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QUASITRIVIAL SEMIMODULES V

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In the paper, critical semimodules over congruence-simple semirings are studied.

This part is an immediate continuation of [4] (but see also [1], [2] and [3]). The notation introduced in the preceding parts is used. Here, critical semimodules over congruence-simple semirings are studied. All the results collected here are fairly basic and we will not attribute them to any particular source.

1. Auxiliary results (A)

In this part, let M be an idempotent (left S-)semimodule over a semiring S.

1.1 Lemma. *Let $w \in P(M)$. Put $A_w = \{ x \in M \mid x + w = w \}$ and $B_w = \{ x \mid x + w = x \}$. Then:*
(i) *A_w is a subsemimodule of M and $w \in A_w$.*
(ii) *B_w is an ideal of M (i.e., B_w is a subsemimodule and $B_w + M \subseteq B_w$) and $w \in B$.*
(iii) *$A_w \cap B_w = \{w\}$ and $A_w \cup B_w$ is a subsemimodule of M.*
(iv) *$A_w = M$ iff $w = o_M$.*
(v) *$B_w = M$ iff $w = 0_M$.*

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Proof. All the assertions are easy to check. □

1.2 Lemma. *Let $w \in P(M)$ be such that $w \neq o_M$ and $B_w \subseteq P(M)$. Then $M \setminus A_w$ is an ideal of M.*

Proof. Since $w \neq o_M$, we have $C = M \setminus A_w \neq \emptyset$. If $x \in C$, $y \in M$ then $x+w \neq w$ and $x+x+y+w = x+y+w$. Consequently, $x+y+w \neq w$ and $x+y \in C$. Furthermore, if $r \in S$ then $x+w \in B_w \subseteq P(M)$ (use 1.1(ii)) implies $w \neq x+w = r(x+w) = rx+rw = rx+w$. Thus $rx \in C$. □

1.3 Proposition. *Assume that M is not id-quasitrivial but that every proper subsemimodule of M is id-quasitrivial. Then $P(M) \subseteq \{0_m, o_M\}$.*

Proof. Let $w \in P(M)$. If $A_w = M$ then $w = o_M$ by 1.1(iv). If $B_w = M$ then $w = 0_M$ by 1.1(v). Henceforth, assume that $C = M \setminus A_w \neq \emptyset \neq M \setminus B_w$. We have $w \in A_w \cap B_w$ and, by 1.1(i),(ii), both A_w and B_w are proper subsemimodules of M. According to our assumptions, we get $A_w \cup B_w \subseteq P(M)$. Of course, $w \notin C$ and, by 1.2, C is a proper subsemimodule of M. Again, $C \subseteq P(M)$ and it follows that $M = A_w \cup C \subseteq P(M)$, a contradiction. □

1.4 Proposition. *Assume that M is not id-quasitrivial but that every proper subsemimodule of M is id-quasitrivial. Then just one of the following seven cases takes place:*

(1) *$P(M) = \emptyset$, M is strictly minimal and $M = Sx$ for every $x \in M$;*
(2) *$P(M) = \{0_M\}$, $SM = \{0_M\}$, $|M| = 2$ and M is minimal;*
(3) *$P(M) = \{0_M\}$, M is minimal and $M = Sv$ for every $v \in M \setminus \{0_M\}$;*
(4) *$P(M) = \{o_M\}$, $SM = \{o_M\}$, $|M| = 2$ and M is minimal;*
(5) *$P(M) = \{o_M\}$, M is minimal and $M = Su$ for every $u \in M \setminus \{o_M\}$;*
(6) *$P(M) = \{o_M, 0_M\}$, $M = P(M) \cup \{z\}$, $|M| = 3$ and $SM = P(M) = Sz$ (then M is congruence simple);*
(7) *$P(M) = \{o_M, 0_M\}$, M is almost minimal and $M = Sw$ for every $w \in M \setminus \{o_M, 0_M\}$.*

Proof. If $P(M) = \emptyset$ then M has no proper subsemimodules at all and (1) is clear. Assume, henceforth, that $P(M) \neq \emptyset$.

First, let $P(M) = \{w\}$ be one-element. By 1.3, either $w = o_M$ or $w = 0_M$. Anyway, $Sw = \{w\}$ and $w \in N = \{x \in M \mid Sx = \{w\}\}$. Clearly, N is a subsemimodule of M. Assume, for a moment, that $N \neq M$. Then $N \subseteq P(M)$, and hence $N = \{w\}$. Moreover, $Sy = M$ for every $y \in M \setminus \{w\}$ and either (3) or (5) is true. Now, assume that $N = M$. Then $SM = \{w\}$, and hence any subsemigroup K of the additive semigroup $M(+)$ is a subsemimodule. Since M is idempotent and $w \in \{o_M, 0_M\}$, the set $\{x, w\}$ is a subsemimodule of M for every $x \in M$. However, $\{w\}$ is the only proper subsemimodule of M. Consequently, $|M| = 2$ and either (2) or (4) is true.

