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A Reproducing Kernel and Toeplitz Operators in the Quantum Plane

Stephen Bruce Sontz

Abstract. We define and analyze Toeplitz operators whose symbols are the elements of the complex quantum plane, a non-commutative, infinite dimensional algebra. In particular, the symbols do not come from an algebra of functions. The process of forming operators from non-commuting symbols can be considered as a second quantization. To do this we construct a reproducing kernel associated with the quantum plane. We also discuss the commutation relations of creation and annihilation operators which are defined as Toeplitz operators. This paper extends results of the author for the finite dimensional case.

1 Introduction

Based on the formalism developed in [3], we have introduced and studied in a pair of papers (see [9], [10]) a reproducing kernel and its associated Toeplitz operators which have symbols in a non-commutative algebra which is a finite dimensional truncated version of the complex quantum plane called a paragrassmann algebra. We extend those results now to the case of the complex quantum plane, which is an infinite dimensional, non-commutative algebra. Creation and annihilation operators are defined as certain Toeplitz operators, and their commutation relations are discussed.

This is much like a quantization scheme according to a common intuition of what those words should mean: “operators instead of functions”. However, here one must modify this catch phrase to say “operators instead of elements in a non-commutative algebra”. This is so because here the symbols are not elements in an algebra isomorphic to an algebra of functions, since the latter is commutative. So, as we remarked in [10], the quantization scheme discussed here is more akin to what in physics is known as a *second quantization*, where one goes from one quantum

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theory to another quantum theory, rather than a *first quantization*, where one goes from a classical theory to a quantum theory.

The paper is organized as follows: The next section introduces the basic definitions and properties. Section 3 is about the reproducing kernel while in Section 4 we define and study Toeplitz operators, including the creation and annihilation operators. Section 5 is about the commutation relations of the creation and annihilation operators. The concluding remarks in Section 6 give some brief indications for possible future research.

2 Definitions and such

We study here the *complex quantum plane* defined as the algebra

$$\mathbb{C}Q_q(\theta, \bar{\theta}) := \mathbb{C}\{\theta, \bar{\theta}\} / \langle \theta\bar{\theta} - q\bar{\theta}\theta \rangle$$

where $\mathbb{C}\{\theta, \bar{\theta}\}$ is the free algebra over \mathbb{C} on the two generators θ and $\bar{\theta}$ while $\langle \theta\bar{\theta} - q\bar{\theta}\theta \rangle$ is the two sided ideal generated by the element $\theta\bar{\theta} - q\bar{\theta}\theta$ for some $q \in \mathbb{C} \setminus \{0\}$. This is a non-commutative algebra provided that $q \neq 1$. It has a vector space basis $AW := \{\theta^j \bar{\theta}^k \mid j, k \in \mathbb{N}\}$, known as the *anti-Wick basis*, and so is infinite dimensional. In Ref. [3] the authors call this the anti-normal ordering, which is synonymous with anti-Wick ordering. This agrees with the definition of quantum plane in [5] (putting the field $k = \mathbb{C}$ there) and with the quantum q -plane in [6], except for notation. However, we will not be studying co-actions of quantum groups on this quantum space as is often done, but rather how its elements serve as the symbols for Toeplitz operators.

Moreover, we also define a *conjugation* (also called a **-operation*) in $\mathbb{C}Q_q(\theta, \bar{\theta})$ on the basis AW by putting

$$(\theta^j \bar{\theta}^k)^* := \theta^k \bar{\theta}^j \quad (1)$$

and then by extending *anti-linearly* to linear combinations with coefficients in \mathbb{C} . This is easily shown to be an involution, i.e., $f^{**} = f$ for all $f \in \mathbb{C}Q_q(\theta, \bar{\theta})$. This conjugation makes θ and $\bar{\theta}$ into a pair of variables, each being the conjugate of the other. We will see that this **-operation* relates well with the operation of taking the adjoint of a Toeplitz operator. However, we are not saying (nor do we need) that this **-operation* converts $\mathbb{C}Q_q(\theta, \bar{\theta})$ into a **-algebra*, that is $(fg)^* = g^* f^*$ for all $f, g \in \mathbb{C}Q_q(\theta, \bar{\theta})$. We do note without giving proof that this is a **-algebra* if and only if $q \in \mathbb{R} \setminus \{0\}$.

We let $w = \{w_j \mid j \in \mathbb{N}\}$ be a sequence of strictly positive real numbers, that is, $w_j > 0$. These will be referred to as weights. We use these weights to define an inner product on $\mathbb{C}Q_q(\theta, \bar{\theta})$ as the sesquilinear extension (anti-linear in the first entry, linear in the second) of

$$\langle \theta^a \bar{\theta}^b, \theta^c \bar{\theta}^d \rangle_w := w_{a+d} \delta_{a+d, b+c} = w_{a+d} \delta_{a-b, c-d} \quad \text{for all } a, b, c, d \in \mathbb{N}, \quad (2)$$

with δ being the Kronecker delta. Notice that the condition $a - b = c - d$ is necessary and sufficient for the inner product in (2) to be non-zero. Clearly, given a pair $a, b \in \mathbb{N}$ there are infinitely many pairs $c, d \in \mathbb{N}$ such that $c - d = a - b$ and also satisfying $(c, d) \neq (a, b)$. Therefore AW is not even an orthogonal basis, let alone an orthonormal basis.

We wish to note, although without giving the relatively straightforward proof, that there is this compatibility between the inner product (2) and the conjugation (1), namely: $\langle f, g \rangle_w^* = \langle f^*, g^* \rangle_w$ for all $f, g \in \mathbb{C}Q_q(\theta, \bar{\theta})$, where the $*$ -operation on the left side is complex conjugation in \mathbb{C} . We note that we also have the identity $\langle f, g \rangle_w^* = \langle g, f \rangle_w$.

The definition (2) is partly motivated by the inner product introduced in [3] and studied in [9] and [10]. There one has the paragrassmann algebra defined by

$$PG_{l,q}(\theta, \bar{\theta}) = \mathbb{C}(\theta, \bar{\theta}) / \langle \theta\bar{\theta} - q\bar{\theta}\theta, \theta^l, \bar{\theta}^l \rangle$$

with $l \geq 2$ an integer. This is a quotient (as an algebra) of $\mathbb{C}Q_q(\theta, \bar{\theta})$ by the nilpotency relations $\theta^l = 0$ and $\bar{\theta}^l = 0$. In that case, using the notation in [9], the inner product used there satisfies

$$\langle \theta^a \bar{\theta}^b, \theta^c \bar{\theta}^d \rangle_w = \langle \theta^{a+d}, \theta^{b+c} \rangle = w_{a+d} \delta_{a+d, b+c} \chi_l(a+d). \tag{3}$$

Here χ_l is the characteristic function for the set of integers $\{0, 1, \dots, l-1\}$. Its presence is due to the nilpotency relations. Equation (3) was not the actual definition of this inner product, although it could have been. Instead the definition of this inner product was given in terms of a Berezin type integral, thereby presenting $PG_{l,q}(\theta, \bar{\theta})$ as something quite analogous to a classical L^2 space. It seems to be impossible to write (2) as a Berezin type integral, since now there are no ‘top classes’ in the theory. However, it might be useful to express (2) as some sort of generalized L^2 inner product.

Now another motivation for the inner product (2) is seen in the well known example of the Hilbert space

$$\mathcal{H} := L^2(\mathbb{C}, \pi^{-1} e^{-|z|^2} dx dy) \tag{4}$$

where the monomials $z^j \bar{z}^k$ form a basis (linearly independent set such that the closure of their algebraic span is the entire Hilbert space). Then using a result that goes back at least as far to Bargmann’s paper [2] in the second equality, for $a, b, c, d \in \mathbb{N}$ this basis satisfies

$$\langle z^a \bar{z}^b, z^c \bar{z}^d \rangle_{\mathcal{H}} = \langle z^{a+d}, z^{b+c} \rangle_{\mathcal{H}} = (a+d)! \delta_{a+d, b+c},$$

where we are using here the standard L^2 inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ in \mathcal{H} . Hence we can think of w_j as some sort of deformation of $j!$, the usual factorial of $j \in \mathbb{N}$. Notice that an immediate consequence is that $\langle z^a, \bar{z}^d \rangle_{\mathcal{H}} = 0$ if either $a > 0$ or $d > 0$, while for $a = b = c = d = 0$ we have $\langle 1, 1 \rangle_{\mathcal{H}} = 1$. In turn this implies for f holomorphic and g anti-holomorphic that

$$\langle f, g \rangle_{\mathcal{H}} = f(0)^* g(0).$$

In particular such a pair of f and g is orthogonal if and only if either $f(0) = 0$ or $g(0) = 0$. This example has an interesting consequence. Suppose that we take the weights in the quantum plane to be $w_j = j!$ for all $j \in \mathbb{N}$. Then the inner product on the quantum plane $\mathbb{C}Q_q(\theta, \bar{\theta})$ is positive definite since in this case

$\mathbb{C}Q_q(\theta, \bar{\theta})$ is unitarily isomorphic to a dense subspace D of the Hilbert space \mathcal{H} for any $q \in \mathbb{C} \setminus \{0\}$. In fact the map $U: \mathbb{C}Q_q(\theta, \bar{\theta}) \rightarrow \mathcal{H}$ given on the basis AW by $U(\theta^i \bar{\theta}^j) := z^i \bar{z}^j$ is an isometry. Actually D is the commutative subalgebra $\mathbb{C}[z, \bar{z}]$ of complex polynomials in two commuting variables, and so the unitary isomorphism

$$U: \mathbb{C}Q_q(\theta, \bar{\theta}) \xrightarrow{\cong} D = \mathbb{C}[z, \bar{z}]$$

is not an algebra isomorphism for $q \neq 1$. Also the completion of $\mathbb{C}Q_q(\theta, \bar{\theta})$ with respect to the corresponding metric is unitarily isomorphic to the Hilbert space \mathcal{H} , again for any $q \in \mathbb{C} \setminus \{0\}$. And hence there are cases where the inner product defined by (2) is positive definite. Motivated in part by this example we call θ a *holomorphic variable* and $\bar{\theta}$ an *anti-holomorphic variable*. (Compare also with the usage of these terms in [9] and [10].)

