c}$*, then there is a Lebesgue non-measurable set, i.e.* **LM** $\rightarrow$ **InC**.

In the next we will mention that parallel theorem on the Baire Property is not provable in **ZF** + **wAC**.

Theorem 15 *If* **wCH** *holds true, then the following are equivalent:*

WR *the set of reals* $\mathbb{R}$ *can be well-ordered;*

¬**InC** $\aleph_1$ *and* $\mathfrak{c}$ *are comparable, i.e* $\aleph_1 \leq \mathfrak{c}$*;*

LDe *there exists a selector for the Lebesgue decomposition.*

Proof. **LDe** $\rightarrow$ ¬**InC** : A selector of the Lebesgue decomposition is a set of reals of cardinality $\aleph_1$.

WR $\rightarrow$ **LDe** : If the set of reals $\mathbb{R}$ were well-ordered, then we can define a selector for the Lebesgue decomposition.

¬**InC** $\rightarrow$ **WR** : If $\mathfrak{c}$ and $\aleph_1$ were comparable, then **wCH** implies that $\aleph_1 = \mathfrak{c}$. □

By an elementary cardinal arithmetic we already know that if $\aleph_1$ and $\mathfrak{c}$ are incomparable, then $\mathfrak{c} < 2^{\aleph_1}$. Thus, we get **InC** $\rightarrow$ **In3**. Further it is easy to verify that **wCH** $\rightarrow$ **In3**, since $\aleph_1 < 2^{\aleph_1}$.

Theorem 16 *If every uncountable set of reals contains a perfect subset, then there is no set X such that* $\aleph_0 < |X| < \mathfrak{c}$*, i.e.* **PSP** $\rightarrow$ **wCH**.

Proof. If an uncountable set of reals contains a perfect subset, i.e. a subset of cardinality $\mathfrak{c}$, then there exists no uncountable set of cardinality smaller than $\mathfrak{c}$. □

Corollary 17 *If every uncountable set of reals contains a perfect subset, then* $\aleph_1$ *and* $\mathfrak{c}$ *are incomparable, i.e.* **PSP** $\rightarrow$ **InC**.

Proof. We already know that **PSP** $\rightarrow$ ¬**BS** $\rightarrow$ ¬**WR** and according to Theorems 15, 16 we are done. □

2. Consistency and Models

We assume that a reader is acquainted with the axiomatic set theory. We will suppose that **ZF** is consistent although it is impossible to show it. According to Theorem 15 is **wCH** equivalent to **CH** in the theory **ZFC** or **ZF**+**WR**. Thus, in any model of **ZF** + **CH** we have

$$\mathbf{wCH} \nrightarrow \neg\mathbf{WR}, \mathbf{wCH} \nrightarrow \mathbf{InC}, \mathbf{wCH} \nrightarrow \neg\mathbf{LDe},$$

$$\mathbf{In3} \nrightarrow \neg\mathbf{WR}, \mathbf{In3} \nrightarrow \mathbf{InC}, \mathbf{In3} \nrightarrow \neg\mathbf{LDe}.$$

The Axiom of Determinacy[9], denoted **AD**, was proposed as an alternative to the Axiom of Choice by J. Mycielski and H. Steinhaus [6], but it is not possible to prove the consistency of **ZF** + **AD** with respect to **ZF**. Note that the consistency strength of **AD** is indicated as much high in due to results by Solovay and mainly by T. Jech [3]. We remind some consequences of **AD**

Theorem 18 (J. Mycielski, R. Solovay) *If* **AD** *holds true, then*
a) **wAC**, **PSP**, **LM**, **BP** *hold true,*
b) **AC** *fails,*
c) *there exists a surjection of* $\mathscr{P}(\omega)$ *onto* $\mathscr{P}(\omega_1)$*, i.e.* $2^{\aleph_1} \ll \mathfrak{c}$.

By R. Solovay [10] and S. Shelah [8] the following theories are equiconsistent[10]:
(a) **ZFC** + **IC**;[11]
(b) **ZFC** + every Σ^1_3-set of reals is Lebesgue measurable;
(c) **ZF** + **DC** + **LM**.

We already know that **wAC** implies that $\aleph_1$ is a regular cardinal, therefore by Shelah's argument in his Remark (1), Chapter 5 of [8], the theory

$$\mathbf{ZF} + \mathbf{wAC} + \mathbf{LM}$$

is equiconsistent with the previous theories. S. Shelah proved that the consistency of **ZF** implies the consistency of **ZF** + **DC** + **BP**, i.e. the theories
(d) **ZFC**;
(e) **ZF** + **DC** + **BP**

are equiconsistent. Therefore the consistency strength of **ZF** + **wAC** + **LM** is strictly greater than that of **ZF** + **wAC** + **BP**. By Solovay's model the consistency of **ZF** + + **wAC** + **LM** is greater than that of **ZF** + **wAC** + **PSP**. Thus, a natural question arises.

Is consistency of an existence of an inaccessible cardinal necessary for **PSP**?