Next, let $|P(M) \geq 2$. According to 1.3, we have $P(M) = \{0_M, o_M\}$. If M is almost minimal then (7) is true. On the other hand, if M is not almost minimal then $Sz \subseteq P(M)$ for some $z \in M \setminus P(M)$. The set $L = \{x \in M \mid Sx \subseteq P(M)\}$ is a subsemimodule of M, $z \in L$ and it means that $L = M$. Thus $SM = P(M)$. Besides, the set $P(M) \cup \{z\}$ is a subsemimodule of M and it means that $M = P(M) \cup \{z\}$. If $Sz = \{o_M\}$ then the set $\{o_M, z\}$ is a subsemimodule of M, a contradiction. Similarly, if $Sz = \{0_M\}$. Thus $Sz = P(M)$ and (6) is true. □

1.5 Remark. Let M be as in 1.4. Then M is minimal, provided that any of the subcases (1),...,(5) holds, If (6) is true then M is not almost minimal. If (7) is true then M is almost minimal. In all the cases, M contains at most four different subsemimodules.

2. Auxiliary results (B)

Let M be a semimodule over a semiring S.

2.1 Lemma. *Assume that $o_S \in S$ and $o_M \in M$. Then:*
(i) *If $x \in M$ is such that $o_M \in Sx$ then $o_M = o_S x$.*
(ii) *If $o_M \in Sy$ for every $y \in M$ then $o_S M = \{o_M\}$.*

Proof. (i) We have $o_M = rx$ for some $r \in S$, and hence $o_S x = (r+o_S)x = rx+o_S x = = o_M + o_S x = o_M$.
(ii) This follows immediately from (i). □

2.2 Lemma. *Assume that $o_S \in S$ and o_S is right multiplicatively absorbing in S. If $o_M \in M$ and $o_M = r_1x_1 + \dots r_nx_n$ for some $n \geq 1$, $r_i \in S$ and $x_i \in M$ then $So_M = \{o_M\}$ (i.e., $o_M \in P(M)$).*

Proof. For $r = r_1 + \cdots + r_n$, we have $ro_M = r_1o_M + \cdots + r_no_M = r_1(x_1 + o_M) + + \cdots + r_n(x_n + o_M) = r_1x_1 + \cdots + r_nx_n + r_1o_M + \cdots + r_no_M = o_M + ro_M = o_M$. Now, $o_So_M = (r + o_S)o_M = ro_M + o_So_M = o_M + o_So_M = o_M$ and $so_M = s(o_So_M) = = (so_S)o_M = o_So_M = o_M$ for every $s \in S$. □

2.3 Lemma. *Assume that $o_S \in S$, $o_M \in M$ and $o_M \in Sx$ for every $x \in M$. Then $o_S M = \{o_M\}$. If, moreover, o_S is right multiplicatively absorbing in S then $So_M = = \{o_M\}$.*

Proof. Combine 2.1 and 2.2. □

2.4 Lemma. *Assume that M is faithful and $|rM| = 1$ for some $r \in S$. Then r is left multiplicatively absorbing in S.*

Proof. We have $(rs)x = r(sx) = rx$ for every $s \in S$ and $x \in M$. Since M is faithful, we get $rs = r$. □

2.5 Lemma. *Assume that $o_S \in S$, M is faithful, $o_M \in M$ and $o_M \in Sx$ for every $x \in M$. Then $o_S M = \{o_M\}$ and o_S is left multiplicatively absorbing in S. If o_S is right multiplicatively absorbing then o_S is bi-absorbing and $S o_M = \{o_M\}$.*

Proof. Combine 2.3 and 2.4. □

2.6 Lemma. *Assume that $0_S \in S$ and $0_M \in M$. Then:*
(i) *If $x \in M$ is such that $0_M \in Sx$ then $0_M = 0_S x$.*
(ii) *If $0_M \in Sy$ for every $y \in M$ then $0_S M = \{0_M\}$.*

Proof. (i) We have $0_M = rx$ for some $r \in S$, and hence $0_S x = 0_M + 0_S x = rx + 0_S x = (r + 0_S)x = rx = 0_M$.

(ii) This follows immediately from (i). □

2.7 Lemma. *Assume that $0_S \in S$ and 0_S is right multiplicatively absorbing in S. If $0_M \in M$ and $0_M = r_1x_1 + \cdots + r_nx_n$ for some $n \geq 1$, $r_i \in S$ and $x_i \in M$ then $S0_M = \{0_M\}$ (i.e., $0_M \in P(M)$).*

Proof. For $r = r_1 + \cdots + r_n$, we have $0_M = r_1x_1 + \cdots + r_nx_n = r_1(x_1 + 0_M) + \cdots + (r_n(x_n + 0_M) = r_1x_1 + \cdots + r_nx_n + r_10_M + \cdots + r_n0_M = 0_M + r0_M = r0_M$. Now, $0_M = r0_M = (r + 0_S)0_M = r0_M + 0_S0_M = 0_M + 0_S0_M = 0_S0_M$ and $s0_M = s(0_S0_M) = (s0_S)0_M = 0_S0_M = 0_M$ for every $s \in S$. □