However there are also cases for which the inner product (2) is not positive definite. To see how this can happen, we first note some elementary calculations:

$$\begin{aligned} \langle 1, 1 \rangle_w &= w_0, \\ \langle \theta \bar{\theta}, 1 \rangle_w &= \langle 1, \theta \bar{\theta} \rangle_w = w_1, \\ \langle \theta \bar{\theta}, \theta \bar{\theta} \rangle_w &= w_2. \end{aligned} \tag{5}$$

As an aside, we note that 1 is a normalized state (the ‘ground state’) if and only if $w_0 = 1$. Let $\alpha \in \mathbb{R}$ be a real number to be specified in more detail later. Then

$$\langle 1 + \alpha \theta \bar{\theta}, 1 + \alpha \theta \bar{\theta} \rangle_w = w_0 + 2\alpha w_1 + \alpha^2 w_2, \tag{6}$$

a quadratic polynomial in α which has distinct real roots if and only if its discriminant is positive, that is, $w_1^2 - w_0 w_2 > 0$. Picking weights that satisfy this condition we see that the inner product in (6) will be zero for two distinct values of $\alpha \in \mathbb{R}$ and negative for values strictly between those two values. (Recall that $w_2 > 0$.) In short, the inner product will not be positive definite in such a case. This example also shows that $w_1^2 - w_0 w_2 < 0$ is a necessary condition for the inner product to be positive definite.

The remarks in the previous paragraphs show that the situation for the quantum plane is rather different from the finite dimensional theory, where the inner product is never positive definite, but always non-degenerate, as shown in [9]. We now wish to establish a necessary and sufficient condition on the weights w_k so that the inner product $\langle \cdot, \cdot \rangle_w$ defined in (2) is non-degenerate. Here it is:

Theorem 1. *The inner product (2) is non-degenerate on $\mathbb{C}Q_q(\theta, \bar{\theta})$ if and only if for every integer $R \geq 1$ and every $n \in \mathbb{Z}$ we have that*

$$\{ W_{R,s} \in \mathbb{C}^R \mid s \geq |n| \}^\perp = 0,$$

where $W_{R,s} = (w_{r+s-|n|})_{|n| \leq r \leq |n|+R-1}$ is a vector in \mathbb{C}^R for every $s \geq |n|$.

Proof. To facilitate this argument we define a partition of the basis AW so that elements in disjoint subsets of the partition are orthogonal with respect to the inner product (2). So for each integer $n \in \mathbb{Z}$ we define

$$P_n := \{ \theta^a \bar{\theta}^b \mid a \geq 0, b \geq 0, a - b = n \}.$$

Then we have $P_n \perp P_m$ for all $n, m \in \mathbb{Z}$ satisfying $n \neq m$ as well as

$$AW = \bigcup_{n \in \mathbb{Z}} P_n,$$

a disjoint union. So we have an algebraic orthogonal decomposition

$$\mathbb{C}Q_q(\theta, \bar{\theta}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{P}_n,$$

where $\mathcal{P}_n := \text{span}_{\mathbb{C}} P_n$. (Here we let $\text{span}_{\mathbb{C}} S$ denote the operation of forming the algebraic subspace over \mathbb{C} generated by the indicated set S . So, we are taking here only *finite* linear combinations of elements in S .) It follows that the inner product (2) is non-degenerate if and only if it is non-degenerate on each of the summands \mathcal{P}_n .

It will be convenient for us to define the *max-degree* of each basis element in AW by

$$\text{maxdeg}(\theta^a \bar{\theta}^b) := \max(a, b) \geq 0.$$

Then P_n contains exactly one element of max-degree $|n| + k$ for $k = 0, 1, 2, \dots$ (and no other elements). For example, for the integers $n \leq 0$ we have

$$P_n = \{ \bar{\theta}^{(-n)}, \theta \bar{\theta}^{(-n+1)}, \dots, \theta^k \bar{\theta}^{(-n+k)}, \dots \}.$$

A similar expression holds for $n > 0$. We denote the unique element of P_n of max-degree r by ε_r for each integer $r \geq |n|$. The reader can check that for $n \geq 0$ we have $\varepsilon_r = \theta^r \bar{\theta}^{r-n}$, while for $n < 0$ we have $\varepsilon_r = \theta^{r+n} \bar{\theta}^r$, where $r \geq |n|$ in both cases. We omit n from the notation ε_r .

Taking the pair of elements $\varepsilon_r, \varepsilon_s \in P_n$ for a given $n \in \mathbb{Z}$ and $r, s \geq |n|$ and then computing their inner product gives (as the reader can check) that

$$\langle \varepsilon_r, \varepsilon_s \rangle_w = w_{r+s-|n|}.$$

In the example (5) given earlier the two elements 1 and $\theta \bar{\theta}$ lie in P_0 and satisfy $\text{maxdeg } 1 = 0$ and $\text{maxdeg } \theta \bar{\theta} = 1$. So $\varepsilon_0 = 1$ and $\varepsilon_1 = \theta \bar{\theta}$ in P_0 .

Suppose that $n \in \mathbb{Z}$ is given. We then consider the inner product (2) restricted to P_n . Take an arbitrary element $f \in P_n$ with $f \neq 0$. We write

$$f = \sum_{r \geq |n|} a_r \varepsilon_r,$$

where each $a_r \in \mathbb{C}$, but only finitely many are non-zero. But at least one of these coefficients a_r is non-zero, since $f \neq 0$. The inner product is non-degenerate on P_n if and only there exists $g \in P_n$ (depending on f , of course) such that $\langle g, f \rangle_w \neq 0$. We expand g as

$$g = \sum_{s \geq |n|} b_s \varepsilon_s$$

for complex coefficients b_s only finitely many of which are non-zero. Then we evaluate

$$\langle g, f \rangle_w = \sum_{r \geq |n|, s \geq |n|} a_r b_s^* \langle \varepsilon_s, \varepsilon_r \rangle_w = \sum_{s \geq |n|} b_s^* \left(\sum_{r \geq |n|} a_r w_{r+s-|n|} \right). \quad (7)$$

For example, if $w_k = 1$ (or any other constant value) for all $k \geq 0$, then taking f above such that $\sum_r a_r = 0$ but some $a_r \neq 0$ gives us an element $f \neq 0$ satisfying $\langle g, f \rangle_w = 0$ for all g . So in this particular case the inner product is degenerate.

Notice that the expression in parentheses on the right in (7) is given to us, while the coefficients b_s are ours to choose as we please *provided that* only finitely many of them are non-zero. So we define

$$v_s(f) := \sum_{r \geq |n|} a_r w_{r+s-|n|} \in \mathbb{C} \quad (8)$$

for every $s \geq |n|$. (Recall that n is a given integer so we do not include it in the notation $v_s(f)$. The sum is well defined since only finitely many of the a_r 's are non-zero.) If just one of these numbers is non-zero, say $v_{s_0}(f) \neq 0$, then we can put $b_s = 0$ for all $s \neq s_0$ and $b_{s_0} = 1$. And therefore (7) is non-zero. And such a choice indeed has only finitely many (namely, one) of the b_s 's different from zero. The element g corresponding to this choice of the b_s 's is $g = \varepsilon_{s_0}$, which satisfies $\langle g, f \rangle_w \neq 0$. Therefore in this case $\{f\}^\perp \neq \mathcal{P}_n$. (Recall that we have restricted the inner product to \mathcal{P}_n .)

So if the inner product is degenerate on \mathcal{P}_n (which means that $\{h\}^\perp = \mathcal{P}_n$ for some $0 \neq h \in \mathcal{P}_n$), then there must exist some $f \neq 0$ (actually, $f = h$ works) such that $v_s(f) = 0$ for all $s \geq |n|$. Conversely, if there exists some $f \neq 0$ such that $v_s(f) = 0$ for all $s \geq |n|$, then for every g we have $\langle g, f \rangle_w = 0$ by (7) and so the inner product is degenerate on \mathcal{P}_n . We now re-write the definition (8) for $v_s(f)$ as

$$v_s(f) = \sum_{|n| \leq r \leq |n|+R-1} a_r w_{r+s-|n|} \in \mathbb{C} \quad (9)$$

for some integer $R \geq 1$. Notice that the existence of R is given to us implicitly as part of the information about f , since only finitely many of the a_r 's are non-zero. R is not unique, but that is not important for this argument.

