Youwei Zhang

Solvability of a generalized third-order left focal problem at resonance in Banach spaces

SOLVABILITY OF A GENERALIZED THIRD-ORDER LEFT FOCAL PROBLEM AT RESONANCE IN BANACH SPACES

YOUWEI ZHANG, Zhangye

Abstract. This paper deals with the generalized nonlinear third-order left focal problem at resonance

$$\begin{cases} (p(t)u''(t))' - q(t)u(t) = f(t, u(t), u'(t), u''(t)), & t \in \,]t_0, T[, \\ m(u(t_0), u''(t_0)) = 0, \\ n(u(T), u'(T)) = 0, \\ l(u(\xi), u'(\xi), u''(\xi)) = 0, \end{cases}$$

where the nonlinear term is a Carathéodory function and contains explicitly the first and second-order derivatives of the unknown function. The boundary conditions that we study are quite general, involve a linearity and include, as particular cases, Sturm-Liouville boundary conditions. Under certain growth conditions on the nonlinearity, we establish the existence of the nontrivial solutions by using the topological degree technique as well as some recent generalizations of this technique. Our results are generalizations and extensions of the results of several authors. An application is included to illustrate the results obtained.

Keywords: Fredholm operator; coincidence degree; left focal problem; nontrivial solution; resonance

MSC 2010: 34B15, 47J05

1. INTRODUCTION

The effect of resonance in a mechanical equation is very important to engineers, nearly every mechanical equation will exhibit some resonance and can with the application of even a very small external pulsed force be stimulated to do just that. Engineers usually work hard to eliminate resonance in some ways through a mechanical equation, as they perceive it to be counter-productive. In fact, it is impossible

to prevent all resonance. Mathematicians have provided more sophisticated research results of resonance in equations or systems. After seeing the regularity, i.e., the existence or nonexistence of solutions and properties of solutions for a mechanical equation or system, we may limit or control its influence, for example, through introduction of damping and source terms, adding a given local or nonlocal boundary conditions, and so on. Recently, there has been increasing interest in questions of solvability of boundary value problems for differential equations at resonance, and many excellent results have been obtained on the existence of solutions, provided the nonlinearity depends on the first-order derivative. Many authors have employed the Leray-Schauder continuation theorem and the coincidence degree theory of Mawhin to establish some existence results of solutions, we refer to Gupta [4], Kosmatov [5], Liu [7], Mawhin et al. [10], Rachůnková [13] and the references therein. At the same time, third-order focal boundary value problems are an area of theoretical exploration in many applied fields, especially in mathematical analysis, mechanics and numerous subjects related to it. It has provided a sound framework for a number of differential models of great importance in applications. Much attention has been paid to discussing a class of the boundary value problem [1], [3], the book [1] discusses at length the existence of positive solutions to the two-point right focal boundary value problem

$$
\begin{cases}
(-1)^{3-k} u'''(t) = f(t, u(t)), & t \in [0, 1], \\
u^j(0) = 0, & 0 \leqslant j \leqslant k - 1, \quad u^{(j)}(1) = 0, \quad k \leqslant j \leqslant 2,
\end{cases}
$$

when $k \in \{1, 2\}$. Based on the fairly general existence theorems for solutions of right focal boundary value problem, some authors have established the properties of solutions in deeper levels, such as the monotonicity of the solutions, fixed-sign solutions et al.; we can refer to [2], [8], [15].

In literature, very little work has been done on the nontrivial solutions to the third-order left focal boundary value problems at resonance in Banach spaces, in which the nonlinearity is involved with the lower order derivatives explicitly and boundary conditions are quite general. In this paper, we give a first application of the topological degree techniques to left focal boundary problems for a generalized third-order equation at resonance in Banach spaces, by demonstrating a technique that takes advantage of the flexibility of the fixed point theorem in obtaining at least one nontrivial solution for

$$
(\mathcal{P}) \quad
\begin{cases}
(p(t)u''(t))' - q(t)u(t) = f(t, u(t), u'(t), u''(t)), & t \in \,]t_0, T[, \\
m(u(t_0), u''(t_0)) = 0, \\
n(u(T), u'(T)) = 0, \\
l(u(\xi), u'(\xi), u''(\xi)) = 0,
\end{cases}
$$

