Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 51 (2010), No. 2, 9--15

Persistent URL: http://dml.cz/dmlcz/143653

An Easily Computable Invariant of Trilinear Alternating Forms

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Received March 22, 2010

We show that number of systems of k paralel triples in the definition of a trilinear alternating form with respect to a basis B is modulo 2 an invariant of the form in the case the underlying vector space of dimension $3k$ is over the two-element field. Values of this invariant can thus be computed only from the values on the basis vectors. If its value is equal to 1, the form is nondegenerate (regular). Moreover, it is possible to extend this invariant to the case $\dim V = 3k + 1$.

1. Introduction

Let $f : V^3 \to F$ be a trilinear form on a vector space V over a field F, $\dim V = n < < \infty$. The form f is called *alternating* if $f(u, v, w) = 0$ whenever two of the input vectors are equal. Two forms f and g on V are *equivalent* if there exists an automorphism of V satisfying $f(u, v, w) = g(\phi(u), \phi(v), \phi(w))$ for all $u, v, w \in V$. Classification of classes of this equivalence seems to be a very difficult problem (unlike in the bilinear case) even for small dimensions of V and not much has been done in this respect. This classification was done for the case $n \leq 7$ in [1] for a large family of fields including all finite fields and Gurevitch [5], D. Djokovic [6] and L. Noui [7] solved the case $n = 8$ for $F = \mathbf{C}$, $F = \mathbf{R}$ and F algebraically closed field of arbitrary characteristics, respectively. There are also results concerning invariants of the forms on dimension 6 ([2]) and dimension 9 ([4]).

In this paper, the case of forms over the two-element field is studied, because the motivation for this research comes from the theory of doubly even binary codes,

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2000 Mathematics Subject Classification. 15A69

Key words and phrases. Trilinear alternating form, invariant, two-element field, parallel lines

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of which trilinear alternating forms over the two-element field appears as important invariants. There is also a connection to the so called code loops, see [3].

One of the possible approaches to the classification of trilinear forms is to use invariants. In this paper we introduce an invariant in the case the dimension of the underlying vector space is divisible by 3 and from this invariant another one for forms on dimension $3k + 1$ is derived. Both these invariants are quite easy to calculate, especially if the number of triples of basis vectors satisfying $f(b_i, b_j, b_k)$ is low. On the other hand, there are only two possible values of these invariants. But in the case $3 \mid dimV$ the form is nondegenerate whenever the value of the invariant is 1. This seems to be a very effective way to prove the nondegenerateness of the form.

Tables in the paper contain representatives of all classes of equivalence for dimensions 6 and 7 together with the values of the invariants. These tables are based on the classification in [1].

2. Invariant on dimension $3k$

Throughout this paper, let V be an n-dimensional vector space over the field $\mathbf{GF}(2)$ and f be a trilinear alternating form. Then f satisfies the equality

$$f(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}) = \operatorname{sgn}(\sigma) f(v_1, v_2, v_3)$$

for every permutation $\sigma \in S_3$, which in the case of characteristics two collapses into symmetry.

We shall write $f[u]$ for the bilinear form $f(u, -, -)$ and similarly $f[u, v]$ shall denote the linear form $f(u, v, -)$.

Let f be a trilinear form on V. The set

$$\{v \in V; f[v] = 0\}$$

is called the *radical* of f and will be denoted by $\mathrm{Rad}f$. If $\mathrm{Rad}f$ is trivial (contains only the zero vector), then f is called *nondegenerate*.

Let $B = \{b_1, \ldots, b_n\}$ be a basis of an n-dimensional vector space V over a field F and let $B^* = \{b_1^*, \ldots, b_n^*\}$ be its dual basis (defined as usually by $b_i^*(b_j) = \delta_{ij}$). Given B and B^* as above, a trilinear alternating form f can be expressed as

$$f_B = \sum_{1 \le i < j < k \le n} f_{ijk} b_i^* \wedge b_j^* \wedge b_k^*.$$

Denote by Δ the set

$$\Delta = \{(i, j, k) \mid 1 \le i < j < k \le n, f_{ijk} \neq 0\}.$$

Then we can write $f_B = \sum_\Delta f_{ijk} b_i b_j b_k$ or even

$$f_B = \sum_\Delta f_{ijk} \underline{ijk}.$$

Notice that $f(b_i, b_j, b_k) = f_{ijk}$ holds for all $1 \le i, j, k \le n$.

