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SOME COMMON FIXED POINT RESULTS FOR RATIONAL TYPE CONTRACTION MAPPINGS IN PARTIALLY ORDERED METRIC SPACES

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Abstract. The purpose of this paper is to establish some common fixed point results for f-nondecreasing mappings which satisfy some nonlinear contractions of rational type in the framework of metric spaces endowed with a partial order. Also, as a consequence, a result of integral type for such class of mappings is obtained. The proved results generalize and extend some of the results of J. Harjani, B. Lopez, K. Sadarangani (2010) and D. S. Jaggi (1977).

Keywords: common fixed point; rational type contraction mapping; compatible mapping; weakly compatible mapping; ordered metric space

MSC 2010: 46T99, 41A50, 47H10, 54H25

1. Introduction and preliminaries

The Banach contraction mapping is one of the pivotal results of analysis. It is a very popular tool for solving existence problems in different fields of mathematics. There are a lot of generalizations of the Banach contraction principle in literature (see [1]–[11]).

Ran and Reurings [11] extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations, while Nieto and Rodríguez-López [10] extended their result and applied their main theorems to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Bhaskar and Lakshmikantham [2] introduced the concept of mixed monotone mappings and obtained some coupled fixed point results. Also, they applied their results to a first order differential equation with periodic boundary conditions. Recently, many researchers have obtained fixed point and common fixed point results in metric spaces and partially ordered metric spaces. The purpose of this paper is to establish some common fixed point results for a rational type contraction mappings in metric spaces endowed with a partial order.

First, we recall some basic definitions.

Let M be a nonempty subset of a metric space (X,d), a point $x \in M$ is a *common fixed (coincidence) point* of f and T if $x = fx = Tx$ ($fx = Tx$). The set of fixed points (respectively, coincidence points) of f and T is denoted by $F(f,T)$ (respectively, $C(f,T)$). The mappings $T, f\colon M \to M$ are called *commuting* if $Tfx = fTx$ for all $x \in M$; *compatible* if $\lim d(Tfx_n, fTx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim Tx_n = \lim fx_n = t$ for some t in M; *weakly compatible* if they commute at their coincidence points, i.e., if $fTx = Tfx$ whenever $fx = Tx$.

Suppose $(X, \leqslant)$ is a partially ordered set and $T, f\colon X \to X$. T is said to be *monotone f-nondecreasing* if for all $x, y \in X$,

$$fx \leqslant fy \quad \text{implies} \quad Tx \leqslant Ty. \tag{1.1}$$

If f is the identity mapping, then T is *monotone nondecreasing.*

A subset W of a partially ordered set X is said to be *well ordered* if every two elements of W are comparable.

2. Main results

Theorem 2.1. *Let $(X, \leqslant)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Suppose that T and f are continuous self-mappings on X, $T(X) \subseteq f(X)$, T is a monotone f-nondecreasing mapping and*

$$d(Tx,Ty) \leqslant \alpha\Big(\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}\Big) + \beta(d(fx,fy)) \tag{2.1}$$

for all $x, y \in X$ for which $f(x)$ and $f(y)$ are comparable, and for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$.

If there exists $x_0 \in X$ such that $f(x_0) \leqslant T(x_0)$ and T and f are compatible, then T and f have a coincidence point.

Proof. Let $x_0 \in X$ be such that $f(x_0) \leqslant T(x_0)$. Since $T(X) \subseteq f(X)$, we can choose $x_1 \in X$ so that $fx_1 = Tx_0$. Since $Tx_1 \in f(X)$, there exists $x_2 \in X$ such that $fx_2 = Tx_1$. By induction, we can construct a sequence $\{x_n\}$ in X such that $fx_{n+1} = Tx_n$ for every $n \geqslant 0$.

Since $f(x_0) \leqslant T(x_0) = f(x_1)$, T is a monotone f-nondecreasing mapping, $T(x_0) \leqslant T(x_1)$. Similarly, since $f(x_1) \leqslant f(x_2)$, we have $T(x_1) \leqslant T(x_2)$, and $f(x_2) \leqslant f(x_3)$. Continuing, we obtain

$$T(x_0) \leqslant T(x_1) \leqslant T(x_2) \leqslant \ldots \leqslant T(x_n) \leqslant T(x_{n+1}) \leqslant \ldots.$$

We suppose that $d(T(x_n), T(x_{n+1})) > 0$ for all n. If not then $T(x_{n+1}) = T(x_n)$ for some n, $T(x_{n+1}) = f(x_{n+1})$, i.e. T and f have a coincidence point x_{n+1}, and so we have the result.