We give a positive answer to this question

Theorem 19 *If* **PSP** *holds true and* $\aleph_1$ *is a regular cardinal, then* $\aleph_1$ *is an inaccessible cardinal in the constructible universe* **L**.

Proof. Assume that $\aleph_1$ is not inaccessible in **L**. If $\aleph_1$ is a regular cardinal, hence being a successor, $(\mu^+)^{\mathbf{L}} = \aleph_1{}^{\mathbf{V}}$, so $\aleph_1{}^{\mathbf{L}[a]} = \aleph_1$ for some real a, which codes a well-ordering of ω of the ordinal type μ (for more details see [8], Remark 4.1 A). Thus, there exists a set $X \subseteq {}^{\omega}2$ of cardinality $\aleph_1$ and by Corollary 17 then there exists an uncountable set of reals which does not contain a perfect set. □

[9] **AD** states that every two-person games of length ω in which both players choose integers is determined; that is, one of the two players has a winning strategy.

[10] Equiconsistent in the sense that each of the theories (a)-(c) has a model in another one.

[11] **IC** denote statement "there exists a strongly inaccessible cardinal", i.e. a limit regular cardinal κ such that for any $\lambda < \kappa$ we have $2^{\lambda} < \kappa$.

So we obtain that the theory **ZF**+$\aleph_1$ is regular +**PSP** is equiconsistent with the theories (a)–(c). Note that the theories (d)–(e) are equiconsistent with the theory **ZF** + + **wCH**. Thus, the consistency of **ZF** + **wAC** + **PSP** is strictly greater than that of **ZF** + **wAC** + **wCH**.

S. Shelah [8] showed that Theorem 14 on the Baire Property is not provable in the theory **ZF** + **DC**. He constructed a model possessing **BP** in which there exists a set of reals of cardinality $\aleph_1$. Thus, we get **BP** $\nrightarrow$ **InC**. Since **BP** implies trivially ¬**WR**, we obtain ¬**WR** $\nrightarrow$ **InC**, and according to Theorem 15 we get **BP** $\nrightarrow$ **wCH**. By Theorem 16 we know that **PSP** $\rightarrow$ **wCH**, and therefore **BP** $\nrightarrow$ **PSP**. However, according to Theorem 14 we have **BP** $\nrightarrow$ **LM**.

By similar arguments we can easily verify that

$$\begin{array}{llll}
\neg\mathbf{BS} \nrightarrow \mathbf{wCH}, & \neg\mathbf{BS} \nrightarrow \mathbf{InC}, & \neg\mathbf{BS} \nrightarrow \mathbf{LM}, & \neg\mathbf{BS} \nrightarrow \mathbf{PSP},\\
\neg\mathbf{Lk} \nrightarrow \mathbf{wCH}, & \neg\mathbf{Lk} \nrightarrow \mathbf{InC}, & \neg\mathbf{Lk} \nrightarrow \mathbf{LM}, & \neg\mathbf{Lk} \nrightarrow \mathbf{PSP},\\
\neg\mathbf{FU} \nrightarrow \mathbf{wCH}, & \neg\mathbf{FU} \nrightarrow \mathbf{InC}, & \neg\mathbf{FU} \nrightarrow \mathbf{LM}, & \neg\mathbf{FU} \nrightarrow \mathbf{PSP}
\end{array}$$

according to Theorems 2, 12 and 13, respectively.

J. Mycielski [5] has mentioned the following result by E. Specker [11] without any proof. We present

Lemma 20 *If there is no selector for the Lebesgue decomposition and $\aleph_1$ is a regular cardinal, then $\aleph_1$ is an inaccessible cardinal in the constructible universe* **L**.

Proof. Assume that $\aleph_1$ is not inaccessible in **L**. If $\aleph_1$ is a regular cardinal, then ${\aleph_1}^{\mathbf{L}[a]} = \aleph_1$ for some real a. Moreover, for a pairing function $\pi : \omega \times \omega \to \omega$ and any $A \subseteq \omega, A \in \mathbf{L}[a]$

$$\left\langle \omega, \pi^{-1}(A) \right\rangle \text{ is well-ordered in } \mathbf{L}[a] \text{ if and only if } \left\langle \omega, \pi^{-1}(A) \right\rangle \text{ is well-ordered in } \mathbf{V}.$$

If $\pi^{-1}(A)$ is a well-ordering of ω of the ordinal type ξ in $\mathbf{L}[a]$, then there exists $f \in$ $\in \mathbf{L}[a]$ such that $f : \omega \xrightarrow[\text{onto}]{1-1} \xi \in \mathbf{On}$. Since $f \in \mathbf{V}$, $\left\langle \omega, \pi^{-1}(A) \right\rangle$ is also well-ordered in **V**. If $\left\langle \omega, \pi^{-1}(A) \right\rangle$ is not well-ordered in $\mathbf{L}[a]$, then there exists a decreasing chain in $\mathbf{L}[a]$, so in **V** too.

