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Skeletal maps and I-favorable spaces

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We show that the class of all compact I-favorable spaces and skeletal maps is an adequate pair in the sense of Shchepin. In particular, it is the smallest class which contains all compact metric spaces and it is closed under skeletal maps and limits of σ-complete inverse systems.

1. Introduction

A continuous surjection $f : X \to Y$ is said to be *skeletal* if the closure of $f[U]$ has a non-empty interior for each non-empty open set $U \subseteq X$. Such maps have been considered by many authors for several purposes, e.g. J. Mioduszewski and L. Rudolf [12] applied skeletal maps in the theory of Katetov's H-closed extensions of topological spaces. There are a few possibilities to introduce skeletal maps - not equivalent in general, but equivalent under some restrictions. For example, assume that X is quasi-regular, i.e. each non-empty open set contains the closure of a non-empty open subset, and $f : X \to Y$ is closed and continuous. The map f is skeletal if and only, if images of regularly closed subsets are regularly closed. Indeed, suppose that f is not skeletal. Let $V \subseteq X$ be a non-empty open set such that $\mathrm{Int}_Y \mathrm{cl}_Y f[V] = \emptyset$. Since f is continuous, we infer $\mathrm{Int}_Y f[\mathrm{cl}_X V] = \emptyset$, a contradiction. On the other hand, suppose that $V \subseteq X$ is a non-empty open set such that

$$W = f[\mathrm{cl}_X V] \setminus \mathrm{cl}_Y \mathrm{Int}_Y f[\mathrm{cl}_X V] \neq \emptyset.$$

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Thus, the open set

$$U = V \cap f^{-1}(Y \setminus \mathrm{cl}_Y \mathrm{Int}_Y f[\mathrm{cl}_X V])$$

is non-empty and $f[U] \subseteq W$. Since f is a closed map, the set W is nowhere dense and $\mathrm{Int}_Y f[U] = \emptyset$. Since X is quasi-regular, f is not skeletal.

Let $\mathbb{A}$ and $\mathbb{B}$ be Boolean algebras. If every maximal antichain in $\mathbb{A}$ is maximal in $\mathbb{B}$, then $\mathbb{A}$ is called a regular subalgebra of $\mathbb{B}$. Equivalently, e. g. in [11, p. 218], this property is known as a completely embedding. If $\mathbb{A}$ is a regular subalgebra of $\mathbb{B}$, then by the Stone duality one obtains a skeletal map from the Stone space of $\mathbb{B}$ onto the Stone space of $\mathbb{A}$. Thus, skeletal maps and compact spaces (we are assuming that any compact space is Hausdorff) are parallel to Boolean algebras and its regular subalgebras, compare [5] or [8].

E. V. Shchepin introduced the notion of adequate pair and explained its significance in [13], compare [14]. He found following adequate pairs, see [13]:

- Absolute retracts and soft maps;
- Dugundji spaces and 0-soft maps, see [13, Th. 8];
- κ-metrizable spaces and open maps, compare [14].

The aim of this note is to show that the pair of all compact *I*-favorable spaces and skeletal maps is another adequate pair. In particular, the class of all compact I-favorable spaces is the smallest class which:

- contains all metric compact spaces;
- is closed under skeletal maps;
- is closed under limits of σ-complete inverse systems with skeletal bounding maps.

A directed set Σ is said to be *σ-complete* if any countable chain of its elements has the least upper bound in Σ. An inverse system $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is said to be a *σ-complete*, whenever Σ is σ-complete and

$$X_\sigma = \varprojlim \{X_{\sigma_n}, \pi^{\sigma_{n+1}}_{\sigma_n}\},$$

for any chain $\{\sigma_n : n \in \omega\}$ such that $\sigma = \sup\{\sigma_n : n \in \omega\}$. The readers are referred to [4, p. 135–144] for details on inverse systems.