2.8 Lemma. *Assume that $0_S \in S$, $0_M \in M$ and $0_M \in Sx$ for every $x \in M$. Then $0_S M = \{0_M\}$. If, moreover, 0_S is right multiplicatively absorbing in S then $S0_M = \{0_M\}$.*

Proof. Combine 2.6 and 2.7. □

2.9 Lemma. *Assume that $0_S \in S$, M is faithful, $0_M \in M$ and $0_M \in Sx$ for every $x \in M$. Then $0_S M = \{0_M\}$ and 0_S is left multiplicatively absorbing in S. If 0_S is right multiplicatively absorbing in S then 0_S is multiplicatively absorbing and $S0_M = \{0_M\}$.*

Proof. Combine 2.8 and 2.4. □

2.10 Lemma. *Let the semiring S be additively idempotent. Put* $\mathrm{Id}(M) = \{x \in M \mid 2x = x\}$. *Then:*
(i) *If $M = Sv$ for at least one $v \in M$ then* $\mathrm{Id}(M) = M$ *and M is idempotent.*
(ii) *If M is not idempotent then $SM \subseteq$* $\mathrm{Id}(M)$ *and* $\mathrm{Id}(M)$ *is a proper subsemimodule of M.*

Proof. We have $rx = (r + r)x = rx + rx$ for all $r \in S$ and $x \in M$. □

3. Critical semimodules (A)

In this section, let S be a congruence-simple semiring.

3.1 Proposition. *Assume that S is not left quasitrivial. Let M be a critical semimodule. Then M is faithful, congruence-simple and not almost quasitrivial. Moreover, $R(M) = Q(M) = P(M) \neq M$, $M = Sv$ for every $v \in M \setminus P(M)$ and just one of the followng four cases takes place:*

(1) *$P(M) = \emptyset$ and M is strictly minimal;*
(2) *$0_M \in M$, $P(M) = \{0_M\}$ and M is minimal;*
(3) *$o_M \in M$, $P(M) = \{o_M\}$ and M is minimal;*
(4) *$0_M \in M$, $o_M \in M$, $P(M) = \{0_M, o_M\}$ and M is almost minimal.*

Proof. Taking into account [4, 5.5], we have to show that (at least) one of the four cases takes place. We know that M is not (almost) quasitrivial, and hence it is not id-quasitrivial either. On the other hand, every proper subsemimodule of M is quasitrivial (see [4, 5.3]), and hence every proper subsemimodule of M is contained in $Q(M) = P(M)$. It follows that every proper subsemimodule of M is id-quasitrivial. The rest follows from 1.4. □

3.2 Proposition. *Let M be a congruence-simple semimodule that is not quasitrivial. If M is minimal or almost minimal then M is critical.*

Proof. See [4, 5,6,5.8]. □

3.3 Corollary. *Assume that S is not left quasitrivial. A semimodule M is critical if and only if M is congruence-simple, minimal or almost minimal and not quasitrivial.* □

3.4 Proposition. *Let a semimodule M be minimal or almost minimal. If M is not quasitrivial there there is a congruence ϱ of M such that the factrosemimodule M/ϱ is critical.*

Proof. See [4, 5.7,5.9]. □

3.5 Proposition. *The following conditions are equivalent:*

(i) *The semiring S is finite.*
(ii) *There is at least one finite critical semimodule* (see [4, 5.3]).

Proof. (i) implies (ii). If the (left S-)semimodule ${}_SS$ is faithful then the result is clear. Assume, henceforth, that ${}_SS$ is not faithful. Then S is left quasitrivial and, in view of [4, 3.6], we have to distinguish the following four cases:
(a) S is a zero multiplication ring $|S| = p$ is a prime number. Without loss of generality, we can assume that $S(+) = \mathbb{Z}_p(+)$, $\mathbb{Z}_p = \{0, 1, \dots, p-1\}$. Put $M(+) = \mathbb{Z}_{p^2}(+)$,

$\mathbb{Z}_{p^2} = \{0, 1, \dots, p^2 - 1\}$, and define an S-scalar multiplication $*$ on M by $a * x = pax$ for all $0 \le a \le p - 1$ and $0 \le x \le p^2 - 1$. One checks readily that M becomes a left S-semimodule and that M is critical.
(b) $S = \{0, 1\}$, $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1 + 1 = 1$ and $ab = b$ for all $a, b \in S$. By [4, 4.12], there is an idempotent critical semimodule M such that $|M| \le 5$.
(c) $S = \{0, 1\}$, $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1 + 1 = 1$ and $|SS| = 1$ (either $SS = \{0\}$ or $SS = \{1\}$). Again, by [4, 4.12], there is an idempotent critical semimodule M such that $|M| \le 5$.
(d) $S = \{0, 1\}$ and $S + S = \{0\} = SS$. Take $\alpha \notin S$, put $M = S \cup \{\alpha\}$, $M + M = \{0\}$ and extend the interior multiplication of S to an S-scalar multiplication on M by $a\alpha = a$ for all $a \in S$. Then M becomes a critical left S-semimodule.
(ii) implies (i). S imbeds into the full endomorphism semiring $\mathrm{End}(M(+))$. □

4. Critical semimodules (B)

In this section, let S be an additively idempotent congruence-simple semiring. Assume, furthermore, that S is not left quasitrivial. The latter assumption is equivalent to: Either $|S| \ge 3$ or $|S| = 2$, the multiplication of S is not constant and $ab \ne b$ for some $a, b \in S$.