362

where $m(\cdot, \cdot)$, $n(\cdot, \cdot)$ and $l(\cdot, \cdot, \cdot)$ denote the linear relations of $u(t_0)$ and $u''(t_0)$, $u(T)$ and $u'(T)$, $u(\xi)$, $u'(\xi)$ and $u''(\xi)$, respectively, $\xi \in \,]t_0, T[$, $p \in C^1([t_0, T],]0, +\infty[)$, $q \in L^1([t_0, T], \mathbb{R})$, $f: [t_0, T] \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function. The problem $(\mathcal{P})$ happens to be at resonance in the sense that the associated half-linear homogeneous boundary value problem

$$
\begin{cases}
(p(t)u''(t))' = 0, \quad t \in \,]t_0, T[, \\
m(u(t_0), u''(t_0)) = 0, \\
n(u(T), u'(T)) = 0, \\
l(u(\xi), u'(\xi), u''(\xi)) = 0
\end{cases}
$$

has $u(t) = u(t_0) + u'(t_0)(t - t_0) + p(t_0)u''(t_0) \int_{t_0}^{t} \frac{t-s}{p(s)} \, ds$ as a nontrivial solution. The boundary conditions that we study are quite general, involve a linearity and include, as particular cases, Sturm-Liouville boundary conditions. Set $u''(t_0) = c(p(t_0))^{-1} < \infty$, where c is a constant. Then for the boundary conditions $m(u(t_0), u''(t_0)) = 0$ and $n(u(T), u'(T)) = 0$ there exist linear mappings $\widetilde{m}$ and $\tilde{n}$ such that $u(t_0) = \widetilde{m}((p(t_0))^{-1})$ and $u'(t_0) = \tilde{n}((p(t_0))^{-1})$. Thus the problem $(\mathcal{P})$ will be at resonance when

$$
l\left(\widetilde{m}((p(t_0))^{-1}) + \tilde{n}((p(t_0))^{-1})(\xi - t_0) + c \int_{t_0}^{\xi} \frac{\xi - \tau}{p(\tau)} \, d\tau, \right.
$$

$$
\left. \tilde{n}((p(t_0))^{-1}) + c \int_{t_0}^{\xi} \frac{d\tau}{p(\tau)}, \; \frac{c}{p(\xi)} \right) = 0.
$$

This implies that $q(t)u(t) + f(t, u(t), u'(t), u''(t)) \in L^1[t_0, T]$. If $l(\widetilde{m}((p(t_0))^{-1}) + \tilde{n}((p(t_0))^{-1})(\xi - t_0) + c \int_{t_0}^{\xi} \frac{\xi-\tau}{p(\tau)} \, d\tau \neq 0$, then the problem has $u(t) \equiv 0$ as its only solution. So we say that the problem $(\mathcal{P})$ happens to be at resonance when $l(\widetilde{m}((p(t_0))^{-1}) + \tilde{n}((p(t_0))^{-1})(\xi - t_0) + c \int_{t_0}^{\xi} \frac{\xi-\tau}{p(\tau)} \, d\tau = 0$, for the case that the linear mapping $Lu(t) = (p(t)u'')'(t)$ is non-invertible, the so-called resonance case. Otherwise, we have the so-called non-resonance case. By applying the coincidence degree theorem of Mawhin, this paper will establish some new and more general results for the existence of a nontrivial solution to the generalized nonlinear third-order left focal problem at resonance. The results are new even for the abstract spaces, our results improve and generalize some known results.

The rest of the paper is organized as follows. Section 2 provides some background material for discussing the problem $(\mathcal{P})$. Some lemmas, a priori estimates and criteria for the existence of nontrivial solutions to the problem $(\mathcal{P})$ are established in Section 3, and an application of our main results is given in Section 4.

2. Preliminaries

In what follows, we provide some background material from Banach spaces and preliminary results.

Definition 2.1. Let X and Z be normed spaces. A linear operator $L \colon \operatorname{Dom} L \subset X \to Z$ is called a Fredholm operator if the following two conditions hold:

(i) $\operatorname{Ker} L$ has a finite dimension;

(ii) $\operatorname{Im} L$ is closed and has a finite codimension.