Suppose $\mathscr{X}$ is a class of compact spaces and Φ is a class of maps. Recall that the pair $(\mathscr{X}, \Phi)$ is an *adequate pair* whenever it fulfills conditions of closure and decomposability, see [13]. The condition of closure assumes: If spaces $X_\alpha \in \mathscr{X}$ and bounding maps $p^\beta_\alpha \in \Phi$, then the limit of a continuous inverse sequence $\{X_\alpha, p^\beta_\alpha, \Sigma\}$ belong to $\mathscr{X}$ and all projections π_β belong to Φ. The condition of decomposability assumes: Each non-metrizable space $X \in \mathscr{X}$ is the limit of a continuous inverse sequence $\{X_\alpha; p^\beta_\alpha; \alpha < \beta < \mathrm{w}(X)\}$, where $\mathrm{w}(X_\alpha) < \mathrm{w}(X)$ and $X_\alpha \in \mathscr{X}$ and $p^\beta_\alpha \in \Phi$.

2. On *I*-favorable spaces

Let X be a topological space equipped with a topology $\mathscr{T}$. Suppose that there exists a function

$$\sigma : \bigcup\{\mathscr{T}^n : n \geq 0\} \to \mathscr{T}$$

such that for each sequence $B_0, B_1, \ldots$ consisting of non-empty elements of $\mathscr{T}$ with $B_0 \subseteq \sigma(\emptyset)$ and $B_{n+1} \subseteq \sigma((B_0, B_1, \ldots, B_n))$, where $n \in \omega$, the union $B_0 \cup B_1 \cup B_2 \cup \ldots$ is dense in X. Then, the space X is called I-*favorable* and the function σ is called a *winning strategy*. In fact, one can take a π-base (or a base) instead of a topology in the definition of a winning strategy. I-favorable spaces were introduced by P. Daniels, K. Kunen and H. Zhou [3]. These spaces were studied in [9] and [10], too. Compact I-favorable spaces have a flavor similar to semi-Cohen Boolean algebras, compare [1], [6] or [8].

Lemma 1 *A skeletal image of I-favorable space is I-favorable.*

Proof. Let $f : X \to Y$ be a skeletal map. Suppose a function $\sigma_X : \bigcup\{\mathscr{T}_X^n : n \geq$ $\geq 0\} \to \mathscr{T}_X$ witnesses that X is I-favorable. Put

$$\sigma_Y(\emptyset) = \operatorname{Int} \operatorname{cl} f[\sigma_X(\emptyset)].$$

If $V_0 \subseteq \sigma_Y(\emptyset)$, then put $B_0 = f^{-1}(V_0) \cap \sigma_X(\emptyset)$. Suppose that non-empty open sets $V_0 \subseteq \sigma_Y(\emptyset)$ and $V_n \subseteq \sigma_Y((V_0, V_1, \ldots, V_{n-1}))$ are chosen and non-empty open sets $B_0, B_1, \ldots, B_{n-1}$ are defined, too. Put

$$B_n = f^{-1}(V_n) \cap \sigma_X(B_0, B_1, \ldots, B_{n-1})$$

and

$$\sigma_Y((V_0, V_1, \ldots, V_n)) = \operatorname{Int} \operatorname{cl} f[\sigma_X(B_0, B_1, \ldots, B_n)].$$

The function σ_Y witnesses that Y is I-favorable. □

The above lemma is parallel to [1, Theorem 4.1]. It follows that the class of compact I-favorable spaces is closed under skeletal maps. From now on, the class of all compact I-favorable spaces will be denoted by $\mathscr{X}$ and the class of all skeletal maps by Φ.

3. On closure

Now, we prove that the class $\mathscr{X}$ is closed under limits of σ-complete inverse systems with bounding maps from Φ, i.e. it fulfills the condition of closure. It is well known that: If $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is an inverse system such that bounding maps π^σ_ϱ are skeletal and projections π_σ are onto, then any projection π_σ is skeletal, compare [2, Lemma 3] or [10, Proposition 8]. So, we shall improve [10, Theorem 13], not assuming that spaces X_σ have countable π-bases.