So we can consider $A_R(f) := (a_r^*)_{|n| \leq r \leq |n|+R-1}$ as a vector in \mathbb{C}^R . Similarly, $W_{R,s} := (w_{r+s-|n|})_{|n| \leq r \leq |n|+R-1}$ is considered as a vector in \mathbb{C}^R . Recall that n is fixed since we are working in \mathcal{P}_n . However, $s \geq |n|$ is arbitrary. We will now use the standard Hermitian inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}^R}$ on \mathbb{C}^R . Then equation (9) is the same as

$$v_s(f) = \langle A_R(f), W_{R,s} \rangle_{\mathbb{C}^R}.$$

Now $\{W_{R,s}\}_{s \geq |n|}$ is an infinite sequence of vectors in the finite dimensional vector space \mathbb{C}^R . Since $f = \sum_{r \geq n} a_r \varepsilon_r$ is an arbitrary non-zero element in \mathcal{P}_n with

$$|n| \leq \max(\{r \mid a_r \neq 0\}) \leq |n| + R - 1,$$

it follows that $A_R(f)$ is an arbitrary non-zero vector in \mathbb{C}^R . Therefore the following statements are equivalent provided that $n \in \mathbb{Z}$ is given:

- The inner product is degenerate on \mathcal{P}_n .
- For some $f \in \mathcal{P}_n$ with $f \neq 0$, we have $v_s(f) = 0$ for all $s \geq |n|$.
- For some sequence $\{a_r \mid r \geq |n|\}$, not identically zero but with only finitely many terms not equal to zero, we have $v_s = 0$ for all $s \geq |n|$, where we define $v_s := \sum_{r \geq |n|} a_r w_{r+s-|n|}$ for $s \geq |n|$.
- There exist some integer $R \geq 1$ and some vector $A \in \mathbb{C}^R \setminus \{0\}$ such that for all $s \geq |n|$ we have $\langle A, W_{R,s} \rangle_{\mathbb{C}^R} = 0$.
- There exists some integer $R \geq 1$ so that $\{W_{R,s} \in \mathbb{C}^R \mid s \geq |n|\}^\perp \neq 0$.

Equivalently, the inner product is non-degenerate on \mathcal{P}_n if and only if for every integer $R \geq 1$ we have

$$\{W_{R,s} \in \mathbb{C}^R \mid s \geq |n|\}^\perp = 0.$$

We have already established that the inner product (2) is non-degenerate on $\mathbb{C}Q_q(\theta, \bar{\theta})$ if and only if it is non-degenerate on \mathcal{P}_n for every integer $n \in \mathbb{Z}$. And so this finishes the proof. \square

Remark 1. This result gives an algebraic necessary and sufficient condition on the weights w_k for their associated inner product to be non-degenerate. While it looks clumsy, it should be amenable to verification in applications. Intuitively, the condition that an infinite sequence in a finite dimensional vector space spans the vector space seems to be a generic condition. And so countably many such conditions should also be generic. Theorem 1 contrasts with the result for the paragrassmann algebra in [9], where we proved that the inner product (3) is non-degenerate for all positive weights.

Inside the subalgebra

$$\text{Pre}(\theta) := \text{span}_{\mathbb{C}}\{\theta^j \mid j \in \mathbb{N}\} \cong \mathbb{C}[\theta] \subset \mathbb{C}Q_q(\theta, \bar{\theta})$$

generated by all powers of the holomorphic variable θ , we have as a particular case of the definition (2) that

$$\langle \theta^j, \theta^k \rangle_w = \delta_{j,k} w_j$$

for all $j, k \in \mathbb{N}$. So the inner product restricted to the ‘holomorphic’ subspace $\text{Pre}(\theta)$ is positive definite. This means that $\text{Pre}(\theta)$ is a pre-Hilbert space. Moreover, an orthonormal basis of $\text{Pre}(\theta)$ is given by

$$\phi_j(\theta) := \frac{1}{w_j^{1/2}} \theta^j \quad \text{for } j \in \mathbb{N}.$$

Similar comments hold for the anti-holomorphic subalgebra $\text{Pre}(\bar{\theta})$ defined in a completely analogous way:

$$\text{Pre}(\bar{\theta}) := \text{span}_{\mathbb{C}}\{\bar{\theta}^j \mid j \in \mathbb{N}\} \cong \mathbb{C}[\bar{\theta}] \subset \mathbb{C}Q_q(\theta, \bar{\theta})$$

We let

$$\mathcal{B}(\theta) = \mathcal{B} := \text{comp}_{\mathbb{C}} \text{Pre}(\theta)$$

denote the *holomorphic space* (or the *Segal-Bargmann space*) of the quantum plane. By the operation $\text{comp}_{\mathbb{C}}$ we mean the completion of the indicated pre-Hilbert space. The set $\{\phi_j(\theta) \mid j \in \mathbb{N}\}$ is also an orthonormal basis for $\mathcal{B}(\theta)$. Unlike the finite dimensional case studied in [3], [9] and [10], the Segal-Bargmann space $\mathcal{B}(\theta)$ here is not necessarily an algebra. However, it does contain the dense subspace $\text{Pre}(\theta) \cong \mathbb{C}[\theta]$ which is an algebra, namely the algebra of polynomials in θ . But the multiplication map for $\mathbb{C}[\theta]$ is not necessarily continuous in the topology induced by the norm associated to the inner product (2) and, if that is the case, then it is not extendible by continuity to $\mathcal{B}(\theta)$.

Analogously, we define the *anti-holomorphic space* (or the *anti-Segal-Bargmann space*) of the quantum plane by

$$\mathcal{B}(\bar{\theta}) := \text{comp}_{\mathbb{C}} \text{Pre}(\bar{\theta}).$$

These two spaces $\mathcal{B}(\theta)$ and $\mathcal{B}(\bar{\theta})$ should be understood as ‘almost’ disjoint. Their ‘intersection’ is the one dimensional subspace spanned by $1 = \theta^0 = \bar{\theta}^0$.

3 Reproducing kernel

As a first step towards the definition of Toeplitz operators, we shall find a reproducing kernel for the Segal-Bargmann space. First of all we will need to define a functional calculus for the Segal-Bargmann space. As is well-known, there always is a functional calculus for polynomials $f \in \mathbb{C}[x]$ associated to any element in any associative algebra. Here we write

$$f = \sum_{j=0}^m a_j x^j \in \mathbb{C}[x]$$

with coefficients $a_j \in \mathbb{C}$ and then use the standard definition

$$f(\theta) := \sum_{j=0}^m a_j \theta^j.$$

But there are some elements in $\mathcal{B}(\theta)$ that are not so representable, since they are infinite sums of elements in the orthogonal basis $\{\theta^j\}$. However, any element $u \in \mathcal{B}(\theta)$ can be expanded as an infinite sum with respect to the orthonormal basis $\{\phi_j(\theta)\}$ giving

$$u = \sum_{j=0}^{\infty} a_j \phi_j(\theta) = \sum_{j=0}^{\infty} a_j w_j^{-1/2} \theta^j$$

with $a_j \in \mathbb{C}$ and $\sum_j |a_j|^2 < \infty$. Equivalently, for all $u \in \mathcal{B}(\theta)$ we have

$$u = \sum_{j=0}^{\infty} f_j \theta^j$$

with $f_j \in \mathbb{C}$ and $\sum_j |f_j|^2 w_j < \infty$. So associated to any sequence of positive real numbers $w = \{w_j \mid j \geq 0\}$ we define a weighted little l^2 space:

$$l^2(w) := \left\{ f = \{f_j \mid j \in \mathbb{N}\} \mid \sum_j |f_j|^2 w_j < \infty \right\}.$$

Then the full *functional calculus* of θ is the linear mapping

$$\Phi: l^2(w) \rightarrow \mathcal{B}(\theta)$$

defined by

$$\Phi(f) = \Phi(\{f_j\}) := \sum_{j=0}^{\infty} f_j \theta^j.$$

So Φ is a unitary isomorphism of Hilbert spaces. We also use the more suggestive notation $f(\theta) := \Phi(f)$ for all $f \in l^2(w)$.

Now the reproducing kernel $K(\theta, \eta)$ is supposed to satisfy the *reproducing kernel formula*, namely

$$f(\theta) = \langle K(\theta, \eta), f(\eta) \rangle_w \in \mathcal{B}(\theta) \tag{10}$$

for all $f \in l^2(w)$ and where $\eta \in \mathcal{B}(\eta)$ is another ‘independent copy’ of a holomorphic variable. The intuitive idea behind the inner product in (10) is that it should only take η into consideration while letting θ have a free ride as a ‘passenger’. The usual structure of reproducing kernel functions in spaces of holomorphic functions suggests that we should have

$$K(\theta, \eta) \in \mathcal{B}(\bar{\theta}) \otimes \mathcal{B}(\eta),$$

the standard tensor product of Hilbert spaces. This expresses the intuition that $K(\theta, \eta)$ should be anti-holomorphic in θ and holomorphic in η . So we want to define an inner product $\langle L(\theta, \eta), f(\eta) \rangle_w$ for all $L(\theta, \eta) \in \mathcal{B}(\bar{\theta}) \otimes \mathcal{B}(\eta)$ and all $f \in l^2(w)$. Actually, we will start off this discussion by suppressing the Hilbert space structures and simply considering $f(\eta) = \sum_k f_k \eta^k$, a formal infinite sum, and

$$L(\theta, \eta) = \sum_{ij} \lambda_{ij} \bar{\theta}^i \otimes \eta^j,$$

another formal infinite sum (that is, no convergence requirements). We now make the following formal calculation in order to motivate a definition:

$$\begin{aligned} \langle L(\theta, \eta), f(\eta) \rangle_w &= \sum_{ijk} \lambda_{ij}^* f_k \langle \bar{\theta}^i \otimes \eta^j, \eta^k \rangle_w \\ &= \sum_{ijk} \lambda_{ij}^* f_k \langle \eta^j, \eta^k \rangle_w \theta^i \\ &= \sum_{ijk} \lambda_{ij}^* f_k \delta_{j,k} w_j \theta^i \\ &= \sum_i \left(\sum_j \lambda_{ij}^* f_j w_j \right) \theta^i. \end{aligned} \tag{11}$$

The inner sum in (11) over $j \geq 0$ is an infinite sum of complex numbers for every $i \geq 0$ and so will not be considered as a formal infinite sum. But to consider it as an absolutely convergent series, say, we will have to impose conditions on the coefficients λ_{ij} and f_k of the above formal expressions. (The weights w_j are considered as given.) After all the inner sums in (11) have been well defined we are left with a formal expression, namely a formal power series in the variable θ . This can be used as such. Or, if one prefers, some more conditions can be imposed so that this series converges in some topological vector space, which could be $\mathcal{B}(\theta)$ with one of its many topological structures (norm topology, weak topology, etc.).