Moreover order type of $\pi^{-1}(A)$ in $\mathbf{L}[a]$ is same as in **V** and a selector of the Lebesgue decomposition in $\mathbf{L}[a]$ is a selector of the Lebesgue decomposition in **V** as well. Since the Axiom of Choice **AC** holds true in $\mathbf{L}[a]$ we are done. □

Since $\aleph_1$ is not inaccessible in **L** in the Shelah's above mentioned model, we obtain **BP** $\nrightarrow$ ¬**LDe** and therewith **LDe** $\nrightarrow$ **WR**.

The relationships between assertions that can be proved in the theory **ZF** are summarized in a Diagram 1.

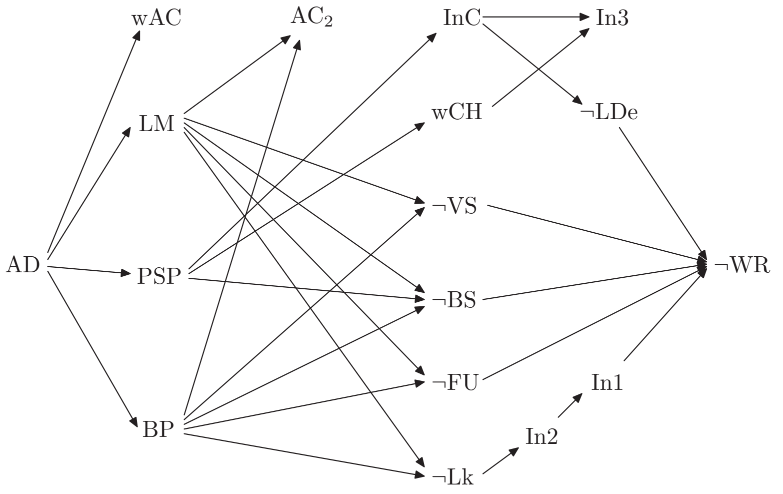

Diagram 1

According to the existence of models of mentioned in Part 2, we have a Diagram 2 in which none of the indicated implications is provable in the theory **ZF**.

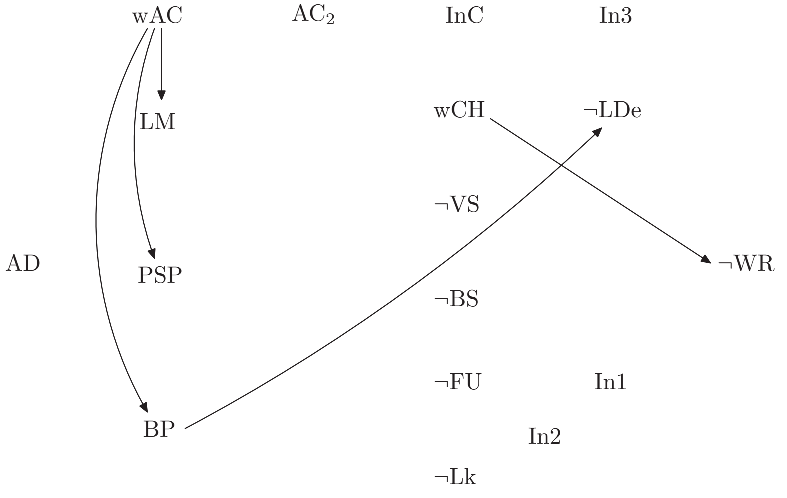

Diagram 2

The arrows that follow from a transitive law are missing in the Diagram 1 and 2 from a typographical reason (they are mentioned above in the text).

Acknowledgment. I would like to thank prof. Lev Bukovský for valuable comments.

References

[1] Bernstein, F.: *Zur Theorie der trigonomischen Reihen*, Sitzungber. Sachs. Akad. Wiss. Leipzig **60** (1908), 325–338.

[2] Bukovský, L.: *The Structure of the Real Line,* (to appear).

[3] Jech, T.: Set Theory, Third edition, Springer-Verlag, Berlin 2006.

[4] Kunen, K. and Vaughan, J. E.: Handbook of Set-Theoretic Topology, North Holland, Amsterdam 1984.

[5] Mycielski, J.: *On the Axiom of Determinateness*, Fund. Math. **53** (1964), 205–224.

[6] Mycielski, J. and Steinhaus, H.: *A mathematical axiom contradicting the axiom of choice*, Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys. **10** (1962), 1–3.

[7] Raisonnier, J.: *A mathematical proof of S. Shelah's theorem on the measure problem and related results*, Israel J. Math. **48** (1984), 48–56.

[8] Shelah, S.: *Can you take Solovay inaccessible away?*, Israel J. Math. **48** (1984), 1–47.

[9] Sierpiński, W.: *Démonstration de l'egalité* $2^{\mathfrak{m}} - \mathfrak{m} = 2^{\mathfrak{m}}$ *pour les nombres cardinaux transfinis*, Fund.Math. **34** (1947), 113–118.

[10] Solovay, R. M.: *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. of Math. **92** (1970), 1–56.

[11] Specker, E.: *Zur Axiomatik der Mengenlehre*, Zeitschrift f. math. Logic and Grundlagen der Math. **3** (1957), 173–210.

[12] Vitali, G.: Sul problema della misura dei gruppi di punti di una retta, Bologna 1905.