L is a Fredholm operator, its Fredholm index is the integer $\operatorname{Ind} L = \dim \operatorname{Ker} L - \operatorname{codim} \operatorname{Im} L$. In the present paper, we are interested in a Fredholm operator of index zero, i.e. $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L$. From Definition 2.1 we know that there exist continuous projectors $P \colon X \to X$ and $Q \colon Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$, $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$, and the operator $L_{\operatorname{Dom} L \cap \operatorname{Ker} P} \colon \operatorname{Dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible. We denote the inverse of $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P}$ by $K_P \colon \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$. The generalized inverse of L denoted by $K_{P,Q} \colon Z \to \operatorname{Dom} L \cap \operatorname{Ker} P$ is defined by $K_{P,Q} \colon K_P(I - Q)$.

Definition 2.2. Let $L \colon \operatorname{Dom} L \subset X \to Z$ be a Fredholm operator, E a metric space, and $N \colon E \to Z$ an operator. Operator N is called L-compact on E if $QN \colon E \to Z$ and $K_{P,Q} N \colon E \to X$ are compact on E. In addition, we say that N is L-completely continuous if it is L-compact on every bounded $E \subset X$.

We recall when that the function $f \colon [t_0, T] \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the Carathéodory conditions.

Definition 2.3. We say that the mapping $f \colon [t_0, T] \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the Carathéodory conditions with respect to $L^1[t_0, T]$, where $L^1[t_0, T]$ denotes the set of all Lebesgue-integrable functions on $[t_0, T]$, if the following conditions are satisfied:

(i) for each $(\mu, \nu, \vartheta) \in \mathbb{R}^3$, the mapping $t \to f(t, \mu, \nu, \vartheta)$ is Lebesgue measurable on $[t_0, T]$;

(ii) for a.e. $t \in [t_0, T]$, the mapping $(\mu, \nu, \vartheta) \to f(t, \mu, \nu, \vartheta)$ is continuous on $\mathbb{R}^3$;

(iii) for each $r > 0$, there exists $\alpha_r \in L^1([t_0, T], \mathbb{R})$ such that for a.e. $t \in [t_0, T]$ and every μ such that $|\mu| \leqslant r$, we have $|f(t, \mu, \nu, \vartheta)| \leqslant \alpha_r(t)$.

Theorem 2.1 ([9]). *Let $\Omega \subset X$ be an open bounded set, L a Fredholm operator of index zero, and N L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(1) $Lu \neq \lambda N u$ for every $(u, \lambda) \in ((\operatorname{Dom} L \setminus \operatorname{Ker} L) \cap \partial \Omega) \times [t_0, T]$;

(2) $Nu \notin \operatorname{Im} L$ for every $u \in \operatorname{Ker} L \cap \partial\Omega$;

(3) $\deg(QN|_{\operatorname{Ker} L \cap \partial\Omega}, \ \Omega \cap \operatorname{Ker} L, 0) \neq 0$ with $Q\colon Z \to Z$ a continuous projector such that $\operatorname{Ker} Q = \operatorname{Im} L$.

Then the equation $Lu = Nu$ admits at least one nontrivial solution in $\operatorname{Dom} L \cap \overline{\Omega}$.

Note that the problem of existence of nontrivial solutions in a convex set for abstract equations at resonance has been considered by Nieto [12] and Santanilla [14]. The authors have presented sufficient conditions for the existence of solutions to the equation

$$Lu = Nu$$

in a cone when the nonlinearity N is bounded, where $L\colon \operatorname{dom} L \subset I \to Z$ is a Fredholm operator of index zero, $N\colon X \to Z$ is linear or nonlinear and possesses a compactness property relative to L, and X, Z are Banach spaces.

3. Main results

Throughout we will denote the Banach space $X = C^1[t_0, T]$ with the norm $\|u\| = \max\{\|u\|_{C^1}, \|u'\|_{C^1}, \|u''\|_{C^1}\}$, where $\|\cdot\|_{C^1} = \sup\limits_{t \in]t_0, T[} |\cdot(t)|$. Let $Z = L^1[t_0, T]$ with the norm $\|z\|_{L^1} = \int_{t_0}^{T} |z(t)| \, \mathrm{d}t$. We use the Sobolev space

$$W^{3,1}[t_0, T] = \{u\colon [t_0, T] \to \mathbb{R}\colon u(t), u'(t), u''(t)$$
$$\text{are a.c. on } [t_0, T] \text{ with } u''' \in L^1[t_0, T]\}.$$