Theorem 2 *If* $\mathscr{S} = \{X_\alpha, \pi^\alpha_\varrho, \Sigma\}$ *is a* σ*-complete inverse system which consists of compact I-favorable spaces and skeletal bounding maps* π^α_ϱ*, then the inverse limit* $\varprojlim \mathscr{S}$ *is a compact I-favorable space.*

Proof. Let $\mathscr{B}$ be a base for $\varprojlim \mathscr{S}$ which consists of all sets $\pi^{-1}_\tau(V)$, where $\tau \in \Sigma$ and each V is an open subset of X_τ. For each $\alpha \in \Sigma$, let σ_α be a winning strategy in the open-open game played on X_α. Fix $\tau_0 \in \Sigma$ and infinite pairwise disjoint sets A_n such that $\bigcup\{A_n : n \in \omega\} = \omega$ and $n \in A_k$ implies $k \le n$. Put $\sigma_\omega(\emptyset) = \pi^{-1}_{\tau_0}(\sigma_{\tau_0}(\emptyset)) \subseteq \varprojlim \mathscr{S}$. Suppose that $n \in A_k$ and all $\sigma_\omega((B_0, B_1, \ldots, B_{n-1}))$ has been already defined such that $\pi^{-1}_{\tau_m}(V_m) = B_m \subseteq \sigma_\omega((B_0, B_1, \ldots, B_{m-1}))$ for $0 \le m \le n$. Thus indexes $\tau_0 < \tau_1 < \ldots < \tau_n$ are fixed. If n is the least element of A_k, then $\tau_k \le \tau_n$. Put

$$\sigma_\omega((B_0, B_1, \ldots, B_n)) = \pi^{-1}_{\tau_k}(\sigma_{\tau_k}(\emptyset)).$$

If $\{i_0, i_1, \ldots, i_j\} = A_k \cap \{0, 1, \ldots, n\}$ and $\tau_{i_0} < \tau_{i_1} < \ldots < \tau_{i_j} \le \tau_n$, then let

$$D_{i_0} = \operatorname{Int} \pi_{\tau_k}(B_{i_0}), D_{i_1} = \operatorname{Int} \pi_{\tau_k}(B_{i_1}), \ldots, D_{i_j} = \operatorname{Int} \pi_{\tau_k}(B_{i_j}) \subseteq X_{\tau_k}$$

and

$$\sigma_\omega((B_0, B_1, \ldots, B_n)) = \pi^{-1}_{\tau_k}(\sigma_{\tau_k}((D_{i_0}, D_{i_1}, \ldots, D_{i_j}))) \subseteq \varprojlim \mathscr{S}.$$

For other cases, put $\sigma_\omega((B_0, B_1, \ldots, B_n)) \in \mathscr{B}$ arbitrarily. The strategy $\sigma_\omega : \bigcup\{\mathscr{B}^n : n \ge 0\} \to \mathscr{B}$ is just defined.

Verify that σ_ω is a winning strategy. Let $\tau_0 < \tau_1 < \ldots$ and $B_0, B_1, \ldots$ be sequences such that $\pi^{-1}_{\tau_0}(V_0) = B_0 \subseteq \sigma_\omega(\emptyset)$ and

$$\pi^{-1}_{\tau_{n+1}}(V_{n+1}) = B_{n+1} \subseteq \sigma_\omega((B_0, B_1, \ldots, B_n)),$$

where all $\tau_k \in \Sigma$ and each $B_k \in \mathscr{B}$. If $\tau \in \Sigma$ is the least upper bound of $\{\tau_k : k \in \omega\}$, then

$$\pi^{-1}_\tau(\pi_\tau(B_k)) = \pi^{-1}_\tau(\pi_\tau(\pi^{-1}_{\tau_k}(V_k))) = \pi^{-1}_\tau((\pi^\tau_{\tau_k})^{-1}(V_k))) = B_k.$$

