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ARTINIAN COFINITE MODULES OVER COMPLETE NOETHERIAN LOCAL RINGS

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Abstract. Let $(R,\mathfrak{m})$ be a complete Noetherian local ring, I an ideal of R and M a nonzero Artinian R-module. In this paper it is shown that if $\mathfrak{p}$ is a prime ideal of R such that $\dim R/\mathfrak{p} = 1$ and $(0 :_M \mathfrak{p})$ is not finitely generated and for each $i \geqslant 2$ the R-module $\operatorname{Ext}^i_R(M, R/\mathfrak{p})$ is of finite length, then the R-module $\operatorname{Ext}^1_R(M, R/\mathfrak{p})$ is not of finite length. Using this result, it is shown that for all finitely generated R-modules N with $\operatorname{Supp}(N) \subseteq V(I)$ and for all integers $i \geqslant 0$, the R-modules $\operatorname{Ext}^i_R(N, M)$ are of finite length, if and only if, for all finitely generated R-modules N with $\operatorname{Supp}(N) \subseteq V(I)$ and for all integers $i \geqslant 0$, the R-modules $\operatorname{Ext}^i_R(M, N)$ are of finite length.

Keywords: Artinian module; cofinite module; Krull dimension; local cohomology

MSC 2010: 13E10, 13D45, 14B15

1. Introduction

Throughout this paper, let R denote a commutative Noetherian local ring (with identity) and I an ideal of R. For an R-module M, the i^{th} local cohomology module of M with respect to I is defined as

$$H^i_I(M) = \varinjlim_{n\geqslant 1} \operatorname{Ext}^i_R(R/I^n, M).$$

We refer the reader to [7] or [4] for more details about local cohomology. In [8], Hartshorne defined an R-module L to be I-*cofinite* if $\operatorname{Supp}(L) \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, L)$ is a finitely generated module for all i. The concept of cofinite modules have been studied by several authors; see, for example, Hartshorne [8],

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Huneke and Koh [9], Delfino [5], Delfino and Marley [6], Yoshida [16], Bahmanpour and Naghipour [2], Abazari and Bahmanpour [1], Kawasaki [11], [12], Bahmanpour, Naghipour and Sedghi [3], Melkersson [15], [14]. More recently, using the main result of [3], in [10] Irani and Bahmanpour have proved that for any ideal I of a Noetherian ring R and any I-cofinite R-module M of dimension $d \leqslant 1$, the R-modules $\operatorname{Ext}^i_R(M,N)$ are finitely generated, for all integers $i \geqslant 0$ and all finitely generated R-modules N with support in $V(I)$. The main goal of this paper is to verify the converse of this result. In this direction as the main result of this paper we shall prove the following theorem:

Theorem 1.1. *Let $(R,\mathfrak{m})$ be a complete Noetherian local ring and I an ideal of R. Let M be an Artinian R-module. Then the following are equivalent:*

(i) *For all finitely generated R-modules N with $\operatorname{Supp}(N) \subseteq V(I)$ and for all integers $i \geqslant 0$, the R-modules $\operatorname{Ext}^i_R(N,M)$ are of finite length.*

(ii) *For all finitely generated R-modules N with $\operatorname{Supp}(N) \subseteq V(I)$ and for all integers $i \geqslant 0$, the R-modules $\operatorname{Ext}^i_R(M,N)$ are of finite length.*

One of our tools for proving Theorem 1.1 is the following:

Theorem 1.2. *Let $(R,\mathfrak{m})$ be a complete Noetherian local ring and M a nonzero Artinian R-module. Let $\mathfrak{p}$ be a prime ideal of R such that $\dim R/\mathfrak{p} = 1$ and $(0 :_M \mathfrak{p})$ is not finitely generated. If for all $i \geqslant 2$ the R-module $\operatorname{Ext}^i_R(M,R/\mathfrak{p})$ is of finite length, then the R-module $\operatorname{Ext}^1_R(M,R/\mathfrak{p})$ is not of finite length.*