For the problem $(\mathcal{P})$, we define the mapping L from $\operatorname{Dom} L \subset X$ to Z by

$$\operatorname{Dom} L = \{u \in W^{3,1}[t_0, T]\colon m(u(t_0), \ u''(t_0)) = 0, \ n(u(T), u'(T)) = 0$$
$$l(u(\xi), u'(\xi), u''(\xi)) = 0\},$$

(3.1) $\quad Lu(t) = (p(t)u''(t))', \quad u \in \operatorname{Dom} L,$

and the nonlinear mapping $N\colon X \to Z$ by

(3.2) $\qquad Nu(t) = f(t, u(t), u'(t), u''(t)) + q(t)u(t), \quad t \in]t_0, T[.$

Obviously, the equation $(p(t)u''(t))' = f(t, u(t), u'(t)) + q(t)u(t)$ admits a solution which is equivalent to the solution of the mapping equation $Lu = Nu$. So we concentrate on the existence of solutions to the equation $Lu = Nu$ with the boundary condition of the problem $(\mathcal{P})$.

Lemma 3.1. $L\colon \operatorname{Dom} L \subset X \to Z$ *is a Fredholm mapping of index zero. Furthermore, there exist real numbers* a_i $(i = 1, 2, \ldots, 5)$. *The continuous linear projector operator* $Q\colon Z \to Z$ *can be defined by*

$$Qz = \frac{1}{\Lambda_1}\left(a_1 \int_{t_0}^{T} \frac{T-\tau}{p(\tau)} \int_{t_0}^{\tau} z(s)\,\mathrm{d}s\,\mathrm{d}\tau + a_2 \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} z(s)\,\mathrm{d}s\,\mathrm{d}\tau\right.$$

$$\left. + a_3 \int_{t_0}^{\xi} \frac{\xi-\tau}{p(\tau)} \int_{t_0}^{\tau} z(s)\,\mathrm{d}s\,\mathrm{d}\tau + a_4 \int_{t_0}^{\xi} \frac{1}{p(\tau)} \int_{t_0}^{\tau} z(s)\,\mathrm{d}s\,\mathrm{d}\tau + a_5\right),$$

where

$$\Lambda_1 = a_1 \int_{t_0}^{T} \frac{(T-\tau)(\tau-t_0)}{p(\tau)}\,\mathrm{d}\tau + a_2 \int_{t_0}^{T} \frac{\tau-t_0}{p(\tau)}\,\mathrm{d}\tau + a_3 \int_{t_0}^{\xi} \frac{(\xi-\tau)(\tau-t_0)}{p(\tau)}\,\mathrm{d}\tau$$

$$+ a_4 \int_{t_0}^{\xi} \frac{\tau-t_0}{p(\tau)}\,\mathrm{d}\tau + a_5 \neq 0.$$

The linear mapping $K_P\colon \operatorname{Im}L \to \operatorname{Dom} L \cap X$ *can be written as*

$$K_P z(t) = \int_{t_0}^{t} \frac{t-\tau}{p(\tau)} \int_{t_0}^{\tau} z(s)\,\mathrm{d}s\,\mathrm{d}\tau, \quad z \in \operatorname{Im}L.$$

Furthermore, $\|K_P z\| = \max\{\zeta_1, \zeta_2, \zeta_3\}\|z\|_{L^1}$, $z \in \operatorname{Im}L$, *where*

$$\zeta_1 = \int_{t_0}^{T} \frac{T-\tau}{p(\tau)}\,\mathrm{d}\tau, \quad \zeta_2 = \int_{t_0}^{T} \frac{1}{p(\tau)}\,\mathrm{d}\tau, \quad \zeta_3 = \sup_{t \in\,]t_0, T[} \frac{1}{p(t)}.$$

P r o o f. It is clear that $\operatorname{Ker} L = \mathbb{R}$. Let $u \in \operatorname{Dom} L$, $z \in \operatorname{Im}L$. The linear equation on $[t_0, T]$ is

$$(p(t)u''(t))' = z(t).$$

For a.e. $t \in [t_0, T]$, twice Lebesgue-integrating the above differential equation from t_0 to t yields