Let M be a critical semimodule (see [4, 5.3,5.4]). Then M is faithful, congruence-simple and not almost quasitrivial. In view of 3.1, just one of the following four cases takes place:

(α) M is strictly minimal (then $R(M) = Q(M) = P(M) = \emptyset$);
(β) $R(M) = Q(M) = P(M) = \{0_M\}$ and M is minimal;
(γ) $R(M) = Q(M) = P(M) = \{o_M\}$ and M is minimal;
(δ) $R(M) = Q(M) = P(M) = \{o_M, 0_M\}$ and M is almost minimal.

In all the four cases, $M = Sv$ for every $v \in M \setminus P(M)$. In particular, the mapping $s \mapsto sv$ is a projective homomorphism of the left S-semimodule ${}_SS$ onto ${}_SM$. Thus M is finite if and only if S is finite. Furthermore, M is idempotent. If $0_S \in S$ then $0_M \in M$. If $o_S \in S$ then $o_M \in M$.

4.1 Lemma. *Let $o_S \in S$. Then:*
(i) $o_M \in M$.
(ii) $o_S x = o_M$ *for every* $x \in (M \setminus P(M)) \cup (M \setminus \{0_M\})$.

Proof. (i) ${}_SM$ is a homomorphic image of ${}_SS$.
(ii) If $x \in M \setminus P(M)$ then $M = Sx$, $o_M = rx$ and $o_M = o_S x$ by 2.1(i). If $x \in M \setminus \{0_M\}$ then either $x \notin P(M)$ and $o_M = o_S x$, or $x \in P(M)$ and then $x = o_M$ and $o_S x = x = = o_M$. □

4.2 Corollary. *If $o_S \in S$ and $0_M \notin P(M)$ then $o_S M = \{o_M\}$.* □

4.3 Lemma. *An element $r \in S$ is right multiplicatively absorbing in S if and only if $rM \subseteq P(M)$.*

Proof. If $rM \subseteq P(M)$ then $srx = rx$ for all $x \in S$ and $x \in M$. Since M is faithful, we get $sr = r$. Conversely, if r is right multiplicatively absorbing then $srx = rx$ for all $s \in S$ and $x \in M$. Thus $rx \in P(M)$. □

4.4 Proposition. *Assume that $o_S \in S$ (e.g., if S is finite then $o_S = \sum s$, $s \in S$). Then:*
(i) *$o_M \in M$ and $o_S(M \setminus \{0_M\}) = \{o_M\}$ (possibly $0_M \notin M$).*
(ii) *If $0_M \notin P(M)$ then $o_S M = \{o\}$.*
(iii) *If M is of type (α) then $o_S M = \{o_M\}$, $S o_M = M$, o_S is left multiplicatively absorbing in S and o_S is not right multiplicatively absorbing in S.*
(iv) *If M is of type (β) then $o_S M = \{o_M, 0_M\}$, $S o_M = M$ and o_S is neither left nor right multiplicatively absorbing in S.*
(v) *If M is of type (γ) then $o_S M = \{o_M\}$, $S o_M = \{o_M\}$ and o_S is bi-absorbing in S.*
(vi) *If M is of type (δ) then $o_S M = \{o_M, 0_M\}$, $S o_M = \{o_M\}$ and o_S is right, but not left multiplicatively absorbing in S.*

Proof. (i) By 4.1, $o_M \in M$. Since $P(M) \subseteq \{o_M, 0_M\}$, we have $o_S(M \setminus \{0_M\}) = \{o_M\}$.
(ii) See 4.2.
(iii) We have $P(M) = \emptyset$, and so $o_S M = \{o_M\}$ by (ii). Since M is strictly minimal, we have $S o_M = M$. Furthermore, $o_S rx = o_M = o_S x$ for all $r \in S$ and $x \in M$. Since M is faithful, we get $o_S r = o_S$, i.e., o_S is left multiplicatively absorbing. By 4.3, o_S is not right multiplicatively absorbing.
(iv) We have $P(M) = \{o_M\}$. Thus $o_S 0_M = 0_M$ and $o_S M = \{o_M, 0_M\}$ follows from (i). Since $o_M \notin P(M)$, we have $S o_M = M$. By 4.3, o_S is not right multiplicatively absorbing. Finally, $r o_M = 0_M$ for some $r \in S$ and $o_S r o_M = o_S 0_M = 0_M \neq o_M = o_S o_M$. Thus $o_S r \neq o_S$ and o_S is not left multiplicatively absorbing.
(v) We have $P(M) = \{o_M\}$, and hence $S o_M = \{o_M\} = o_S M$ (see (ii)). By 4.3, o_S is right multiplicatively absorbing. Finally, $o_S rx = o_M = o_S x$ for all $r \in S$ and $x \in M$. Since M is faithful, we have $o_S r = o_S$ and o_S is left multiplicatively absorbing. Thus o_S is bi-absorbing.
(vi) We have $P(M) = \{o_M, 0_M\}$ and $o_S M = P(M)$ follows from (i). Of course, $S o_M = \{o_M\}$ and o_S is right multiplicatively absorbing by 4.3. On the other hand, if $x \in M \setminus P(M)$ then $0_M = rx$ for some $r \in S$ and we have $o_S rx = o_S 0_M = 0_M \neq o_M = o_S x$. Thus $o_S r \neq o_S$ and o_S is not left multiplicatively absorbing. □