Throughout this paper, R will always be a commutative Noetherian ring with nonzero identity and I will be an ideal of R. Recall that, for each R-module M, all integers $j \geqslant 0$ and all prime ideals $\mathfrak{p}$ of R, the j^{th} Bass number of M with respect to $\mathfrak{p}$ is defined as $\mu^j(\mathfrak{p},M) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^j_{R_\mathfrak{p}}(k(\mathfrak{p}),M_\mathfrak{p})$, where $k(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. For an Artinian R-module A we denote by $\operatorname{Att}_R(A)$ the set of attached prime ideals of A. For any ideal $\mathfrak{a}$ of R we denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. We denote the support of each R-module M by $\operatorname{Supp}(M)$. Also, for each R-module M we denote by $\operatorname{Ass}_R(M)$ the set of associated prime ideals of M. Moreover, for each R-module M we denote by $\operatorname{Ann}_R(M)$ the annihilator of M in R. Finally, for each R-module M we denote by $E_R(M)$ the injective envelope (or injective hull) of M.

2. The results

To prove the main results of this paper, we need the following lemmas.

Lemma 2.1. *Let R be a Noetherian ring and $\mathfrak{p}$ a prime ideal of R. Let $M \neq 0$ be an arbitrary R-module such that $0 \neq \mu_R^0(\mathfrak{p}, M) = n < \infty$. Then there exists an exact sequence*

$$0 \to \bigoplus_{i=1}^{n} R/\mathfrak{p} \to M.$$

Proof. Let $E := E_R(M)$. Then we may assume $E = \Big(\bigoplus\limits_{i=1}^{n} E_R(R/\mathfrak{p})\Big) \bigoplus E'$ for some injective R-module E'. Now for each $1 \leqslant i \leqslant n$, let $L_i = \Big(\bigoplus\limits_{j=1}^{n} L_{i,j}\Big) \oplus 0$, where $L_{i,i} = R/\mathfrak{p}$ and $L_{i,j} = 0$ for each $j \in \{1, \ldots, n\} \setminus \{i\}$. Then as L_i is a submodule of E and E is an essential extension of M, it follows from the definition that for each $1 \leqslant i \leqslant n$ we have $L_i \cap M \neq 0$. Therefore $\emptyset \neq \operatorname{Ass}_R(L_i \cap M) \subseteq \operatorname{Ass}_R(L_i) = \{\mathfrak{p}\}$ and hence $\operatorname{Ass}_R(L_i \cap M) = \{\mathfrak{p}\}$. Therefore the R-module $L_i \cap M$ has a submodule L'_i such that $L'_i \cong R/\mathfrak{p}$. Now it is easy to see that $L'_1 + \ldots + L'_n \cong \bigoplus\limits_{i=1}^{n} R/\mathfrak{p}$ and obviously $L'_1 + \ldots + L'_n$ is a submodule of M. This completes the proof. $\square$

Lemma 2.2. *Let R be a Noetherian ring, I a proper ideal of R and A a nonzero Artinian I-cofinite R-module. Then for each nonzero finitely generated R-module N with support in $V(I)$, the R-modules $\operatorname{Ext}_R^i(A, N)$ have finite length for all integers $i \geqslant 0$.*

Proof. See [10, Theorem 2.3]. $\square$

Corollary 2.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and A a nonzero Artinian R-module. Then for each nonzero R-module N of finite length, the R-modules $\operatorname{Ext}_R^i(A, N)$ have finite length for all integers $i \geqslant 0$.*

Proof. Since each Artinian R-module is $\mathfrak{m}$-cofinite the assertion follows immediately from Lemma 2.2. $\square$

The following theorem is our main tool for the proof of the main result of this paper.

Theorem 2.4. *Let $(R,\mathfrak{m})$ be a complete Noetherian local ring and M a nonzero Artinian R-module. Let $\mathfrak{p}$ be a prime ideal of R such that $\dim R/\mathfrak{p}=1$ and $(0:_M\mathfrak{p})$ is not finitely generated. If for each $i\geqslant 2$ the R-module $\mathrm{Ext}^i_R(M,R/\mathfrak{p})$ is of finite length, then the R-module $\mathrm{Ext}^1_R(M,R/\mathfrak{p})$ is not of finite length.*