For example, we can use Hölder’s inequality to get the estimate

$$\sum_j |\lambda_{ij}^* f_j w_j| \leq \left(\sum_j |\lambda_{ij}|^p w_j \right)^{1/p} \left(\sum_j |f_j|^{p'} w_j \right)^{1/p'} \tag{12}$$

for any $1 < p < \infty$, where p' is the usual dual index of p . Consequently, if there exists some $1 < p < \infty$ such that the first sum on the right side of (12) is finite for every $i \geq 0$ and such that the second sum is finite, then we have that the formula (11) defines the inner product $\langle L(\theta, \eta), f(\eta) \rangle_w$ as a formal power series in θ .

We next consider the canonical orthogonal basis of $l^2(w)$ given by

$$\varepsilon_j = (0, \dots, 0, 1, 0, \dots)$$

(all zeros with one single occurrence of 1 in entry $j \in \mathbb{N}$). Then we have

$$\varepsilon_j(\theta) = \Phi(\varepsilon_j) = \theta^j.$$

So a necessary condition for (10) to hold is that

$$\theta^j = \langle K(\theta, \eta), \eta^j \rangle_w \tag{13}$$

for all $j \in \mathbb{N}$. We look for a solution $K(\theta, \eta) = \sum_{kl} a_{kl} \bar{\theta}^k \otimes \eta^l$, a formal series, for unknown coefficients $a_{kl} \in \mathbb{C}$. So we use our formal definition (11) to get

$$\langle K(\theta, \eta), \eta^j \rangle_w = \sum_k a_{kj}^* w_j \theta^k,$$

a formal power series in θ . So (13) holds if and only if

$$\theta^j = \sum_k a_{kj}^* w_j \theta^k \tag{14}$$

for all $j \in \mathbb{N}$. Of course, the left side of (14) is a *finite* series. Clearly, (14) is satisfied if and only if $a_{jk} = \delta_{j,k} / w_j$.

Putting this into the formula for the reproducing kernel gives us

$$\begin{aligned} K(\theta, \eta) &= \sum_{kl} a_{kl} \bar{\theta}^k \otimes \eta^l = \sum_{kl} \frac{\delta_{k,l}}{w_k} \bar{\theta}^k \otimes \eta^l = \sum_k \frac{1}{w_k} \bar{\theta}^k \otimes \eta^k \\ &= \sum_k \phi_k(\bar{\theta}) \otimes \phi_k(\eta). \end{aligned} \tag{15}$$

But this series is not convergent in the norm topology of the Hilbert space $\mathcal{B}(\bar{\theta}) \otimes \mathcal{B}(\eta)$, since the terms satisfy

$$\|\phi_k(\bar{\theta}) \otimes \phi_k(\eta)\| = 1.$$

However, there is another topology on $\mathcal{B}(\bar{\theta}) \otimes \mathcal{B}(\eta)$ for which this series is convergent. This other topology corresponds to the strong operator topology (see [7]) in the space $\text{End}(\text{Pre}(\theta))$ of bounded linear operators mapping $\mathcal{B}(\eta)$ to $\mathcal{B}(\theta)$. Without going into a lot of technical details, let us simply note that the formula (15) induces a unitary isomorphism $S: \mathcal{B}(\eta) \rightarrow \mathcal{B}(\theta)$ given in Dirac notation by

$$S = \sum_k |\phi_k(\theta)\rangle\langle\phi_k(\eta)|$$

which is an infinite sum of rank one operators, each of which has operator norm 1, and so is not convergent in the operator norm topology.

Nonetheless this infinite series of operators is convergent in the strong operator topology. It satisfies $S: \phi_k(\eta) \mapsto \phi_k(\theta)$ for the basis elements and therefore $S: f(\eta) \mapsto f(\theta)$ for $f \in l^2(w)$. This is quite tautological, since this is exactly what the mapping induced by the reproducing kernel, as given by the right side of equation (10), is supposed to do! And so it does. Intuitively, the expression in (15) expresses in this context the formula for the kernel of the Dirac delta as a ‘smooth’ object.

This section may seem like a lot of work to arrive at a result that appears to lack substance. However, the formula (15) will be used in the next section to define Toeplitz operators in a rather natural way. And these Toeplitz operators have some substantial, non-trivial properties. There may be other ways, still to be discovered, for defining these Toeplitz operators. But for the time being we seem to have found a reasonable approach.

Also, it is worth mentioning that the reproducing kernel K in (15) is not a function of two variables in the usual sense of those words. If it were, then $f(\theta)$ would be the ‘value’ of f at the ‘point’ θ . But $f(\theta)$ is an element in $\mathcal{B}(\theta)$ for all $f \in l^2(w)$. And θ itself is an element in the very same space $\mathcal{B}(\theta)$. So the sort of reproducing kernel as given in (15) is not included in the classical theory of reproducing kernel *functions* such as found in [1] and [8]. For example, the usual point-wise estimate, which follows immediately from the Cauchy-Schwarz inequality in the classical case, seems to have no good analogue here. Anyway, the Cauchy-Schwarz inequality does not apply to the general reproducing kernel formula in (10) nor to its special case (13).

But there are some properties of the reproducing kernel (15) that are analogous to standard properties of reproducing kernel functions. (See [1] and [8].) The correct interpretation of the following properties entails defining with some care notations which superficially appear obvious. We will not go into that analysis, but refer the reader to [9] where a similar analysis was made. We now present these properties:

1. Positive definite: $\sum_{n,m=1}^N \lambda_n^* \lambda_m K(\theta_n, \theta_m) \geq 0$ for $\lambda_1, \dots, \lambda_N \in \mathbb{C}$.

2. Complex symmetry: $K(\theta, \eta)^* = K(\eta, \theta)$.
3. Self reproducing: $K(\theta, \eta) = \langle K(\eta, \cdot), K(\theta, \cdot) \rangle_w$.
4. Positivity on the diagonal: $K(\theta, \theta) = \sum_k |\phi_k(\theta)\rangle\langle\phi_k(\theta)| = I_{\mathcal{B}(\theta)} \geq 0$.

Also, there is the question of constructing a space with a given $K(\theta, \eta)$ (satisfying properties 1 and 2) as its reproducing kernel. While this is a well known result in the theory of reproducing kernel *functions*, it appears that the analogous construction can not be made here since we are not dealing with functions.

4 Toeplitz Operators

Much of the above material about the reproducing kernel appears to be somewhat tautological in nature, though with a lot of technical details since here we are dealing with infinite dimensional spaces rather than the finite dimensional theory in [9]. But the real point of the reproducing kernel for us is that it can be extended in a ‘natural’ manner to the quantum plane and as such becomes one of the principle ingredients in defining a non-trivial theory of Toeplitz operators with symbols in the complex quantum plane, a non-commutative algebra for $q \neq 1$. As noted earlier in [10], passing from a symbol in a non-commutative algebra to its Toeplitz operator is an example of second quantization, since it is the quantization of a theory that is itself a non-commutative (i.e., quantum) theory to begin with. Nonetheless, the initial theory is still often referred to as the classical theory.

To start off this discussion we define the inner product of any finite sum or any infinite (formal) sum of the form

$$M(\theta, \eta) = \sum_{jk} m_{jk} \bar{\theta}^j \otimes \eta^k$$

with coefficients $m_{jk} \in \mathbb{C}$ for $j, k \geq 0$ and a basis element $\eta^a \bar{\eta}^b \in \mathbb{C}Q_q(\eta, \bar{\eta})$ in AW by

$$\begin{aligned} \langle M(\theta, \eta), \eta^a \bar{\eta}^b \rangle_w &:= \sum_j \left(\sum_k m_{jk}^* \langle \eta^k, \eta^a \bar{\eta}^b \rangle_w \right) \theta^j \\ &= \sum_j \left(\sum_k m_{jk}^* \langle \eta^{k+b}, \eta^a \rangle_w \right) \theta^j \\ &= \sum_j \left(\sum_k m_{jk}^* \delta_{k+b,a} w_a \right) \theta^j \\ &= w_a \sum_j m_{j,a-b}^* \theta^j \end{aligned} \tag{16}$$

provided that the sum on j converges in $\mathcal{B}(\theta)$, which is equivalent to

$$\sum_j w_j |m_{j,a-b}|^2 < \infty.$$

Or we could simply take (16) to be a formal series. Here we have introduced the convention that $m_{jk} = 0$ if $k < 0$. Then for any given arbitrary element

$$F = \sum_{ab} c_{ab} \eta^a \bar{\eta}^b \in \mathbb{C}Q_q(\eta, \bar{\eta})$$

(which is always a finite sum) such that for each pair (a, b) satisfying $c_{ab} \neq 0$ we have convergence in (16), we define

$$\langle M(\theta, \eta), F \rangle_w := \sum_{ab} c_{ab} \langle M(\theta, \eta), \eta^a \bar{\eta}^b \rangle_w,$$

which is also a finite sum. Notice that this inner product in general takes values in $\mathcal{B}(\theta)$ provided that we impose the convergence conditions, though in some specific cases the inner product could lie in some subspace of $\mathcal{B}(\theta)$.