4.5 Corollary. *Let $o_S \in S$. Then any two critical semimodules have the same type.* □

Assume that $o_S \in S$. With respect to 4.5, we can define the type of S to be the type of M. Thus S is of type

(α) if o_S is left, but not right multiplicatively absorbing;

(β) if o_S is neither left nor right multiplicatively absorbing;
(γ) if o_S is bi-absorbing;
(δ) if o_S is right, but not left multiplicatively absorbing.

5. Critical semimodules (C)

The preceding section is immediately continued. We will assume here that $0_S \in S$.

5.1 Lemma. (i) $0_M \in M$.
(ii) $0_S x = 0_M$ *for every* $x \in (m \setminus P(M)) \cup (M \setminus \{o_M\}$.

Proof. (i) ${}_S M$ is a homomorphic image of ${}_S S$.
(ii) If $x \in M \setminus P(M)$ then $M = Sx$, $0_M = rx$ for some $r \in S$ and $0_M = 0_S x$ by 2.6(i). If $x \in M \setminus \{o_M\}$ then either $x \notin P(M)$ and $0_M = 0_S x$ or $x \in P(M)$ and then $x = 0_M$ and $0_S x = x = 0_M$. □

5.2 Corollary. *If* $o_M \notin P(M)$ *then* $0_S M = \{0_M\}$. □

5.3 Proposition. (i) $0_M \in M$ *and* $0_S(M \setminus \{o_M\}) = \{0_M\}$ *(possibly* $o_M \notin M$*).*
(ii) *If* $o_M \notin P(M)$ *then* $0_S M = \{0_M\}$.
(iii) *If* M *is of type* (α) *then* $0_S M = \{0_M\}$, $S0_M = M$ *and* 0_S *is left multiplicatively absorbing, but not right multiplicatively absorbing.*
(iv) *If* M *is of type* (β) *then* $0_S M = \{0_M\}$, $S0_M = \{0_M\}$ *and* 0_S *is multiplicatively absorbing.*
(v) *If* M *is of type* (γ) *then* $0_S M = \{o_M, 0_M\}$, $S0_M = M$ *and* 0_S *is neither left nor right multiplicatively absorbing in* S.
(vi) *If* M *is of type* (γ) *then* $0_S M = \{o_M, 0_M\}$, $S0_M = \}0_M\}$ *and* 0_S *is right, but not left multiplicatively absorbing.*

Proof. (i) By 5.1, $0_M \in M$. Since $P(M) \subseteq \{o_M, 0_M\}$, we have $0_S(M \setminus \{o_M\}) = \{0_M\}$.
(ii) See 5.2.
(iii) We have $P(M) = \emptyset$, and so $0_S M = \{0_M\}$ by (ii). Since M is strictly minimal, we have $S0_M = M$. Furthermore, $0_S rx = 0_M = 0_S x$ for all $r \in S$ and $x \in M$. Since M is faithful, we get $0_S r = 0_S$, i.e., 0_S is left multiplicatively absorbing. By 4.3, 0_S is not right multiplicatively absorbing.
(iv) We have $P(M) = \{0_M\}$. Thus $0_S M = \{0_M\}$ follows from (ii). Of course, $S0_M = \{0_M\}$. By 4.3, 0_S is right multiplicatively absorbing. Finally, $0_S rx = 0_M = 0_S x$ for all $r \in S$ and $x \in M$. Since M is faithful, we have $0_S r = 0_S$, i.e., 0_S is left multiplicatively absorbing.
(v) We have $P(M) = \{o_M\}$, and hence $0_S M = \{o_M, 0_M\}$ and $S0_M = M$ (use (i)). By 4.3, 0_S is not right multiplicatively absorbing. Finally, $r0_M = o_M$ for some $r \in S$ and $0_S r0_M = 0_S o_M = o_M \neq 0_M = 0_S 0_M$. Thus $0_S r \neq 0_S$ and 0_S is not left multiplicatively absorbing.