Proof. Let $\lambda(\mathfrak{p},M)=\dim_{R/\mathfrak{m}}((\mathfrak{p}+0:_R M)/(0:_R M)\otimes_R R/\mathfrak{m})$. We prove the assertion by induction on $\lambda(\mathfrak{p},M)$. Let $\lambda(\mathfrak{p},M)=0$. In this case $\mathfrak{p}\subseteq 0:_R M$ and so $\mathfrak{p}M=0$. Consider the following exact sequence:

$$0\to R/\mathfrak{p}\to E_R(R/\mathfrak{p})\to T\to 0. \tag{2.4.1}$$

Since M is Artinian it follows from the definition that $\mathrm{Supp}(M)\subseteq\{\mathfrak{m}\}$ and so $\mathrm{Hom}_R(M,R/\mathfrak{p})=\mathrm{Hom}_R(M,E_R(R/\mathfrak{p}))=\mathrm{Hom}_R(M,T/\Gamma_\mathfrak{m}(T))=0$. Now from the exact sequence

$$0\to\Gamma_\mathfrak{m}(T)\to T\to T/\Gamma_\mathfrak{m}(T)\to 0, \tag{2.4.2}$$

we conclude that $\mathrm{Hom}_R(M,T)\simeq\mathrm{Hom}_R(M,\Gamma_\mathfrak{m}(T))$. Since the R-module $E_R(R/\mathfrak{p})$ is injective from the exact sequence (2.4.1) we have

$$\mathrm{Ext}^1_R(M,R/\mathfrak{p})\simeq\mathrm{Hom}_R(M,T)\simeq\mathrm{Hom}_R(M,\Gamma_\mathfrak{m}(T)).$$

On the other hand,

$$\Gamma_\mathfrak{m}(E_R(R/\mathfrak{p}))=H^1_\mathfrak{m}(E_R(R/\mathfrak{p}))=0.$$

Therefore $\Gamma_\mathfrak{m}(T)\simeq H^1_\mathfrak{m}(R/\mathfrak{p})$ and consequently

$$\mathrm{Ext}^1_R(M,R/\mathfrak{p})\simeq\mathrm{Hom}_R(M,H^1_\mathfrak{m}(R/\mathfrak{p})).$$

So it is enough to show that the R-module $\mathrm{Hom}_R(M,H^1_\mathfrak{m}(R/\mathfrak{p}))$ is not of finite length. Suppose that $\mathrm{Hom}_R(M,H^1_\mathfrak{m}(R/\mathfrak{p}))$ is of finite length. Set $L:=\mathrm{Hom}_R(H^1_\mathfrak{m}(R/\mathfrak{p}),E_R(R/\mathfrak{m}))$. In view of [4, Theorem 7.1.3] the R-module $H^1_\mathfrak{m}(R/\mathfrak{p})$ is Artinian. Since R is complete, it follows that L is a finitely generated R-module. Moreover, by [4, Theorem 7.3.2] we have

$$\mathrm{Att}_R(H^1_\mathfrak{m}(R/\mathfrak{p}))=\{\mathfrak{p}\}.$$

Now, as R is a complete local ring, it follows from [4, Exercise 10.2.15(iii)] that

$$\{\mathfrak{p}\}=\mathrm{Att}_R(H^1_\mathfrak{m}(R/\mathfrak{p}))=\mathrm{Att}_R(0:_{H^1_\mathfrak{m}(R/\mathfrak{p})}0)=\mathrm{Ass}_R(L/0L)=\mathrm{Ass}_R(L).$$

Therefore, L is a finitely generated R-module such that $\mathfrak{p} \in \operatorname{Ass}_R(L)$ and $\mathfrak{p}L = 0$. Let $\mu^0(\mathfrak{p}, L) = n$. By Lemma 2.1 there is an exact sequence

$$0 \to \bigoplus_{i=1}^{n} R/\mathfrak{p} \to L \to B \to 0, \tag{2.4.3}$$

which implies the following exact sequence:

$$0 \to \bigoplus_{i=1}^{n} R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \to L_{\mathfrak{p}} \to B_{\mathfrak{p}} \to 0. \tag{2.4.4}$$

From the assumption $\mathfrak{p}L = 0$ we conclude that

$$\begin{aligned} \mu^0(\mathfrak{p}, L) &= \dim_{R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}} \operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, L_{\mathfrak{p}}) \\ &= \dim_{R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}}(L_{\mathfrak{p}}) = n = \dim_{R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}}\left(\bigoplus_{i=1}^{n} R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}\right). \end{aligned}$$