Next we define the operator associated with the reproducing kernel K . This is the extension of the reproducing kernel to the quantum plane that we mentioned earlier.

Definition 1. The linear operator associated to the reproducing kernel of $\text{Pre}(\theta)$, $P_K: \mathbb{C}Q_q(\theta, \bar{\theta}) \rightarrow \mathbb{C}Q_q(\theta, \bar{\theta})$, is defined for all $F(\theta, \bar{\theta}) \in \mathbb{C}Q_q(\theta, \bar{\theta})$ by

$$P_K F(\theta) := \langle K(\theta, \eta), F(\eta, \bar{\eta}) \rangle_w. \tag{17}$$

This definition comes down to a special case of the discussion in the previous paragraph. So we must show that the inner product in (17) is well defined. Also P_K is actually a symmetric projection as we prove next.

Theorem 2. P_K is well defined and is a projection, that is, $P_K^2 = P_K$. Also, P_K is symmetric with respect to the inner product $\langle \cdot, \cdot \rangle_w$.

Remark 2. Since this inner product is not necessarily non-degenerate, we do not always have that the adjoint of P_K exists. Nonetheless, it makes sense to speak of the symmetry of P_K . And in those cases when the inner product is non-degenerate, we do have $P_K^* = P_K$.

Proof. We write $F_{ab}(\theta, \bar{\theta}) := \theta^a \bar{\theta}^b$ for the elements in the basis AW . We extend the notation established above by setting $\theta^n = 0$ and $w_n = 1$ for all integers $n < 0$. As we noted earlier, this basis AW is not orthogonal.

Acting with P_K on the basis elements F_{ab} in AW we obtain

$$\begin{aligned} (P_K F_{ab})(\theta) &= \langle K(\theta, \eta), F_{ab}(\eta, \bar{\eta}) \rangle_w = \langle K(\theta, \eta), \eta^a \bar{\eta}^b \rangle_w \\ &= \sum_j \frac{1}{w_j} \langle \eta^j, \eta^a \bar{\eta}^b \rangle_w \theta^j = \sum_j \frac{1}{w_j} \delta_{j+b, a} w_a \theta^j = \frac{w_a}{w_{a-b}} \theta^{a-b}. \end{aligned} \tag{18}$$

This result corresponds in this case to the convergence in (16) for all a, b . In this particular case, the infinite series collapses to at most one non-zero term, and so we have convergence not only in $\mathcal{B}(\theta)$ but even to an element in its subspace $\text{Pre}(\theta)$.

So by extending linearly to finite sums we see that the definition (17) makes sense. Moreover, (18) shows that $\text{Ran } P_K \subset \text{Pre}(\theta)$. In particular by putting $b = 0$ in (18) we find that $(P_K F_{a,0})(\theta) = F_{a,0}(\theta)$, which says $P_K: \theta^a \mapsto \theta^a$, that is, P_K acts as the identity on $\text{Pre}(\theta)$. So, $P_K^2 = P_K$ and $\text{Ran } P_K = \text{Pre}(\theta)$ follow immediately.

For the symmetry of P_K we calculate various matrix elements for P_K with respect to the elements in the basis AW . First for P_K acting on the right entry we obtain:

$$\begin{aligned} \langle F_{ab}, P_K F_{cd} \rangle_w &= \left\langle \theta^a \bar{\theta}^b, \frac{w_c}{w_{c-d}} \theta^{c-d} \right\rangle_w = \frac{w_c}{w_{c-d}} \delta_{a,b+c-d} w_a H(c-d) \\ &= \frac{w_a w_c}{w_{c-d}} \delta_{a-b,c-d} H(c-d), \end{aligned} \quad (19)$$

where H is the (discrete) Heaviside function $H: \mathbb{Z} \rightarrow \{0, 1\}$ defined by $H(n) := 1$ for $n \geq 0$ and $H(n) := 0$ for $n < 0$.

Next we calculate the matrix elements for P_K acting on the left entry:

$$\begin{aligned} \langle P_K F_{ab}, F_{cd} \rangle_w &= \left\langle \frac{w_a}{w_{a-b}} \theta^{a-b}, \theta^c \bar{\theta}^d \right\rangle_w = \frac{w_a}{w_{a-b}} H(a-b) \delta_{a-b+d,c} w_c \\ &= \frac{w_a w_c}{w_{a-b}} H(a-b) \delta_{a-b,c-d}. \end{aligned} \quad (20)$$

Since the matrix entries (19) and (20) with respect to the elements in the vector space basis AW of $\mathbb{C}Q_q(\theta, \bar{\theta})$ are equal, we can pass to finite linear combinations to get

$$\langle F, P_K G \rangle_w = \langle P_K F, G \rangle_w$$

for all $F, G \in \mathbb{C}Q_q(\theta, \bar{\theta})$, which is the desired symmetry of P_K . \square

Because of the previous proof we can think of P_K as a mapping

$$P_K: \mathbb{C}Q_q(\theta, \bar{\theta}) \rightarrow \text{Pre}(\theta) \subset \mathcal{B}(\theta).$$

For any $g \in \mathbb{C}Q_q(\theta, \bar{\theta})$ we define the linear map $M_g: \text{Pre}(\theta) \rightarrow \mathbb{C}Q_q(\theta, \bar{\theta})$ to be multiplication by g on the right, that is

$$M_g \phi := \phi g$$

for all $\phi \in \text{Pre}(\theta)$. It is straightforward to show that $\text{Ran } M_g \subset \mathbb{C}Q_q(\theta, \bar{\theta})$.

Definition 2. We define the *Toeplitz operator* associated to a symbol $g \in \mathbb{C}Q_q(\theta, \bar{\theta})$ to be

$$T_g = P_K M_g,$$

that is, right multiplication by g followed by the projection operator associated to the reproducing kernel K . We also write

$$T_g: \text{Pre}(\theta) \rightarrow \mathcal{B}(\theta)$$

with the domain of T_g defined by $\text{Dom}(T_g) := \text{Pre}(\theta) \subset \mathcal{B}(\theta)$ to indicate that T_g is a densely defined linear operator acting in (but not on) the Segal-Bargmann space $\mathcal{B}(\theta)$. An equivalent way to write this definition is

$$T_g f(\theta) = \langle K(\theta, \eta), f(\eta) g(\eta, \bar{\eta}) \rangle_w,$$

where $f \in \text{Pre}(\theta)$.

Actually, we have that $\text{Ran } T_g \subset \text{Pre}(\theta)$, but we prefer to consider the codomain to be the larger space $\mathcal{B}(\theta)$ in order to be able to apply the theory of densely defined linear operators acting in a Hilbert space. For example, see [7]. The definition of T_g can be expressed as this composition:

$$\text{Dom}(T_g) = \text{Pre}(\theta) \xrightarrow{M_g} \mathbb{C}Q_q(\theta, \bar{\theta}) \xrightarrow{P_K} \text{Pre}(\theta) \subset \mathcal{B}(\theta)$$

One of the first considerations here is to find necessary and sufficient conditions on g in order that T_g is bounded and so has a unique bounded extension to $\mathcal{B}(\theta)$. And when T_g is bounded, one would like some information, at best a formula but at least an estimate, about the operator norm of T_g . While bounded operators are important, we will also be interested in certain operators that are not bounded.

We have used the common way of defining Toeplitz operators: multiply by a symbol and then project back into the Hilbert space. However, we are making choices here that are somewhat arbitrary. For example, we could have used left multiplication instead of right multiplication. Also the choice of the Segal-Bargmann space is arbitrary too. We could just as well have chosen the anti-Segal-Bargmann space, which also has a reproducing kernel. And having chosen instead that space, we would again have two possible choices for the multiplication operator: left and right. In all, there are four different choices for the definition of Toeplitz operators, and we simply have opted for one of these. The other three choices lead to very similar theories and will not be discussed further.

Next we define the *Toeplitz mapping* $T: g \mapsto T_g$ giving us a linear function

$$T: \mathbb{C}Q_q(\theta, \bar{\theta}) \rightarrow \text{End}(\text{Pre}(\theta)),$$

where $\text{End}(\text{Pre}(\theta))$ is the complex vector space of all linear densely defined operators S acting in the Hilbert space $\mathcal{B}(\theta)$ with $\text{Dom } S = \text{Pre}(\theta)$ and leaving $\text{Pre}(\theta)$ invariant, that is $S(\text{Pre}(\theta)) \subset \text{Pre}(\theta)$. Because of this last condition $\text{End}(\text{Pre}(\theta))$ is closed under composition and so is an algebra. We also call T the *Toeplitz quantization*.

One verifies that $T_1 = I_{\text{Pre}(\theta)}$, the identity, as an immediate consequence of the fact that K is the reproducing kernel of $\text{Pre}(\theta)$. However, even though T is a map from one algebra to another algebra, it is not an algebra morphism. The product on the domain space is determined by $q \in \mathbb{C} \setminus \{0\}$, while the operator T_g is defined using the inner product which depends on the weights w_k . Even when the weights are functions of q (and so are not independent quantities) it is not expected that T preserves products, given what happens with Toeplitz operators in other contexts. Here is a result which shows what is happening in a ‘nice’ case.