(vi) We have $P(M) = \{o_M, 0_M\}$ and $0_S M = P(M)$ follows from (i). Of course, $S0_M = \{0_M\}$ and 0_S is right multiplicatively absorbing by 4.3. On the other hand, if $x \in M \setminus P(M)$ then $o_M = rx$ for some $r \in S$ and we have $0_S rx = 0_S o_S = o_M \neq 0_M = 0_S x$. Thus $0_S r \neq 0_S$ and 0_S is not left multiplicatively absorbing. □

5.4 Corollary. *Any two critical semimodules have the same type.* □

With respect to 5.4, we can define the type of S to be the type of M. Thus S is of type

(α) if 0_S is left but not right multiplicatively absorbing;
(β) if 0_S is multiplicatively absorbing;
(γ) if 0_S is neither left nor right multiplicatively absorbing;
(δ) if 0_S is right, but not left multiplicatively absorbing.

5.5 Lemma. *Assume that M is of type (γ) or (δ). Then $o_M \in P(M)$ and o_M is irreducible.*

Proof. Left $o_M = u + v$ for some $u, v \in M$. Then $o_M = 0_S o_M = 0_S(u + v) = 0_S u + 0_S v$, and hence either $0_S u \neq 0_M$ pr $0_S v \neq 0_M$. If $0_S u \neq 0_M$ then $u = o_M$ by 5.1(ii). Similarly, if $0_S v \neq 0_M$ then $v = o_M$. Thus o_M is irreducible. □

6. Critical semimodules (D)

Let S be an additively idempotent congruence-simple semiring that is not left quasitrivial. Let M be a critical semimodule.

6.1 Lemma. *Let $0_S \notin S$, $o_M \in M$ and $0_M \in M$. Then:*
(i) *M is of type (γ) or (δ).*
(ii) *If $o_S \in S$ then o_S is right multiplicatively absorbing.*

Proof. (i) We have to show that $o_M \in P(M)$. Suppose, on the contrary, that $o_M \notin P(M)$. Then $So_M = M$ and there is $r \in S$ such that $ro_M = 0_M$ and $0_M = ro_M = r(x + o_M) = rx + ro_M = rx + 0_M = rx$ for every $x \in M$. That is $rM = \{0_M\}$. Now, $(r + s)x = rx + sx$, $0_M + sx = sx$ for all $s \in S$ and $x \in M$. Since M is faithful, we get $r + s = s$ and $r = 0_S$, a contradiction.
(ii) Combine (i) and 4.4(v),(vi). □

6.2 Lemma. *Let $0_S \notin S$, $o_S \notin S$, $o_M \in M$ and $0_M \in M$. Then M is of type (δ).*

Proof. By 6.1, M is of type (γ) or (δ). Proceeding by contradiction, assume that M is of type (γ). Then $P(M) = \{o_M\}$, $S0_M = M$, $o_M = r0_M$ for some $r \in S$ and $rx = r(x + 0_M) = rx + r0_M = rx + o_M = o_M$ for every $x \in M$. That is, $rM = \{o_M\}$. Now, $(r + s)x = rx + sx = o_M + sx = o_M = rx$ for all $s \in S$ and $x \in M$. Since M is faithful, we get $r + s = r$ and $r = o_S$, a contradiction. □

6.3 Lemma. *Let $o_S \notin S$, $o_M \in M$ and $0_M \in M$. Then:*
(i) *M is of type (β) or (δ).*
(ii) *If $0_S \in S$ then 0_S is right multiplicatively absorbing.*

Proof. (i) We have to show that $0_M \in P(M)$. Suppose, on the contrary, that $0_M \notin$ $\notin P(M)$. Then $S0_M = M$, $r0_M = o_M$ for some $r \in S$, $rx = r(x + 0_M) = rx + r0_M =$ $= rx + o_M = o_M$ for every $x \in M$, and hence $rM = \{o_M\}$. Now, $(r + s)x = rx + sx =$ $= o_m + sx = o_M = rx$ for all $s \in S$ and $x \in M$. Since M is faithful, we get $r + s = r$ and $r = o_S$, a contradiction.
(ii) Combine (i) and 5.3(iv),(vi). □

6.4 Lemma. *Let $o_S \in S$ be not right multiplicatively absorbing. Then:*
(i) *$0_S \in S$ if and only if $0_M \in M$.*
(ii) *If $0_S \in S$ then 0_S is left multiplicatively absorbing.*
(iii) *M is of type (α) or (β) and $o_M \in M$.*

Proof. Since $o_S \in S$ is not right multiplicatively absorbing, the semimodule M is of type (α) or (β) (use 4.4). Of course, $o_M \in M$. If $0_S \in S$ then $0_M \in M$ and 0_S left multiplicatively absorbing by 5.3. Finally, if $0_M \in M$ then $0_S \in S$ by 6.1. □

6.5 Lemma. *Let $o_S \in S$ be neither left nor right multiplicatively absorbing. Then:*
(i) *$0_S \in S$ and 0_S is multiplicatively absorbing.*
(ii) *M is of type (β) and $o_M \in M$, $0_M \in M$.*