Therefore from the exact sequence (2.4.4) we conclude that $B_{\mathfrak{p}} = 0$. Hence, we have $\operatorname{Supp}(B) \subseteq V(\mathfrak{p}) \setminus \{\mathfrak{p}\} \subseteq V(\mathfrak{m}) = \{\mathfrak{m}\}$ and $\mathfrak{p}B = 0$. Since R is complete, applying the exact functor $D := \operatorname{Hom}_R(-, E_R(R/\mathfrak{m}))$ to the exact sequence (2.4.3) we get the exact sequence

$$0 \to C \to H^1_{\mathfrak{m}}(R/\mathfrak{p}) \to \bigoplus_{i=1}^{n} E_{R/\mathfrak{p}}(R/\mathfrak{m}) \to 0, \tag{2.4.5}$$

where $C := \operatorname{Hom}_R(B, E_R(R/\mathfrak{m}))$ is an R-module of finite length and $E_{R/\mathfrak{p}}(R/\mathfrak{m}) \simeq \operatorname{Hom}_R(R/\mathfrak{p}, E_R(R/\mathfrak{m}))$. Consider the exact sequence

$$\operatorname{Hom}_R(M, H^1_{\mathfrak{m}}(R/\mathfrak{p})) \to \operatorname{Hom}_R\left(M, \bigoplus_{i=1}^{n} E_{R/\mathfrak{p}}(R/\mathfrak{m})\right) \to \operatorname{Ext}^1_R(M, C). \tag{2.4.6}$$

By hypothesis the R-module $\operatorname{Hom}_R(M, H^1_{\mathfrak{m}}(R/\mathfrak{p}))$ is of finite length. Also by Corollary 2.3 the R-module $\operatorname{Ext}^1_R(M, C)$ has finite length. Therefore the exact sequence (2.4.6) implies that the R-module $\operatorname{Hom}_R(M, \bigoplus_{i=1}^{n} E_{R/\mathfrak{p}}(R/\mathfrak{m}))$ is of finite length. On the other hand,

$$\operatorname{Hom}_R\left(M, \bigoplus_{i=1}^{n} E_{R/\mathfrak{p}}(R/\mathfrak{m})\right) \simeq \bigoplus_{i=1}^{n} \operatorname{Hom}_R(M, E_R(R/\mathfrak{m})),$$

since $\mathfrak{p}M = 0$. Therefore the R-module $\operatorname{Hom}_R(M, E_R(R/\mathfrak{m}))$ is of finite length and so the R-module M is of finite length, and so the R-module $0 :_M \mathfrak{p}$ is of finite length which is a contradiction. Now suppose, inductively, that $\lambda(\mathfrak{p}, M) = t \geqslant 1$, and the result has been proved for all values smaller than t. By an argument similar to that in the first step it is enough to prove that the R-module $\mathrm{Hom}_R(M, H^1_{\mathfrak{m}}(R/\mathfrak{p}))$ is not of finite length. We suppose that the R-module $\mathrm{Hom}_R(M, H^1_{\mathfrak{m}}(R/\mathfrak{p}))$ is of finite length and again look for a contradiction. Since $\mathfrak{p}H^1_{\mathfrak{m}}(R/\mathfrak{p}) = 0$ it follows that $\mathrm{Hom}_R(R/\mathfrak{p}, H^1_{\mathfrak{m}}(R/\mathfrak{p})) \simeq H^1_{\mathfrak{m}}(R/\mathfrak{p})$ and so we have

$$\begin{aligned}\mathrm{Hom}_R(M, H^1_{\mathfrak{m}}(R/\mathfrak{p})) &\simeq \mathrm{Hom}_R(M, \mathrm{Hom}_R(R/\mathfrak{p}, H^1_{\mathfrak{m}}(R/\mathfrak{p})) \\ &\simeq \mathrm{Hom}_R(M \otimes_R R/\mathfrak{p}, H^1_{\mathfrak{m}}(R/\mathfrak{p})) \simeq \mathrm{Hom}_R(M/\mathfrak{p}M, H^1_{\mathfrak{m}}(R/\mathfrak{p})).\end{aligned}$$