Theorem 3. *Suppose that we have symbols g_1 and g_2 , but with $g_2 \in \text{Pre}(\theta)$, that is g_2 ‘depends’ only on θ . Then $T_{g_1}T_{g_2} = T_{g_2g_1}$.*

Proof. The point is since $g_2 \in \text{Pre}(\theta)$ we have that $T_{g_2} = P_K M_{g_2} = M_{g_2}$, because multiplication by g_2 leaves $\text{Pre}(\theta)$ invariant. So we calculate

$$T_{g_1}T_{g_2} = P_K M_{g_1} P_K M_{g_2} = P_K M_{g_1} M_{g_2} = P_K M_{g_2g_1} = T_{g_2g_1},$$

where the second to last equality is left to the reader to check. □

Remark 3. In the standard theories of Toeplitz operators, the symbols are functions and so commute. So essentially the same argument in such cases (with the corresponding hypothesis!) gives $T_{g_1}T_{g_2} = T_{g_1g_2}$. The fact that the map T in this context reverses the order of multiplication in this special case is not important as such. The equation $P_K M_g = M_g$ is not true for all symbols g and this is what is behind the fact that T does not respect multiplication. In fact, Theorem 3 implies that $T_{\bar{\theta}}T_{\theta} = T_{\theta\bar{\theta}}$. In the next calculation we actually will use something ever so slightly stronger, namely $T_{\bar{\theta}}T_{\theta} = T_{\theta\bar{\theta}} \neq 0$, but this will become clear later on. So for $q \neq 1$ we have

$$T_{\bar{\theta}\theta} = T_{q^{-1}\theta\bar{\theta}} = q^{-1}T_{\theta\bar{\theta}} = q^{-1}T_{\bar{\theta}}T_{\theta} \neq T_{\bar{\theta}}T_{\theta}.$$

Later on we will also calculate $T_{\theta}T_{\bar{\theta}}$ and see that this is yet another operator also not equal, in general, to $T_{\theta\theta}$.

Theorem 4. *The linear map $T: \mathbb{C}Q_q(\theta, \bar{\theta}) \rightarrow \text{End}(\text{Pre}(\theta))$ is a vector space monomorphism if and only if the inner product (2) is non-degenerate.*

Proof. We are looking for a necessary and sufficient for $\ker T = 0$. So we take $g \in \ker T$, which means that $T_g = 0$. In particular, this is equivalent to $T_g f_d = 0$ for all $d \geq 0$, where $f_d = \theta^d$, an orthogonal basis of $\text{Pre}(\theta) = \text{Dom}(T_g)$. We calculate

$$\begin{aligned} T_g f_d(\theta) &= \langle K(\theta, \eta), f_d(\eta) g(\eta, \bar{\eta}) \rangle_w = \sum_c \frac{1}{w_c} \langle \bar{\theta}^c \otimes \eta^c, \eta^d g(\eta, \bar{\eta}) \rangle_w \\ &= \sum_c \frac{1}{w_c} \langle \eta^c \bar{\eta}^d, g(\eta, \bar{\eta}) \rangle_w \theta^c. \end{aligned}$$

So, $T_g f_d(\theta) = 0$ for all $d \geq 0$ if and only if $\langle \eta^c \bar{\eta}^d, g(\eta, \bar{\eta}) \rangle_w = 0$ for all $c, d \geq 0$ if and only if $g(\eta, \bar{\eta})$ is orthogonal to $\mathbb{C}Q_q(\theta, \bar{\theta})$. So $\ker T = (\mathbb{C}Q_q(\theta, \bar{\theta}))^{\perp}$ and the result follows. \square

Remark 4. One way to interpret this theorem is that it tells us when the symbol of a Toeplitz operator is uniquely determined by the operator. In the finite dimensional theory presented in [9] the inner product is always non-degenerate and the corresponding result proved there is that the Toeplitz quantization is always a monomorphism. Moreover in the context of [9] the domain and codomain vector space of the Toeplitz quantization have the same *finite* dimension; therefore that Toeplitz quantization is automatically a vector space (but not algebra) isomorphism. Here one expects the situation to be more complicated due to the fact that the domain and codomain of T have infinite dimension. To be more precise one expects that T is not surjective, that is, there exist operators which are not Toeplitz. Moreover, in the current context Toeplitz operators are not necessarily bounded as we shall see momentarily.

We calculate next the Toeplitz operators for the basis elements $\theta^i \bar{\theta}^j$ of the symbol space $PG_{l,q}(\theta, \bar{\theta})$.

Theorem 5. *The action of the Toeplitz operator $T_{\theta^i \bar{\theta}^j}$ on the orthonormal basis elements $\phi_a(\theta) = w_a^{-1/2} \theta^a \in \text{Pre}(\theta)$ with $a \geq 0$ is given by*

$$(T_{\theta^i \bar{\theta}^j} \phi_a)(\theta) = \frac{w_{i+a}}{(w_a w_{i+a-j})^{1/2}} \phi_{i+a-j}(\theta). \tag{21}$$

Proof. We evaluate as follows:

$$\begin{aligned} (T_{\theta^i \bar{\theta}^j} \phi_a)(\theta) &= \langle K(\theta, \eta), \phi_a(\eta) \eta^i \bar{\eta}^j \rangle_w \\ &= \left\langle \sum_k \phi_k(\bar{\theta}) \otimes \phi_k(\eta), w_a^{-1/2} \eta^a \eta^i \bar{\eta}^j \right\rangle_w \\ &= w_a^{-1/2} \sum_k w_k^{-1/2} \langle \eta^{j+k}, \eta^{i+a} \rangle_w \phi_k(\theta) \\ &= w_a^{-1/2} \sum_k w_k^{-1/2} \delta_{j+k, i+a} w_{j+k} \phi_k(\theta) \\ &= \frac{w_{i+a}}{(w_a w_{i+a-j})^{1/2}} \phi_{i+a-j}(\theta). \end{aligned}$$

Recall that $\theta^n = 0$ and $w_n = 1$ for $n < 0$. So we also have put $\phi_n(\theta) = 0$ for $n < 0$ in the above calculation. □

This result determines T_g for all symbols $g \in \mathbb{C}Q_q(\theta, \bar{\theta})$ by linearity. Also, this result exhibits $T_{\theta^i \bar{\theta}^j}$ as a weighted shift operator with the degree of the shift being $i - j$. Next to see when this operator is bounded or compact we apply some basic functional analysis to obtain immediately:

Corollary 1. *First, $T_{\theta^i \bar{\theta}^j}$ is a bounded operator if and only if*

$$\|T_{\theta^i \bar{\theta}^j}\|_{op} = \sup \left\{ \frac{w_{i+a}}{(w_a w_{i+a-j})^{1/2}} \mid a \geq 0 \right\} < \infty,$$

where $\|\cdot\|_{op}$ denotes the operator norm. Secondly, $T_{\theta^i \bar{\theta}^j}$ is a compact operator if and only if

$$\lim_{a \rightarrow \infty} \frac{w_{i+a}}{(w_a w_{i+a-j})^{1/2}} = 0.$$

Knowing this, it is now easy to construct examples of Toeplitz operators which are not bounded provided that we are free to choose the weights w_k . Similarly, it is now straightforward to construct Toeplitz operators which are bounded, but not compact, given the same freedom. We also showed earlier that $T_1 = I_{\text{Pre}(\theta)}$, which is bounded but not compact.

We next obtain a consequence which relates the adjoint of a Toeplitz operator with symbol g to the Toeplitz operator with the conjugate symbol g^* .

Theorem 6. *Let $g \in \mathbb{C}Q_q(\theta, \bar{\theta})$ be arbitrary. Then*

$$\langle T_g f_1, f_2 \rangle_w = \langle f_1, T_{g^*} f_2 \rangle_w \tag{22}$$

for all $f_1, f_2 \in \text{Pre}(\theta)$.

Proof. It suffices to prove this for $g = \theta^i \bar{\theta}^j$ where $i, j \geq 0$ and for $f_1 = \phi_a$ and $f_2 = \phi_b$ where $a, b \geq 0$. So we compute each side of (22) for these choices. For the left side we get

$$\begin{aligned} \langle T_{\theta^i \bar{\theta}^j} \phi_a, \phi_b \rangle_w &= \frac{w_{i+a}}{(w_a w_{i+a-j})^{1/2}} \langle \phi_{i+a-j}, \phi_b \rangle_w \\ &= \frac{w_{i+a}}{(w_a w_{i+a-j})^{1/2}} \delta_{i+a-j, b}. \end{aligned} \tag{23}$$

Note that the Kronecker delta is enforcing the condition that $i + a - j = b \geq 0$. Next for the right side we have

$$\begin{aligned} \langle \phi_a, T_{(\theta^i \bar{\theta}^j)^*} \phi_b \rangle_w &= \langle \phi_a, T_{\theta^j \bar{\theta}^i} \phi_b \rangle_w \\ &= \frac{w_{j+b}}{(w_b w_{j+b-i})^{1/2}} \langle \phi_a, \phi_{j+b-i} \rangle_w \\ &= \frac{w_{j+b}}{(w_b w_{j+b-i})^{1/2}} \delta_{a, j+b-i}. \end{aligned} \tag{24}$$

This time the delta imposes the condition $j + b - i = a \geq 0$. So in each case we have the combined conditions $a, b \geq 0$ and $i + a = b + j$. Using these conditions one can see that the expressions in (23) and (24) are equal. \square

Remark 5. This result holds even when the inner product is degenerate. However, even when the inner product is non-degenerate all it says about the adjoint of T_g is that $T_{g^*} \subset (T_g)^*$, that is, the adjoint of T_g is an extension of T_{g^*} . Of course, such details are typical of densely defined operators. We recall that the Toeplitz operators are densely defined operators, all of which have the same dense domain, namely $\text{Pre}(\theta)$. Also, this relation $T_{g^*} \subset (T_g)^*$ shows a compatibility between our definition of the conjugation in $\mathbb{C}Q_q(\theta, \bar{\theta})$ and the adjoint of a Toeplitz operator.