If V is a vector space over the two element field $F = \mathbf{GF}(2)$ then to give a form f means to point out triples $\{i, j, k\}$ satisfying $f(b_i, b_j, b_k) = 1$, i.e., to give the set Δ.

Using similar notation, we state well known classification of bilinear alternating forms:

Proposition 2.1 *Let f be a bilinear alternating form on a vector space V of dimension n. Then there exists a basis $B = \{b_1, \ldots, b_n\}$ and $k \leq n$ such that*

$$f_B = \underline{b_1 b_2} + \underline{b_3 b_4} + \cdots + \underline{b_{k-1} b_k}.$$

For the rest of this section assume that $n = 3k$, $k \in \mathbf{N}$ and that a trilinear form f is given with respect to a basis $B = \{b_1, \ldots, b_n\}$. Denote by $P = P_B$ the set of all partitions $\{\alpha_1, \ldots, \alpha_k\}$ of B, where every α_r, $1 \leq r \leq k$, contains exactly three elements. Call such a partition a 3-*partition*. If $\alpha_r = \{b_i, b_k, b_l\}$, $1 \leq i, k, l \leq n$, we shall denote by α_r^{ij} the set $\{b_j, b_k, b_l\}$ (the element b_i was replaced with b_j, $1 \leq j \leq$ $\leq n$). The expression $f(\alpha_r)$ is a shorthand for $f(b_i, b_k, b_l)$ (since f is alternating, this notation is correct).

Now for f_B define the value $I_n(f_B) \in F$ as follows:

$$I_n(f_B) = \sum_{\{\alpha_1,\ldots,\alpha_k\} \in P} \left(\prod_{p=1}^{k} f(\alpha_p)\right). \tag{1}$$

Notice that the sum runs over all possible 3-partitions of B. Since the computation is done in $\mathbf{GF}(2)$, there are only two possible results. Every member of the sum (1) is a product and thus is equal to 1 only if all $f(\alpha_i)$ in a given 3-partition are equal to 1, i.e., it is a partition of B into "paralel lines" with three elements, where α being a line means that $f(\alpha) = 1$.

Theorem 2.2 *Let f be a trilinear alternating form on a vector space of dimension $3k$ over the two-element field and let B be a basis of V. Then $I_n(f_B)$ defined in (1) does not depend on the basis B.*

Proof. It is clear that a permutation of vectors of the basis does not change the value of I_n. If B and C are two bases of V we can find a sequence of bases $B_1, \ldots, B_{2s}$ such that $B = B_1, C = B_{2s}$, B_{2t} is obtained by permuting vectors of B_{2t-1}, and if $B_{2t} = \{b_1, \ldots, b_n\}$ then $B_{2t+1} = \{b_1 + b_2, b_2, \ldots, b_n\}$. Thus it is enough to prove that the value $I_n(f_{B_{2t+1}})$ is equal to $I_n(f_{B_{2t}})$.

We shall write P_t instead of P_{B_t}. Without loss of generality we can assume that $b_1 + b_2 \in \alpha_1$ for every 3-partition $\{\alpha_1, \ldots, \alpha_k\}$ of B_{2t+1}. Then we can compute:

$$I_n(f_{B_{2t+1}}) = \sum_{\{\alpha_1,\ldots,\alpha_k\} \in P_{2t+1}} \left[\prod_{p=1}^{k} f(\alpha_p)\right] =$$

$$= \sum_{\{\alpha_1,\ldots,\alpha_k\} \in P_{2t}} \left[f(\alpha_1) + f(\alpha_1^{12})\right] \prod_{p=2}^{k} f(\alpha_p)) =$$

$$= \sum_{\{\alpha_1,\ldots,\alpha_k\}\in P_{2t}} \left[f(\alpha_1)\prod_{p=2}^{k} f(\alpha_p)\right] + \sum_{\{\alpha_1,\ldots,\alpha_k\}\in P_{2t}} \left[f(\alpha_1^{12})\prod_{p=2}^{k} f(\alpha_p)\right] =$$

$$= I_n(f_{B_{2t}}) + \sum_{\{\alpha_1,\ldots,\alpha_k\}\in P_{2t}} \left[f(\alpha_1^{12})\prod_{p=2}^{k} f(\alpha_p)\right].$$