Consider

$$\begin{aligned} d(Tx_{n+1}, Tx_n) &\leqslant \alpha\Big(\frac{d(fx_{n+1}, Tx_{n+1})d(fx_n, Tx_n)}{d(fx_{n+1}, fx_n)}\Big) + \beta(d(fx_{n+1}, fx_n)) \\ &= \alpha(d(Tx_n, Tx_{n+1})) + \beta(d(Tx_n, Tx_{n-1})), \end{aligned} \tag{2.2}$$

which implies that

$$d(Tx_{n+1}, Tx_n) \leqslant \frac{\beta}{1-\alpha} d(Tx_n, Tx_{n-1}). \tag{2.3}$$

Using mathematical induction we have

$$d(Tx_{n+1}, Tx_n) \leqslant \Big(\frac{\beta}{1-\alpha}\Big)^n d(Tx_1, Tx_0). \tag{2.4}$$

Put $k = \beta/(1-\alpha) < 1$. Now, we shall prove that $\{Tx_n\}$ is a Cauchy sequence. For $m \geqslant n$, we have

$$\begin{aligned} d(Tx_m, Tx_n) &\leqslant d(Tx_m, Tx_{m-1}) + d(Tx_{m-1}, Tx_{m-2}) + \ldots + d(Tx_{n+1}, Tx_n) \\ &\leqslant (k^{m-1} + k^{m-2} + \ldots + k^n) d(Tx_1, Tx_0) \\ &\leqslant \Big(\frac{k^n}{1-k}\Big) d(Tx_1, Tx_0), \end{aligned} \tag{2.5}$$

which implies that $d(Tx_m, Tx_n) \to 0$ as $m, n \to \infty$. Thus $\{Tx_n\}$ is a Cauchy sequence in a complete metric space X. Therefore there exits $u \in X$ such that $\lim Tx_n = u$. By the continuity of T, we have $\lim\limits_{n\to\infty} T(Tx_n) = Tu$. Since $fx_{n+1} = Tx_n \to u$ and the pair (T, f) is compatible, we have $\lim\limits_{n\to\infty} d(f(Tx_n), T(fx_n)) = 0$. By the triangular inequality, we have

$$d(Tu, fu) \leqslant d(Tu, T(fx_n)) + d(T(fx_n), f(Tx_n)) + d(f(Tx_n), fu).$$

Letting $n \to \infty$ and using the fact that T and f are continuous, we get $d(Tu, fu) = 0$, i.e. $Tu = fu$ and u is a coincidence point of T and f. $\square$

If $\beta = 0$, we have the following result.

Corollary 2.2. *Let $(X, \leqslant)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T and f are continuous self-mappings on X, $T(X) \subseteq f(X)$, T is a monotone f-nondecreasing mapping and*

$$d(Tx, Ty) \leqslant \alpha\Big(\frac{d(fx, Tx)d(fy, Ty)}{d(fx, fy)}\Big) \tag{2.6}$$

for all $x, y \in X$ for which $f(x)$ and $f(y)$ are comparable, and for some $\alpha < 1$.

If there exists $x_0 \in X$ such that $f(x_0) \leqslant T(x_0)$ and T and f are compatible, then T and f have a coincidence point.

In what follows, we prove that Theorem 2.1 is still valid for T not necessarily continuous, assuming the following hypothesis in X:

If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x_n \leqslant x$ for all n.

Theorem 2.3. *Let $(X, \leqslant)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a metric space. Suppose that T and f are self-mappings on X, $T(X) \subseteq f(X)$, T is a monotone f-nondecreasing mapping and*

$$d(Tx, Ty) \leqslant \alpha\Big(\frac{d(fx, Tx)d(fy, Ty)}{d(fx, fy)}\Big) + \beta(d(fx, fy)) \tag{2.7}$$

for all $x, y \in X$ for which $f(x)$ and $f(y)$ are comparable, and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$.

Assume that there exists $x_0 \in X$ such that $f(x_0) \leqslant T(x_0)$ and $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$. Then $x_n \leqslant x$ for all $n \in \mathbb{N}$.

If $f(X)$ is a complete subspace of X, then T and f have a coincidence point.

Further, if T and f are weakly compatible, then T and f have a common fixed point. Moreover, the set of common fixed points of T and f is well ordered if and only if T and f have one and only one common fixed point.

P r o o f. Following the proof of Theorem 2.1, we have that $\{Tx_n\}$ is a Cauchy sequence. As $fx_{n+1} = Tx_n$, so $\{fx_n\}$ is a Cauchy sequence in $(f(X), d)$. Since $f(X)$ is complete, there is $fu \in f(X)$ such that $\lim\limits_{n\to\infty} T(x_n) = \lim\limits_{n\to\infty} f(x_n) = f(u)$. Notice that the sequences $\{T(x_n)\}$ and $\{f(x_n)\}$ are nondecreasing. Then from our assumptions we have $T(x_n) \leqslant f(u)$ and $f(x_n) \leqslant f(u)$ for all n. Keeping in mind that T is montone f-nondecreasing we get $T(x_n) \leqslant T(u)$ for all n. Letting n tend to ∞ we obtain $f(u) \leqslant T(u)$.