Corollary 2. *If $g \in \mathbb{C}Q_q(\theta, \bar{\theta})$ is a self-adjoint element (meaning $g^* = g$), then the Toeplitz operator T_g is a symmetric operator.*

Proof. By Theorem 6 and $g^* = g$ we have

$$\langle T_g f_1, f_2 \rangle_w = \langle f_1, T_g f_2 \rangle_w$$

for all $f_1, f_2 \in \text{Pre}(\theta) = \text{Dom}(T_g)$. And this is exactly what it means for a densely defined operator to be symmetric. (See [7].) \square

Remark 6. If $g^* = g$, then it behooves us to study the self-adjoint extensions of the symmetric operator T_g . This remains an open problem.

Corollary 3. *Every Toeplitz operator T_g is closable. Moreover, its closure satisfies $\bar{T}_g = (T_g)^{**} \subset (T_{g^*})^*$.*

Proof. This follows rather directly from Theorem VIII.1b in [7]. We get from that reference that T_g is closable if and only if $\text{Dom}(T_g)^*$ is a dense subspace. But this is so since $\text{Dom}(T_g)^* \supset \text{Dom } T_{g^*} = \text{Pre}(\theta)$ and $\text{Pre}(\theta)$ is dense. The equality $\bar{T}_g = (T_g)^{**}$ follows from the cited theorem. The inclusion $(T_g)^{**} \subset (T_{g^*})^*$ follows from Theorem 6. \square

We now analyze various particular cases of (21). First for $i = j = 0$ we have

$$(T_1 \phi_a)(\theta) = \frac{w_a}{(w_a w_a)^{1/2}} \phi_a(\theta) = \phi_a(\theta)$$

for all $a \geq 0$, so that $T_1 = I_{\text{Pre}(\theta)}$, the identity map, as already noted above.

For the case $i = j$ of (21) we obtain for all $a \geq 0$ that

$$(T_{\theta^i \bar{\theta}^i} \phi_a)(\theta) = \frac{w_{i+a}}{(w_a w_{i+a-i})^{1/2}} \phi_{i+a-i}(\theta) = \frac{w_{i+a}}{(w_a w_a)^{1/2}} \phi_a(\theta) = \frac{w_{i+a}}{w_a} \phi_a(\theta).$$

Hence the basis $\phi_a(\theta)$ diagonalizes simultaneously the family of symmetric operators $T_{\theta^i \bar{\theta}^i}$ for $i \geq 0$. By Corollary 2 we see that $T_{\theta^i \bar{\theta}^i}$ is symmetric.

Next we consider (21) for the case $j = 0$ and get

$$(T_{\theta^i} \phi_a)(\theta) = \frac{w_{i+a}}{(w_a w_{i+a})^{1/2}} \phi_{i+a}(\theta) = \frac{w_{i+a}^{1/2}}{w_a^{1/2}} \phi_{i+a}(\theta)$$

or, equivalently, $T_{\theta^i} : \theta^a \mapsto \theta^{i+a}$ which itself can be written as $T_{\theta^i} = M_{\theta^i}$. Of course, this also follows from the definition $T_{\theta^i} = P_K M_{\theta^i} = M_{\theta^i}$, since M_{θ^i} leaves $\text{Pre}(\theta)$ invariant and P_K acts as the identity on $\text{Pre}(\theta)$. A subcase here is $T_\theta = M_\theta$, which merits the name *creation operator* since it increases by 1 the degree of the elements in $\text{Pre}(\theta)$, which are exactly the polynomials in θ . Moreover, $T_{\theta^i} = (T_\theta)^i$ also is immediate. (Recall that T_θ leaves $\text{Pre}(\theta)$ invariant, and so $(T_\theta)^i$ is defined.)

So, if T_θ is bounded (resp., compact), then T_{θ^i} is bounded (resp., compact) for all $i \geq 1$. In the Hilbert space introduced by Bargmann in [2], one has $w_a = a!$ and $\theta = z$, so that $T_{\theta^i} = T_{z^i}$ is not bounded for $i \geq 1$ in that space. One might expect that with w_a being some reasonable deformation of the factorial function the corresponding operators T_{θ^i} would also not be bounded. However, the boundedness of these operators depends completely on the choice of weights w_a , nothing else. So for some choices (such as, for example, w_a constant) these operators will be bounded.

Yet another interesting special case of (21) is when $i = 0$. Then we have

$$(T_{\bar{\theta}^j} \phi_a)(\theta) = \frac{w_a}{(w_a w_{a-j})^{1/2}} \phi_{a-j}(\theta) = \left(\frac{w_a}{w_{a-j}} \right)^{1/2} \phi_{a-j}(\theta)$$

or, in terms of the unnormalized monomials,

$$T_{\bar{\theta}^j} : \theta^a \mapsto \frac{w_a}{w_{a-j}} \theta^{a-j}$$

for all $a \geq 0$. In particular, for $j = 1$ we can see that

$$T_{\bar{\theta}} : \theta^a \mapsto \frac{w_a}{w_{a-1}} \theta^{a-1}$$

deserves to be called an *annihilation operator*, since it lowers the degree of any non-constant polynomial by 1 and sends constants to zero. A simple argument shows

that $T_{\bar{\theta}^j} = (T_{\bar{\theta}})^j$. And similar to the above situation, we see that if $T_{\bar{\theta}}$ is bounded (resp., compact), then $T_{\bar{\theta}^j}$ is bounded (resp., compact) for all $j \geq 1$. Again, the space in [2] is an important example for which the operators $T_{\bar{\theta}^j}$ are not bounded. And again, the boundedness of these operators depends solely on the weights.

Using Theorem 3 in the first equality and two properties established above in the second equality, we see that

$$T_{\theta^i \bar{\theta}^j} = T_{\bar{\theta}^j} T_{\theta^i} = (T_{\bar{\theta}})^j (T_{\theta})^i.$$

The last expression here is in anti-Wick order, which by definition means a product of creation and annihilation operators such that all of the creation operators are to the right of all of the annihilation operators. By linearity every Toeplitz operator T_g will then be a sum of terms, each of which is in anti-Wick order. Because of this property one says that the Toeplitz quantization is an *anti-Wick quantization*.

There is another way of viewing the annihilation operator $T_{\bar{\theta}}$. We note that in the case when $w_a = a!$ as in [2], we have that

$$T_{\bar{\theta}}: \theta^a \mapsto \frac{w_a}{w_{a-1}} \theta^{a-1} = \frac{a!}{(a-1)!} \theta^{a-1} = a \theta^{a-1},$$

which is the derivative operator from elementary calculus. So we can think of $T_{\bar{\theta}}$ in this more general context as a deformation of the classical derivative. We call it the *w-deformed derivative* and denote it by ∂_w . If we define the *w*-deformed integers to be $[n]_w := w_n/w_{n-1}$ for every integer $n \geq 1$ and $[0]_w := 0$, then we have

$$\partial_w = T_{\bar{\theta}}: \theta^a \mapsto [a]_w \theta^{a-1}.$$

The upshot of this paragraph is merely a change to another notation that is more compatible with notations used elsewhere in the literature, nothing else really.

Notice again that $T_{\bar{\theta}^j} T_{\theta^i} = T_{\theta^i \bar{\theta}^j}$ follows from Theorem 3. We now calculate $T_{\theta^i} T_{\bar{\theta}^j}$ using the individual formulas derived above for T_{θ^i} and $T_{\bar{\theta}^j}$. So,

$$\phi_a \xrightarrow{T_{\bar{\theta}^j}} \left(\frac{w_a}{w_{a-j}} \right)^{1/2} \phi_{a-j} \xrightarrow{T_{\theta^i}} \left(\frac{w_a}{w_{a-j}} \right)^{1/2} \left(\frac{w_{i+a-j}}{w_{a-j}} \right)^{1/2} \phi_{a-j+i}$$

which gives

$$T_{\theta^i} T_{\bar{\theta}^j} \phi_a = \frac{(w_a w_{i+a-j})^{1/2}}{w_{a-j}} \phi_{a-j+i}.$$

This is different from the formula (21) derived above for $T_{\theta^i \bar{\theta}^j}$. In particular, for the case $i = j = 1$ which we left unfinished earlier we have

$$T_{\theta} T_{\bar{\theta}} \phi_a = \frac{w_a}{w_{a-1}} \phi_a = [a]_w \phi_a.$$

For the sake of completeness we note that the operator $N_{\theta} := T_{\theta} T_{\bar{\theta}}$ is called the *w-deformed number operator*. On the other hand from equation (21) we have that

$$T_{\bar{\theta}} T_{\theta} \phi_a = T_{\theta \bar{\theta}} \phi_a = \frac{w_{a+1}}{w_a} \phi_a = [a+1]_w \phi_a.$$

5 Canonical Commutation Relations

This final section is a continuation of the two calculations just made at the end of the last section. First, we define the q -commutator of any two elements a and b in any (associative, say) algebra over \mathbb{C} by

$$[a, b]_q := ab - qba,$$

where $q \in \mathbb{C} \setminus \{0\}$. This is the commutator which is appropriate for the study of q -deformations.