Proof. Since $o_S \in S$ is neither left nor right multiplicatively absorbing, the semimodule M is of type (β) by 4.4. Of course, $o_M \in M$, $0_M \in M$ and we have $P(M) = \{0_M\}$. By 6.4, $0_S \in S$ and 0_S is multiplicatively absorbing by 5.3(iv). □

6.6 Lemma. *Let $0_S \in S$ be not right multiplicatively absorbing. Then:*
(i) *M is of type $\alpha)$ or (γ) and $0_M \in M$.*
(ii) *$o_S \in S$ if and only if $o_M \in M$.*
(iii) *If $o_S \in S$ then o_S is left multiplicatively absorbing.*

Proof. By 5.3, M is of type (α) or (γ) and $0_M \in M$. If $o_S \in S$ then $o_M \in M$ and o_S is left multiplicatively absorbing by 4.4 Conversely, if $o_M \in M$ then $o_S \in S$ by 6.3. □

6.7 Lemma. *Let $0_S \in S$ be neither left nor right multiplicatively absorbing. Then:*
(i) *M is of type (γ) and $o_M \in M$, $0_M \in M$.*
(ii) *$o_S \in S$ is bi-absorbing.*

Proof. By 5.3, M is of type (γ) and we have $0_M \in M$. Of course, $P(M) = \{o_M\}$, and hence $o_S \in S$ by 6.6. By 4.4(v), o_S is bi-absorbing. □

6.8 Proposition. *Assume that the semimodule M is of type* (α). *Then:*
(i) *If $o_S \in S$ then o_S is left multiplicatively absorbing, o_S is not right multiplicatively absorbing, $o_M \in M$, $S\,o_M = M$ and $o_S M = \{o_M\}$.*
(ii) *If $0_S \in S$ then 0_S is left multiplicatively absorbing, 0_S is not right multiplicatively absorbing, $0_M \in M$, $0_M = M$ and $0_S M = \{0_M\}$.*
(iii) *If $o_S \in S$ and $0_M \in M$ then $0_S \in S$.*
(iv) *If $0_S \in S$ and $o_M \in M$ then $o_S \in S$.*
(v) *Assume that either $o_S \in S$ or $0_S \in S$. If N is a critical semimodule then N is of type* (α).

Proof. See 4.4(i),(iii), 5.3(i),(iii), 6.4(i), 6.6(ii), 4.5 and 5.4. □

6.9 Proposition. *Assume that the semimodule M is of type* (β). *Then:*
(i) *If $o_S \in S$ then o_S is neither left nor right multiplicatively absorbing, $o_M \in M$, $S\,o_M = M$, $0_M \in M$ and $o_S M = \{o_M, 0_M\}$.*
(ii) *If $0_S \in S$ then 0_S is multiplicatively absorbing, $0_M \in M$ and $S\,0_M = \{0_M\} = 0_S M$.*
(iii) *If $o_S \in S$ then $0_S \in S$.*
(iv) *If $o_M \in M$ then $0_M \in M$.*
(v) *Assume that either $o_S \in S$ or $0_S \in S$. If N is a critical semimodule then N is of type* (β).

Proof. See 4.4(i),(iv), 5.3(iv), 6.1, 4.5 and 5.4. □

6.10 Proposition. *Assume that the semimodule M is of type* (γ). *Then:*
(i) *If $o_S \in S$ then o_S is bi-absorbing and $S\,o_M = \{o_M\} = o_S M$.*
(ii) *If $0_S \in S$ then 0_S is neither left nor right multiplicatively absorbing in S, $0_M \in M$, $S\,0_M = M$ and $0_S M = \{o_M, 0_M\}$.*
(iii) If $0_S \in S$ then $o_S \in S$.
(iv) If $0_M \in M$ then $o_S \in S$.
(v)(Assume that either $o_S \in S$ or $0_S \in S$. If N is a critical semimodule then N is of type (γ).

Proof. See 4.4(v), 5.3(i),(v), 6.3, 4.5 and 5.4. □

6.11 Proposition. *Assume that the semimodule M is of type* (δ). *Then:*
(i) *If $o_S \in S$ then o_S is right multiplicatively absorbing, o_S is not left multiplicatively absorbing and $o_S M = \{o_M, 0_M\}$.*
(ii) *If $0_S \in S$ then 0_S is right, but not left multiplicatively absorbing and $0_S M = = \{o_M, 0_M\}$.*
(iii) *Assume that either $o_S \in S$ or $0_S \in S$. If N is a critical semimodule then N is of type* (δ).