(3.3) $\qquad u(t) = u(t_0) + u'(t_0)(t - t_0)$

$$+ p(t_0)u''(t_0) \int_{t_0}^{t} \frac{t-\tau}{p(\tau)}\,\mathrm{d}\tau + \int_{t_0}^{t} \frac{t-\tau}{p(\tau)} \int_{t_0}^{\tau} z(s)\,\mathrm{d}s\,\mathrm{d}\tau.$$

The equation (3.3) satisfies $m(u(t_0), u''(t_0)) = 0$, $n(u(T), u'(T)) = 0$, $l(u(\xi), u'(\xi), u''(\xi)) = 0$ if and only if there exist real numbers a_i $(i = 1, 2, \ldots, 5)$ such that

(3.4) $\quad a_1 \int_{t_0}^{T} \frac{T-\tau}{p(\tau)} \int_{t_0}^{\tau} z(s)\,\mathrm{d}s\,\mathrm{d}\tau + a_2 \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} z(s)\,\mathrm{d}s\,\mathrm{d}\tau$

$$+ a_3 \int_{t_0}^{\xi} \frac{\xi-\tau}{p(\tau)} \int_{t_0}^{\tau} z(s)\,\mathrm{d}s\,\mathrm{d}\tau + a_4 \int_{t_0}^{\xi} \frac{1}{p(\tau)} \int_{t_0}^{\tau} z(s)\,\mathrm{d}s\,\mathrm{d}\tau + a_5 = 0.$$

On the other hand, if (3.4) holds, then for some $z \in \text{Im} L$, if we take $u \in \text{Dom} L$ as given by (3.3), then $(p(t)u''(t))' = z(t)$ for a.e. $t \in [t_0, T]$. So

$$\text{Im} L = \Bigg\{ z \in L^1[t_0, T]: \; a_1 \int_{t_0}^{T} \frac{T - \tau}{p(\tau)} \int_{t_0}^{\tau} z(s) \, ds \, d\tau + a_2 \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} z(s) \, ds \, d\tau$$

$$+ a_3 \int_{t_0}^{\xi} \frac{\xi - \tau}{p(\tau)} \int_{t_0}^{\tau} z(s) \, ds \, d\tau + a_4 \int_{t_0}^{\xi} \frac{1}{p(\tau)} \int_{t_0}^{\tau} z(s) \, ds \, d\tau + a_5 = 0 \Bigg\}.$$

Further, we define the mapping $Q: Z \to Z$ by

$$Qz = \frac{1}{\Lambda_1} \Bigg(a_1 \int_{t_0}^{T} \frac{T - \tau}{p(\tau)} \int_{t_0}^{\tau} z(s) \, ds \, d\tau + a_2 \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} z(s) \, ds \, d\tau$$

$$+ a_3 \int_{t_0}^{\xi} \frac{\xi - \tau}{p(\tau)} \int_{t_0}^{\tau} z(s) \, ds \, d\tau + a_4 \int_{t_0}^{\xi} \frac{1}{p(\tau)} \int_{t_0}^{\tau} z(s) \, ds \, d\tau + a_5 \Bigg)$$

for $z \in Z$. It is easy to see that $Q: Z \to Z$ is a linear continuous projector. For the mapping L and the continuous linear projector Q, it is not difficult to check that $\text{Im} L = \text{Ker} Q$. Set $z = (z - Qz) + Qz$, thus $z - Qz \in \text{Ker} Q = \text{Im} L$ and $Qz \in \text{Im} Q$, so $Z = \text{Im} L + \text{Im} Q$. If $z \in \text{Im} L \cap \text{Im} Q$, then $z(t) = 0$, hence $Z = \text{Im} L \oplus \text{Im} Q$. From $\text{Ker} L = \mathbb{R}$ we obtain that $\text{Ind} L = \dim \text{Ker} L - \text{codim} \, \text{Im} L = \dim \text{Ker} L - \dim \text{Im} Q = 0$, that is, L is a Fredholm mapping of index zero.