The Toeplitz quantization starts with the ‘classical’ space $\mathbb{C}Q_q(\theta, \bar{\theta})$ of symbols and from them produces operators acting in the ‘quantum’ Segal-Bargmann space $\mathcal{B}(\theta)$. The point here is that before the Toeplitz quantization we have the *homogeneous* q -commutation relation in $\mathbb{C}Q_q(\theta, \bar{\theta})$, namely

$$[\theta, \bar{\theta}]_q = \theta\bar{\theta} - q\bar{\theta}\theta = 0. \quad (25)$$

Speaking roughly without going into the rigorous details, in quantum theory we have creation operators and annihilations operators which come in pairs, say A for an annihilation operator and A^* for its corresponding creation operator. Then a typical commutation relation is something more or less like

$$[A, A^*] = I, \quad \text{the identity.}$$

This is called a *canonical commutation relation*. So in general in a quantum theory we expect *inhomogeneous* canonical commutation relations.

Now the Toeplitz quantization of the q -commutator $[\theta, \bar{\theta}]_q$ is

$$[T_\theta, T_{\bar{\theta}}]_q = T_\theta T_{\bar{\theta}} - qT_{\bar{\theta}} T_\theta.$$

But recall that T_θ is the creation operator and that $T_{\bar{\theta}}$ is the annihilation operator; so this q -commutator has the form $[A^*, A]_q$. And this is not the form of a canonical commutation relation. However, since it is homogeneous and $q \neq 0$ we can trivially rewrite (25) as

$$[\bar{\theta}, \theta]_{q^{-1}} = \bar{\theta}\theta - q^{-1}\theta\bar{\theta} = 0. \quad (26)$$

In fact we have an identification $\mathbb{C}Q_q(\theta, \bar{\theta}) \cong \mathbb{C}Q_{q^{-1}}(\bar{\theta}, \theta)$. What this means is that at the classical level we can not distinguish the q -deformed theory associated to the holomorphic (resp., anti-holomorphic) variable θ (resp., $\bar{\theta}$) from the q^{-1} -deformed theory associated to the holomorphic (resp., anti-holomorphic) variable $\bar{\theta}$ (resp., θ). (The previous sentence does not contain a typographical error. It makes perfect sense to consider $\bar{\theta}$ as a holomorphic variable whose associated anti-holomorphic variable is θ .) Another way of saying this is that as far as our theory is concerned only with the classical level we have no way to distinguish between q -deformations and q^{-1} -deformations nor between holomorphic and anti-holomorphic variables.

However, the quantizations of θ and $\bar{\theta}$ are distinguishable. In this sense Toeplitz quantization breaks a symmetry. And the choice of quantization determines exactly how the symmetry is broken. For example, if we define a Toeplitz quantization as in this paper, but using instead the anti-Segal-Bargmann space $\mathcal{B}(\bar{\theta})$ as the Hilbert

space in which the quantized operators act, then θ quantizes to the annihilation operator while $\bar{\theta}$ quantizes to the creation operator, just the reverse of what we have obtained with the present Toeplitz quantization in the Segal-Bargmann space $\mathcal{B}(\theta)$. These comments indicate that naming a particular order in $\mathbb{C}Q_q(\theta, \bar{\theta})$ the anti-Wick ordering (that is, all creation operators to the right of all annihilation operators) is not justifiable in terms of mathematical structures of $\mathbb{C}Q_q(\theta, \bar{\theta})$ alone. We have simply decided to follow the nomenclature used in [3] as indicated earlier.

Now the Toeplitz quantization of the q^{-1} -commutator $[\bar{\theta}, \theta]_{q^{-1}}$ is

$$[T_{\bar{\theta}}, T_{\theta}]_{q^{-1}} = T_{\bar{\theta}}T_{\theta} - q^{-1}T_{\theta}T_{\bar{\theta}}.$$

And this has the virtue of being of the form $[A, A^*]$. So we require this canonical commutation relation to hold:

$$[T_{\bar{\theta}}, T_{\theta}]_{q^{-1}} = T_{\bar{\theta}}T_{\theta} - q^{-1}T_{\theta}T_{\bar{\theta}} = I_{\text{Pre}(\theta)}, \quad \text{the identity on } \text{Pre}(\theta). \quad (27)$$

This gives us the recursion relation

$$[a + 1]_w - q^{-1}[a]_w = 1$$

for all $a \geq 0$. But we already have $[0]_w = 0$. So the sequence $[a]_w$ is uniquely determined by q (or by q^{-1} if one wishes to consider this as the primary parameter). It is rather straightforward to find an explicit formula for $[a]_w$. The next definition is standard, though not universal. See [3] for a different, more symmetric definition.

Definition 3. Let $r \in \mathbb{C}$. For each integer $n \geq 0$ we define

$$[n]_r := 1 + r + r^2 + \dots + r^{n-1} \quad \text{if } n \geq 1$$

and $[0]_r := 0$. This is called the r -deformation of n .

For example, $[1]_r = 1$ and $[2]_r = 1 + r$. Taking $r = 1$ gives $[n]_r = n$ for every integer $n \geq 0$. This justifies saying that these are deformations of the integers and that r in the deformation parameter. If $r \neq 1$, then we have the alternative expression $[n]_r = \frac{1-r^{n+1}}{1-r}$, which often appears in the literature.

Proposition 1. *The unique solution of the recursion relation*

$$[a + 1]_w - q^{-1}[a]_w = 1$$

for all integers $a \geq 0$ with $[0]_w = 0$ is $[a]_w = [a]_{q^{-1}}$.

Proof. The recursion relation for $[n]_r$ is $[n+1]_r - r[n]_r = 1$, as the reader can easily check. Taking $r = q^{-1}$ shows that the sequences $[a]_w$ and $[a]_{q^{-1}}$ satisfy the same recursion relation. But they both start out with $[0]_w = 0 = [0]_{q^{-1}}$, which ends the proof. \square

Now it is a matter of going from the deformed integers $[a]_w = [a]_{q^{-1}}$ to the weights w_k . Now for every integer $a \geq 1$ we have

$$[a]_{q^{-1}} = [a]_w = \frac{w_a}{w_{a-1}} \quad (28)$$

by definition of $[a]_w$. It turns out that $[0]_w = 0$ carries no information about the weights. Then (28) gives a sequence of identities

$$w_1 = [1]_{q^{-1}}w_0, \quad w_2 = [2]_{q^{-1}}w_1, \quad w_3 = [3]_{q^{-1}}w_2,$$

and so on. The solution for $k \geq 1$ is clearly

$$w_k = [k]!_{q^{-1}}w_0,$$

where the q^{-1} -deformed factorial is defined by

$$[k]!_{q^{-1}} := [k]_{q^{-1}}[k - 1]_{q^{-1}} \cdots [2]_{q^{-1}}[1]_{q^{-1}}$$

and where $w_0 > 0$ is arbitrary. In this way we have defined a unique sequence (up to a multiplicative positive constant) of weights $w_k = w_k(q)$, which are functions of the one parameter q such that

$$[T_{\bar{\theta}}, T_{\theta}]_{q^{-1}} = T_{\bar{\theta}}T_{\theta} - q^{-1}T_{\theta}T_{\bar{\theta}} = I_{\text{Pre}(\theta)}.$$

In particular, $[T_{\bar{\theta}}, T_{\theta}]_{q^{-1}}$ is bounded. By putting the deformation parameter q equal to 1 and normalizing $1 \in \mathbb{C}Q_q(\theta, \bar{\theta})$ by putting $w_0 = 1$, we recover the weights $w_k = k!$ of the Hilbert space \mathcal{H} defined in (4). Recall that the Segal-Bargmann space based on the phase space \mathbb{C} in [2] is the closed subspace of \mathcal{H} consisting of the holomorphic functions in \mathcal{H} .

If we wish to have some other operator instead of the identity on the ‘right side’ of the canonical commutation relation, the same method applies to give the corresponding weights.

6 Concluding Remarks

Since the Toeplitz operators introduced here are only densely defined, one has the standard problems in the analysis of such operators. For example, we know they are closable, but can we identify exactly what the closure is? And if a Toeplitz operator is symmetric, then we would like to know what its self-adjoint extensions are. In particular, we would like to know exactly what are the conditions for a Toeplitz operator to be essentially self-adjoint.

We have given necessary and sufficient conditions for the Toeplitz $T_{\theta^i \bar{\theta}^j}$ to be bounded or compact. But the full story remains to be told for T_g where g is an arbitrary symbol, though our results allow us to form sufficient conditions for boundedness and compactness by expanding T_g as a linear combination of $T_{\theta^i \bar{\theta}^j}$ ’s. We expect such conditions to be far from necessary.

Another possibility for further research is to define coherent states in this context, much as was done in [3] in a similar finite dimensional case. This would allow the introduction of a coherent state transform and a coherent state quantization. (Also see [4].) This would relate the material in this paper with yet another aspect of mathematical physics. Also it might be of interest to study in more detail the classical space $\mathbb{C}Q_q(\theta, \bar{\theta})$ from a physics point of view as a sort of non-commutative phase space.

Given the positive result in the finite dimensional case presented in [9] it seems reasonable to conjecture that $\mathbb{C}Q_q(\theta, \bar{\theta})$ also has its own reproducing kernel, at least in the case when its inner product is non-degenerate. We also leave this as a problem for another day.

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