We have to prove that the second term is equal to zero. Use P instead of P_{2t}. If $b_2 \in \alpha_1$ then $f(\alpha_1^{12}) = 0$ (the form is alternating) and thus we can suppose that the summation runs only over the set Q of partitions satisfying $b_2 \notin \alpha_1$. Let $A = \{\alpha_1, \ldots, \alpha_k\}$ be a partition and suppose that $b_2 \in \alpha_2$. Define partition $A' = \{\alpha_1^{12}, \alpha_2^{21}, \alpha_3, \ldots, \alpha_k\}$. Since $(A')' = A$ we can choose sets Q' and Q'' such that:

$$Q' \cup Q'' = Q, Q' \cap Q'' = \emptyset \text{ and } A \in Q'' \Rightarrow A' \in Q'.$$

Now, we can finish the proof:

$$\sum_{\{\alpha_1,\ldots,\alpha_k\}\in P} \left[f(\alpha_1^{12})\prod_{p=2}^{k} f(\alpha_p)\right] =$$

$$= \sum_{A\in Q'} \left[f(\alpha_1^{12}) f(\alpha_2)\prod_{p=3}^{k} f(\alpha_p)\right] + \sum_{A\in Q'} \left[f((\alpha_2^{21})^{12}) f(\alpha_1^{12})\prod_{p=3}^{k} f(\alpha_p)\right] =$$

$$= \sum_{A\in Q'} \left[f(\alpha_1^{12}) f(\alpha_2)\prod_{p=3}^{k} f(\alpha_p) + f(\alpha_2) f(\alpha_1^{12})\prod_{p=3}^{k} f(\alpha_p)\right] = 0,$$

because all the terms in the brackets are equal to zero (the field has the characteristics equal to 2). □

Since the dimension of V is given by the the form considered, we shall write $I(f)$ instead of $I_{3k}(f_B)$.

Corollary 2.3 *Let V be a vector space of dimension $n = 3k$ over* $\mathbf{GF}(2)$ *and let f be a trilinear form. If $I(f) = 1$ then f is nondegenerate.*

Proof. Let f be a degenerate form. Choose a basis B such that $b_1 \in \mathrm{Rad} f$. Then we have $f(b_1, b_j, b_k) = 0$ for all $1 \leq j, k \leq n$ and thus every member of the sum (1) is equal to zero, because the element $f(\alpha_r)$, $1 \in \alpha_r$, is. □

The next example shows that this invariant can also be effectively used when the form is given with respect to a "rich" basis. We show that the presented form is nondegenerate.

Example Let $B = \{b_1, \ldots, b_6\}$ be a basis of a vector space V. Define a trilinear alternating form f by:

$$f(b_i, b_j, b_k) = 1 \text{ iff } \{i, j, k\} \neq \{1, 2, 3\}.$$

There are $\binom{6}{3}/2 = 10$ 3-partitions of 6 points, nine of them contribute to the I_6 and the partition $\{1, 2, 3\}, \{4, 5, 6\}$ does not. Thus $I_6(f)$ is equal to $9 \equiv 1 \pmod 2$.

Similar approach can be used for general m-linear alternating forms on dimensions $n = mk$, but the m-linear forms, $m > 3$, are even less known (and studied) then the trilinear ones. In the bilinear case, this invariant yields 1 iff the form is the only (up to equivalence) nondegenerate form (see 2.1) and thus can be of some use.

Example Let f be a form on V of dimension 6 over $\mathbf{GF}(2)$. Using the result in [1] (the numbers of forms coincide), f is equivalent to exactly one of the forms listed in Table 1 together with the values of I_6. Notice that the forms f_0, f_1, f_2 are degenerate. The last column contains the values of the invariant I_6 for these forms.

TABLE 1. Values of I_6

	f	$I_6(f)$
f_0	0	0
f_1	$\underline{123}$	0
f_2	$\underline{123} + \underline{345}$	0
f_3	$\underline{123} + \underline{456}$	1
f_4	$\underline{123} + \underline{345} + \underline{156}$	0
f_{10}	$\underline{123} + \underline{234} + \underline{345} + \underline{246} + \underline{156}$	1

3. Extension to dimension $3k + 1$

Let f be a form on V and let W be a $3k$-dimensional subspace of V. Denote by g the restriction of f to W. We shall write $I_{3k}(W)$ (or even $I(W)$) instead of $I_{3k}(g)$.