Proof. See 4.4(vi), 5.3(vi), 4.5 and 5.4. □

6.12 Remark. Put $S^{\mathrm{op}} = T$ $(= T(+,*), a * b = ba)$.
(i) Let M be a right S-semimodule. Put $a \circ x = xa$ for all $a \in S$ and $x \in M$. Then $a\circ(x+y) = (x+y)a = xa+ya = a\circ x+a\circ y$, $(a+b)\circ x = x(a+b) = xa+xb = a\circ x+b\circ x$ and $a \circ (b \circ x) = a \circ xb = (xb)a = x(ba) = x(a*) = (a * b) \circ x$. Thus $M(*, \circ)$ is a left T-semimodule.
(ii) Let $M(+, \circ)$ be a left T-semimodule. Put $xa = a \circ x$ for all $a \in S$. Again, $(xa)b = b \circ (a \circ x) = (b * a) \circ x = x(b * a) = x(ab)$. It means that M becomes a right S-semimodule.
(iii) Combining (i) and (ii), we get a biunique correspondence between right S-semimodules and left T-semimodules.
(iv) $T^{\mathrm{op}} = S$, and hence there is a biunique correspondence between right T-semimodules and left S-semimodules as well.
(iv) Assume that S is neither left nor right quasitrivial. Let N be a critical right S-semimodule. Denote by $\overline{N}$ the corresponding left T-semimodule. A subset K of N is a subsemimodule of N if and only if K is a subsemimodule of $\overline{N}$. Clearly, $\overline{N}$ is a faithful T-semimodule and $P(\overline{N}) = P(N)$. Consequently, $\overline{N}$ is critical and of the same type as N.

6.13 Proposition. *Assume that S is neither left nor right quasitrivial (e.g., if $|S| \geq 3$) and that either $o_S \in S$ or $0_S \in S$. Let M be a critical left S-semimodule and N be a critical right S-semimodule. Then:*
(i) *M is of type (α) iff N is of type (δ).*
(ii) *M is of type (β) iff N is of type (β).*
(iii) *M is of type (γ) iff N is of type (γ).*

Proof. (i) First, let M be of type (α). If $o_S \in S$ ($0_S \in S$, resp.) then o_S (0_S, resp.) is left, but not right multiplicatively absorbing in S (see 6.8)i),(ii)). Put $T = S^{\mathrm{op}}$. If $o_S \in S$ ($0_S \in S$, resp.) then $o_T \in T$ ($0_T \in T$, resp.) and if o_S (0_S, resp.) is left multiplicatively absorbing in S then o_T (0_T, resp.) is right multiplicatively absorbing in T. By 6.11, the left T-semimodule $\overline{N}$ is of type (δ). By 6.12(v), the right S-semimodule N is of type (δ) as well.
(ii) and (iii). Combine 6.9, 6.10 and 6.12. □

7. Critical semimodules (E)

Let S be a finite additively idempotent and congruence-simple semiring such that $|S| \geq 3$. Then $o_S = \sum S \in S$ and S is neither left nor right quasitrivial.

7.1 Remark. Let M be a critical left S-semimodule and N be a critical right S-semimodule. The type of M (N, resp.) is uniquely determined and M is of type (α) ((β), (γ), (δ), resp.) if and only if N is of type (δ) ($(\beta$, (γ), (α), resp.). We will say that S is of type (I) ((II), (III), (IV), resp.). The semiring S is of this type if and only if the opposite semiring S^{op} is of the type (IV) ((II), (III), (I), resp.). We have $o_M = \sum M \in M$ and $o_N = \sum N \in N$.

(i) If S is of type (I) then $o_S M = \{o_M\}$ and $No_S = \{o_N, 0_N\} = P(N)$. If S is of type (II) then $o_S M = \{o_M, 0_M\}$ and $No_S = \{o_N, 0_N\}$. If S is of type (III) then $o_S M = \{o_M\} = S o_M$ and $No_S = \{o_N\} = o_N S$. If S is of type (IV) then $o_S M = \{o_M, 0_M\}$ and $o_S = \{o_N\}$.
(ii) S is of type (I) ((IV), resp.) if and only if o_S is left (right, resp.) and not right (left, resp.) multiplicatively absorbing. S is of type (I) if and only if o_S is neither left nor right multiplicatively absorbing. S is of type (III) if and only if o_S is bi-absorbing.
(iii) If S is of type (II) then $0_S \in S$. If S is of type (I) ((IV), resp.) then $0_M \in M$ ($0_N \in N$, resp.) implies $0_S \in S$.
(iv) Assume that $0_S \in S$. Then $0_M \in M$ and $0_N \in N$. S is of type (I) if and only if 0_S is left but not right multiplicatively absorbing. Then $0_S M = \{0_M\}$, $S 0_M = M$, $0_N S = \{0_N\}$ and $N 0_S = \{o_N, 0_N\}$. S is of type (II) if and only if 0_S is multiplicatively absorbing. Then $S 0_M = \{0_M\} = 0_S M$ and $0_N S = \{0_N\} = N 0_S$. S is of type (III) if and only if 0_S is neither left nor right multiplicatively absorbing. Then $S 0_M = M$, $0_S M = \{o_M, 0_M\}$, $0_N S = N$ and $N 0_S = \{o_N, 0_N\}$. Finally, S is of type (IV) if and only if 0_S is right but not left multiplicatively absorbing. Then $S 0_M = \{0_M\}$, $0_S M = \{o_M, 0_M\} = P(M)$, $0_N S = N$ and $N 0_S = \{0_N\}$.

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