Take an arbitrary base set $(\pi^\tau_{\tau_k})^{-1}(W) \subseteq X_\tau$, where W is an open subset in X_{τ_k}. Such sets form a base for X_τ, since the inverse system is σ-complete and [4, 2.5.5. Proposition]. If σ_{τ_k} is a winning strategy on X_{τ_k}, then there exists $j \in A_k$ such that $W \cap \operatorname{Int} \pi_{\tau_k}(B_j) \ne \emptyset$. Hence

$$(\pi^\tau_{\tau_k})^{-1}(W) \cap \pi_\tau(B_j) \ne \emptyset,$$

Indeed, suppose that $(\pi^\tau_{\tau_k})^{-1}(W) \cap \pi_\tau(B_j) = \emptyset$. Then

$$\emptyset = \pi^\tau_{\tau_k}[(\pi^\tau_{\tau_k})^{-1}(W) \cap \pi_\tau(B_j)] = W \cap \pi^\tau_{\tau_k}[\pi_\tau(B_j)] \supseteq W \cap \operatorname{Int} \pi_{\tau_k}(B_j),$$

a contradiction. Thus, the union $\pi_\tau(B_0) \cup \pi_\tau(B_1) \cup \pi_\tau(B_2) \cup \ldots$ is dense in X_τ. But π_τ is a skeletal map, hence

$$\pi^{-1}_\tau(\pi_\tau(B_0) \cup \pi_\tau(B_1) \cup \pi_\tau(B_2) \cup \ldots) = B_0 \cup B_1 \cup B_2 \cup \ldots$$

has to be dense in $\varprojlim \mathscr{S}$. □

We have proved that the pair $(\mathscr{X}, \Phi)$ fulfills the condition of closure. A version of Theorem 2 - for semi-Cohen Boolean algebras, one can find in [1, p. 197].

4. On decomposability

Recall that $\mathscr{X}$ is the class of all limits of σ-complete inverse systems of compact metric space with skeletal bounding maps, see [10]. It remains to prove that the pair $(\mathscr{X}, \Phi)$ fulfills the condition of decomposability. We will use Frink's characterization of completely regular spaces, see [7] or [4, p. 72], likewise in the proof of [10, Theorem 12]. At first, discuss concisely some notions and facts from [10]. Let $\mathscr{P}$ be a family of open subsets of X. If the following holds:

$$\forall_{V \in \mathscr{P}} (x \in V \Leftrightarrow y \in V);$$

then $y \in [x]_{\mathscr{P}}$. Let $X/\mathscr{P}$ be the family of all classes $[x]_{\mathscr{P}}$. The topology on $X/\mathscr{P}$ is generated by sets $\{[x]_{\mathscr{P}} : x \in V\}$, where $V \in \mathscr{P}$. If $\mathscr{P}$ is closed under finite intersection, then the map $q(x) = [x]_{\mathscr{P}}$ is continuous. Let $\mathscr{P}_{seq}$ be the family of all sets W which satisfy the following condition: There exist sequences of sets

$$\{U_n : n \in \omega\} \subseteq \mathscr{P} \text{ and } \{V_n : n \in \omega\} \subseteq \mathscr{P}$$

such that

$$U_k \subseteq (X \setminus V_k) \subseteq U_{k+1},$$

for any $k \in \omega$, and

$$\bigcup \{U_n : n \in \omega\} = W.$$

If a ring $\mathscr{P}$ of open subsets of X is closed under a winning strategy and $\mathscr{P} \subseteq \mathscr{P}_{seq}$, then $X/\mathscr{P}$ is a completely regular space and the map $q : X \to X/\mathscr{P}$ is skeletal.