Proposition 3.1 *Let f be a form on V of dimension $n = 3k + 1, k \in \mathbf{N}$ over the two-element field. If there is a hyperplane W such that $I_{3k}(W) = 1$, then there exist exactly 2^{n-1} hyperplanes satisfying $I_{3k}(X) = 1$.*

Proof. Choose a hyperplane $Y \neq W$ and let $B = \{b_1, \ldots, b_{3k-1}, w, y\}$ be a basis of V such that $\{b_1, \ldots, b_{3k-1}\} \in W \cap Y$, $w \in W$ and $y \in Y$. Denote by Y' the hyperplane $\langle\{b_1, \ldots, b_{3k-1}, w + y\}\rangle$. Using the definition of I_{3k} we obtain:

$$I_{3k}(Y') = I_{3k}(W) + I_{3k}(Y). \tag{2}$$

Since $(Y')'$ is equal to Y, there is a bijection between the set of hyperplanes $A_0 = \{Y; Y \neq W, I(Y) = 0\}$ and the set $A_1 = \{X; X \neq W, I(X) = 1\}$. Now it is enough to realize, that number of hyperplanes not equal to W is $2^n - 2$. □

Based on Proposition 3.1, we can define an invariant $I_{3k+1}(f)$ for a form on a vector space V of dimension $n = 3k + 1$:

- $I_{3k+1}(f) = 1$ iff there exists a hyperplane (and thus 2^{n-1} hyperplanes) $W \leq V$ with $I_{3k}(W) = 1$, and

TABLE 2. Values of I_7

	form	I_7
f_5	$\underline{123} + \underline{345} + \underline{567}$	0
f_6	$\underline{123} + \underline{145} + \underline{167} + \underline{357}$	0
f_7	$\underline{123} + \underline{167} + \underline{246} + \underline{357}$	1
f_8	$\underline{123} + \underline{145} + \underline{167}$	0
f_9	$\underline{123} + \underline{145} + \underline{167} + \underline{246} + \underline{357}$	1
f_{11}	$\underline{123} + \underline{234} + \underline{345} + \underline{246} + \underline{156} + \underline{367}$	1

- $I_{3k+1}(f) = 0$ otherwise.

Let $B = \{b_1, \dots, b_{3k+1}\}$ be a basis of V. Denote by B_i the hyperplane $B_i = \langle b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_{3k+1}\rangle$ and by f_i the restriction of f to B_i. It is possible to identify a hyperplane W with the nonzero linear form f_W with kernel W and then the equation (2) says that I_{3k} is zero for all nonzero forms $f_W \in V^*$ or there is a hyperplane $X^* \leq V^*$ such that $I_{3k}(W) = 1$ iff $f_W \in W^*$. Since f_{B_i} are linear independent in V^*, if there is a hyperplane W with $I_{3k}(W) = 1$ then there exist at least one i such that $f_{B_i} \in W^*$. We can conclude:

Theorem 3.2 *Let f be a trilinear alternating form on a vector space of dimension $n = 3k + 1$ over the two-element field and let B be a basis of V. Then*

(1) *$I_n(f_B)$ does not depend on the basis B.*

(2) *$I_n(f) = 1$ iff $I_{n-1}(f_{B_i}) = 1$ for at least one $i \in \{1, \dots, n\}$.*

Example Let f be a nondegenerate form on V of dimension 7 over $\mathbf{GF}(2)$. Using again the classification in [1], f is equivalent to exactly one of the forms listed in Table 2. Values of I_7 are easily seen from the representatives and are in the second column. The degenerate forms on dimension 7 are (up to one dimensional radical) forms in Table 1. Since $I_7(f)$ is equal to one iff there is a hyperplane W with $I_6(W) = 1$, the values I_7 of these degenerate forms are the values given in Table 1.

Corollary 3.3 *Using the notation from Tables 1 and 2, nondegenerate forms f_5, f_6 and f_8 have no hyperplane equivalent to either f_3 or f_{10}.*

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