The argument now proceeds like that used in the first step with the R-module $M_1 = M/\mathfrak{p}M$. Since $\mathfrak{p}M_1 = 0$, so by a similar argument for M, the R-module

$$\mathrm{Hom}_R(M_1, H^1_{\mathfrak{m}}(R/\mathfrak{p}))$$

is of finite length, and consequently the R-module M_1 is of finite length. By [4, Corollary 7.2.12] and [4, Exercise 7.2.6], we deduce that $\mathrm{Att}_R(M) \cap V(\mathfrak{p}) = \mathrm{Att}_R(M/\mathfrak{p}M) \subseteq \{\mathfrak{m}\}$. In particular, $\mathfrak{p} \nsubseteq \bigcup\limits_{q \in \mathrm{Att}_R(M) \setminus \{\mathfrak{m}\}} q$. Now by assumption $\lambda(\mathfrak{p}, M) = t \geqslant 1$, so there exist $x_1, \ldots, x_t$ in $\mathfrak{p}$ such that

$$\mathfrak{p} + \mathrm{Ann}_R M/\mathrm{Ann}_R M = (x_1, \ldots, x_t) + \mathrm{Ann}_R M/\mathrm{Ann}_R M.$$

Since $\mathfrak{p} \nsubseteq \bigcup\limits_{q \in \mathrm{Att}_R(M) \setminus \{\mathfrak{m}\}} q$,

$$\mathfrak{p} + \mathrm{Ann}_R M = (x_1, \ldots, x_t) + \mathrm{Ann}_R M \nsubseteq \bigcup_{q \in \mathrm{Att}_R(M) \setminus \{\mathfrak{m}\}} q,$$

and therefore using the fact that $\mathrm{Ann}_R M \subseteq \bigcap\limits_{q \in \mathrm{Att}_R(M) \setminus \{\mathfrak{m}\}} q$, it follows that

$$(x_1, \ldots, x_t) \nsubseteq \bigcup_{q \in \mathrm{Att}_R(M) \setminus \{\mathfrak{m}\}} q.$$

Consequently, in view of [13, Exercise 16.8], there exists $y_1 \in (x_2, \ldots, x_t)$ such that $z_1 \notin \bigcup\limits_{q \in \mathrm{Att}_R(M) \setminus \{\mathfrak{m}\}} q$, where $z_1 = x_1 + y_1$. Clearly $z_1 \in \mathfrak{p}$ and $(x_1, \ldots, x_t) = (z_1, x_2, \ldots, x_t)$, hence

$$\mathfrak{p} + \mathrm{Ann}_R M/\mathrm{Ann}_R M = (z_1, x_2, \ldots, x_t) + \mathrm{Ann}_R M/\mathrm{Ann}_R M.$$

Now we have

$$\mathrm{Att}_R(M/z_1M) \subseteq \mathrm{Att}_R(M) \cap V(Rz_1) \subseteq \{\mathfrak{m}\}.$$

By [4, Corollary 7.2.12] M/z_1M is of finite length. The exact sequence

$$0 \to z_1M \to M \to M/z_1M \to 0 \tag{2.4.7}$$

induces an exact sequence

$$\mathrm{Ext}^i_R(M, R/\mathfrak{p}) \to \mathrm{Ext}^i_R(z_1M, R/\mathfrak{p}) \to \mathrm{Ext}^{i+1}_R(M/z_1M, R/\mathfrak{p})$$

for each $i \geqslant 2$. By assumption, $\mathrm{Ext}^i_R(M, R/\mathfrak{p})$ for all $i \geqslant 2$ is of finite length. Also $\mathrm{Ext}^{i+1}_R(M/z_1M, R/\mathfrak{p})$ is of finite length and so for all $i \geqslant 2$ the R-module $\mathrm{Ext}^i_R(z_1M, R/\mathfrak{p})$ is of finite length. The exact sequence

$$0 \to 0 :_M z_1 \to M \to z_1M \to 0 \tag{2.4.8}$$

induces an exact sequence

$$\mathrm{Ext}^i_R(M, R/\mathfrak{p}) \to \mathrm{Ext}^i_R(0 :_M z_1, R/\mathfrak{p}) \to \mathrm{Ext}^{i+1}_R(z_1M, R/\mathfrak{p})$$