Let $\{X_\alpha : \alpha \in \Sigma\}$ be a family of topological spaces, where $(\Sigma, <)$ is an upward directed set. Suppose that there are given continuous functions $p^\beta_\alpha : X_\beta \to X_\alpha$ such that $p^\gamma_\alpha = p^\gamma_\beta \circ p^\beta_\alpha$ whenever $\alpha < \beta < \gamma$. Thus $\mathscr{S} = \{X_\alpha; p^\beta_\alpha; \Sigma\}$ is the inverse system. Let X be a topological space. Assume that there exist surjections $\pi_\beta : X \to X_\beta$ such that $\pi_\beta = p^\alpha_\beta \circ \pi_\alpha$ and for each two different points $x, y \in X$ there exists $\alpha \in \Sigma$ with $\pi_\alpha(x) \neq \pi_\alpha(y)$. Then there exists a one-to-one continuous map $f : X \to \varprojlim \mathscr{S}$ onto a dense subspace of $\varprojlim \mathscr{S}$. Indeed, for any $x \in X$, put $f(x) = \{\pi_\alpha(x)\}$. The function f is a required one, compare [4] or [10, Theorem 11]. Additionally, if X is compact, then X is homeomorphic to $\varprojlim \mathscr{S}$.

Theorem 3 *Any non-metrizable $X \in \mathscr{X}$ is homeomorphic to the inverse limit of a continuous sequence*

$$\{X_\alpha; p^\beta_\alpha; \omega \leq \alpha < \beta < \mathrm{w}(X)\},$$

where $\mathrm{w}(X_\alpha) < \mathrm{w}(X)$ *and* $X_\alpha \in \mathscr{X}$ *and* $p^\beta_\alpha \in \Phi$.

Proof. For each cozero set $W \subseteq X$ fix a continuous function $f_W : X \to [0, 1]$ such that $W = f^{-1}_W((0, 1])$. Put

$$\sigma_{2n}(W) = f^{-1}_W((\frac{1}{n}, 1]) \text{ and } \sigma_{2n+1}(W) = f^{-1}_W([0, \frac{1}{n})).$$

Assume that $\sigma = \sigma_0$. Fix a base $\{V_\alpha : \alpha < \mathrm{w}(X)\}$ consisting of cozero sets of X. If $\omega \le \beta < \mathrm{w}(X)$, then let $\mathscr{P}_\beta \supseteq \{V_\alpha : \alpha < \beta\}$ be the smallest family consisting of cozero sets and closed under finite unions and under finite intersections and closed under all functions σ_n. Thus $|\mathscr{P}_\beta| = |\beta|$ and $\mathscr{P}_\gamma \subseteq \mathscr{P}_\beta$, whenever $\omega \le \gamma \le \beta$. Also, we get $\mathscr{P}_\beta = \bigcup\{\mathscr{P}_\gamma : \gamma < \beta\}$ for a limit ordinal β. Put $X_\beta = X/\mathscr{P}_\beta$ and $q_{\mathscr{P}_\beta}(x) = [x]_{\mathscr{P}_\beta}$, thus maps $q^{\mathscr{P}_\beta}_{\mathscr{P}_\gamma} : X_\beta \to X_\gamma$ are skeletal. The inverse limit

$$\varprojlim\{X_\beta; q^{\mathscr{P}_\beta}_{\mathscr{P}_\gamma}; \omega \le \gamma < \beta < \mathrm{w}(X)\}$$

is homeomorphic to X and each inverse limit

$$\varprojlim\{X_\beta; q^{\mathscr{P}_\beta}_{\mathscr{P}_\gamma}; \omega \le \gamma < \beta < \alpha\}$$

is homeomorphic to X_α. All spaces X_α are skeletal images of X, hence they are I-favorable by Lemma 1. We infer that $\mathrm{w}(X_\beta) < \mathrm{w}(X)$, since the family $\{\{[x]_{\mathscr{P}_\beta} : x \in \in V\} : V \in \mathscr{P}_\beta\}$ is a base for X_β, see [10, Lemma 1]. □

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