Suppose $f(u) < T(u)$ (otherwise we are done). Construct a sequence $\{u_n\}$ as $u_0 = u$ and $fu_{n+1} = Tu_n$ for all n. An argument similar to that in the proof of Theorem 2.1 yields that $\{fu_n\}$ is a nondecreasing sequence and $\lim_{n\to\infty} f(u_n) = \lim_{n\to\infty} T(u_n) = f(v)$ for some $v \in X$. From our assumptions it follows that $\sup_n f(u_n) \leqslant f(v)$ and $\sup_n T(u_n) \leqslant f(v)$.

Notice that

$$f(x_n) \leqslant f(u) < f(u_1) \leqslant \ldots \leqslant f(u_n) \leqslant \ldots \leqslant f(v).$$

Now, if there is $n_0 \geqslant 1$ with $f(x_{n_0}) = f(u_{n_0})$, then $f(x_{n_0}) = f(u) = f(u_{n_0}) = f(u_1) = T(u)$.

Suppose that $f(u_n) \neq f(x_n)$ for all $n \geqslant 1$. Then from the contraction assumption we obtain

$$\begin{aligned} d(fx_{n+1}, fu_{n+1}) &= d(Tx_n, Tu_n) \\ &\leqslant \alpha\Big(\frac{d(fx_n, Tx_n)d(fu_n, Tu_n)}{d(fx_n, fu_n)}\Big) + \beta d(fx_n, fu_n). \end{aligned}$$

Letting n tend to ∞ we get $d(fu, fv) \leqslant (\beta)d(fu, fv)$, which implies that $f(u) = f(v)$ since $\beta < 1$. This implies $f(u) = f(v) = f(u_1) = Tu$. Hence we conclude that u is a coincidence point of T and f.

Now suppose that T and f are weakly compatible. Let $w = T(z) = f(z)$. Then $T(w) = T(f(z)) = f(T(z)) = f(w)$. Consider

$$d(T(z), T(w)) \leqslant \alpha\Big(\frac{d(fz, Tz)d(fw, Tw)}{d(fz, fw)}\Big) + \beta d(fz, fw) \leqslant \beta d(Tz, Tw).$$

This implies that $d(Tz, Tw) = 0$, as $\beta < 1$. Therefore, $T(w) = f(w) = w$.

Now suppose that the set of common fixed points of T and f is well ordered. We claim that the common fixed point of T and f is unique. Assume on the contrary that $Tu = fu = u$ and $Tv = fv = v$ but $u \neq v$. Consider

$$d(u, v) = d(Tu, Tv) \leqslant \alpha\Big(\frac{d(fu, Tu)d(fv, Tv)}{d(fu, fv)}\Big) + \beta d(fu, fv) \leqslant \beta d(u, v).$$

This implies that $d(u, v) = 0$, as $\beta < 1$. Hence $u = v$. Conversely, if T and f have only one common fixed point then the set of common fixed points of f and T being a singleton is well ordered. $\square$

If $\beta = 0$, we have the following result.

Corollary 2.4. *Let $(X, \leqslant)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a metric space. Suppose that T and f are self-mappings on X, $T(X) \subseteq f(X)$, T is a monotone f-nondecreasing mapping and*

$$d(Tx, Ty) \leqslant \alpha\Big(\frac{d(fx, Tx)d(fy, Ty)}{d(fx, fy)}\Big) \tag{2.8}$$

for all $x, y \in X$ for which $f(x)$ and $f(y)$ are comparable, and for some $\alpha < 1$.

Assume that there exists $x_0 \in X$ such that $f(x_0) \leqslant T(x_0)$ and $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$. Then $x_n \leqslant x$.

If fX is a complete subspace of X, then T and f have a coincidence point.

Further, if T and f are weakly compatible, then T and f have a common fixed point. Moreover, the set of common fixed points of T and f is well ordered if and only if T and f have one and only one common fixed point.

Remark 2.1. If $f = I$ (identity mapping) in Theorems 2.1 and 2.3, then we have Theorems 2.2 and 2.3 of Harjani, Lopez and Sadarangani [8].

Other consequences of our results for the mappings involving contractions of integral type are the following.

Denote by Λ the set of functions $\mu\colon [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

(h1) μ is a Lebesgue-integrable mapping on each compact subset of $[0, \infty)$;

(h2) for any $\varepsilon > 0$ we have $\int_0^\varepsilon \mu(t)\,\mathrm{d}t > 0$.

Corollary 2.5. *Let $(X, \leqslant)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T and f are continuous self-mappings on X, $T(X) \subseteq f(X)$, T is a monotone f-nondecreasing mapping and*

$$\int_0^{d(Tx,Ty)} \psi(t)\,\mathrm{d}t \leqslant \alpha \int_0^{\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}} \psi(t)\,\mathrm{d}t + \beta \int_0^{d(fx,fy)} \psi(t)\,\mathrm{d}t \tag{2.9}$$

for all $x, y \in X$ for which $f(x)$ and $f(y)$ are comparable, $\psi \in \Lambda$, and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$.

If there exists $x_0 \in X$ such that $f(x_0) \leqslant T(x_0)$ and T and f are compatible, then T and f have a coincidence point.

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