for each $i \geqslant 2$. This shows that for each $i \geqslant 2$, the R-module $\mathrm{Ext}^i_R(0 :_M z_1, R/\mathfrak{p})$ is of finite length. Since $z_1 \in \mathfrak{p}$, it follows that $0 :_{(0:_M z_1)} \mathfrak{p} = 0 :_M \mathfrak{p}$ is not finitely generated. Now $Rz_1 + (0 :_R M) \subseteq 0 :_R (0 :_M z_1)$, so $\lambda(\mathfrak{p}, 0 :_M z_1) \leqslant \lambda(\mathfrak{p}, M) - 1 = t - 1$. Hence, by induction hypothesis the R-module $\mathrm{Ext}^1_R(0 :_M z_1, R/\mathfrak{p})$ is not of finite length. The exact sequence (2.4.8) induces an exact sequence

$$\mathrm{Ext}^1_R(M, R/\mathfrak{p}) \to \mathrm{Ext}^1_R(0 :_M z_1, R/\mathfrak{p}) \to \mathrm{Ext}^2_R(z_1M, R/\mathfrak{p}).$$

Consequently, as the R-module $\mathrm{Ext}^2_R(z_1M, R/\mathfrak{p})$ is of finite length it follows that the R-module $\mathrm{Ext}^1_R(M, R/\mathfrak{p})$ is not of finite length, hence the R-module $\mathrm{Hom}_R(M, H^1_{\mathfrak{m}}(R/\mathfrak{p}))$ is not of finite length, which is a contradiction. □

An immediate consequence of Theorem 4.2 is the following theorem.

Theorem 2.5. *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and M an Artinian R-module. Let $\mathfrak{p}$ be a prime ideal of R such that* $\dim R/\mathfrak{p} = 1$. *Then the following are equivalent:*

(i) *The R-module M is $\mathfrak{p}$-cofinite.*

(ii) *For all $i \geqslant 0$, the R-module $\mathrm{Ext}^i_R(M, R/\mathfrak{p})$ has finite length.*

Proof. (i) → (ii) Follows from Lemma 2.2.

(ii) → (i) Suppose that M is not $\mathfrak{p}$-cofinite. Then by [14, Proposition 4.1] the R-module $0 :_M \mathfrak{p}$ is not finitely generated. Now by Theorem 2.4 there exists $i \geqslant 1$ such that $\mathrm{Ext}^i_R(M, R/\mathfrak{p})$ is not of finite length, which is a contradiction. □

The following theorem is needed in the proof of the main result of this paper.

Theorem 2.6. *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and I an ideal of R. Let M be an Artinian R-module. Then the following are equivalent:*

(i) *The R-module M is I-cofinite.*

(ii) *For every finitely generated R-module N with $\operatorname{Supp}(N) \subseteq V(I)$ and for all $i \geqslant 0$, the R-module $\operatorname{Ext}^i_R(M, N)$ is of finite length.*

Proof. (i) $\rightarrow$ (ii) Follows from Lemma 2.2.

(ii) $\rightarrow$ (i) Suppose that M is not I-cofinite. Then by [15, Theorem 1.6] there exists $q \in \operatorname{Att}_R(M)$ such that $\dim R/I + q \geqslant 1$. Therefore there exists $\mathfrak{p} \in V(I + q)$ such that $\dim R/\mathfrak{p} = 1$. By [15, Theorem 1.6], M is not $\mathfrak{p}$-cofinite. So by Theorem 2.5 there exists $i \geqslant 0$ such that $\operatorname{Ext}^i_R(M, R/\mathfrak{p})$ is not of finite length. But $R/\mathfrak{p}$ is finitely generated and $\operatorname{Supp}(R/\mathfrak{p}) \subseteq V(I)$, which is a contradiction. □

Now we are ready to state and to prove the main result of this paper.

Theorem 2.7. *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and I an ideal of R. Let M be an Artinian R-module. Then the following are equivalent:*

(i) *For all finitely generated R-modules N with $\operatorname{Supp}(N) \subseteq V(I)$ and all integers $i \geqslant 0$, the R-modules $\operatorname{Ext}^i_R(N, M)$ are of finite length.*

(ii) *For all finitely generated R-modules N with $\operatorname{Supp}(N) \subseteq V(I)$ and all integers $i \geqslant 0$, the R-modules $\operatorname{Ext}^i_R(M, N)$ are of finite length.*

Proof. The assertion follows from Theorem 2.6 and [11, Lemma 1]. □

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