Take $P: X \to X$ as follows:

$$Pu(t) = u(t_0) + u'(t_0)(t - t_0) + p(t_0)u''(t_0) \int_{t_0}^{t} \frac{t - \tau}{p(\tau)} \, d\tau, \quad t \in [t_0, T],$$

and set $u \in X$ in the form

$$u(t) = u(t_0) + u'(t_0)(t - t_0) + p(t_0)u''(t_0) \int_{t_0}^{t} \frac{t - \tau}{p(\tau)} \, d\tau + \int_{t_0}^{t} \frac{t - \tau}{p(\tau)} \int_{t_0}^{\tau} z(s) \, ds \, d\tau$$

$$+ u(t) - u(t_0) - u'(t_0)(t - t_0) - p(t_0)u''(t_0) \int_{t_0}^{t} \frac{t - \tau}{p(\tau)} \, d\tau$$

$$- \int_{t_0}^{t} \frac{t - \tau}{p(\tau)} \int_{t_0}^{\tau} z(s) \, ds \, d\tau.$$

Obviously, $\text{Im} P = \text{Ker} L$ and $X = \text{Ker} L \oplus \text{Ker} P$, hence the generalized inverse $K_P: \text{Im} L \to \text{Dom} L \cap \text{Ker} P$ is defined by

$$K_P z(t) = \int_{t_0}^{t} \frac{t - \tau}{p(\tau)} \int_{t_0}^{\tau} z(s) \, ds \, d\tau,$$

and it follows that $(K_P z)'(t) = \int_{t_0}^{t} \frac{1}{p(\tau)} \int_{t_0}^{\tau} z(s) \, ds \, d\tau$, $(K_P z)''(t) = \frac{1}{p(t)} \int_{t_0}^{t} z(s) \, ds$. It is not difficult to obtain that $\|K_P z\| = \max\{\zeta_1, \zeta_2, \zeta_3\} \|z\|_{L^1}$.

For $z \in \mathrm{Im}\,L$, we have

$$(3.5) \qquad L(K_P z(t)) = \left(p(t) \left(\int_{t_0}^{t} \frac{t-\tau}{p(\tau)} \int_{t_0}^{\tau} z(s)\,\mathrm{d}s\,\mathrm{d}\tau \right)'' \right)' = z(t),$$

and for $u \in \mathrm{Dom}\,L \cap \mathrm{Ker}\,P$,

$$K_P(Lu(t)) = \int_{t_0}^{t} \frac{t-\tau}{p(\tau)} \int_{t_0}^{\tau} (p(s)u''(s))'\,\mathrm{d}s\,\mathrm{d}\tau$$

$$= \int_{t_0}^{t} (t-\tau)u''(\tau)\,\mathrm{d}\tau - p(t_0)u''(t_0) \int_{t_0}^{t} \frac{t-\tau}{p(\tau)}\,\mathrm{d}\tau.$$

In view of $u \in \mathrm{Dom}\,L \cap \mathrm{Ker}\,P$ we have $Pu(t) = u(t_0) + u'(t_0)(t-t_0) + p(t_0)u''(t_0) \times \int_{t_0}^{t} ((t-\tau)/p(\tau))\,\mathrm{d}\tau = 0$, thus

$$(3.6) \qquad K_P(Lu(t)) = u(t), \quad t \in [t_0, T].$$

(3.5) and (3.6) yield $K_P = (L|_{\mathrm{Dom}\,L \cap \mathrm{Ker}\,P})^{-1}$, and the proof is complete. $\qquad \square$

Furthermore,

$$Q(Nu) = \frac{1}{\Lambda_1}\left(a_1 \int_{t_0}^{T} \frac{T-\tau}{p(\tau)} \int_{t_0}^{\tau} Nu(s)\,\mathrm{d}s\,\mathrm{d}\tau + a_2 \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} Nu(s)\,\mathrm{d}s\,\mathrm{d}\tau \right.$$

$$+ a_3 \int_{t_0}^{\xi} \frac{\xi-\tau}{p(\tau)} \int_{t_0}^{\tau} Nu(s)\,\mathrm{d}s\,\mathrm{d}\tau + a_4 \int_{t_0}^{\xi} \frac{1}{p(\tau)} \int_{t_0}^{\tau} Nu(s)\,\mathrm{d}s\,\mathrm{d}\tau + a_5 \left. \right)$$

$$= \frac{1}{\Lambda_1}\left(a_1 \int_{t_0}^{T} \frac{T-\tau}{p(\tau)} \int_{t_0}^{\tau} \left(f(s,u(s),u'(s),u''(s)) + q(s)u(s) \right)\,\mathrm{d}s\,\mathrm{d}\tau \right.$$

$$+ a_2 \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} \left(f(s,u(s),u'(s),u''(s)) + q(s)u(s) \right)\,\mathrm{d}s\,\mathrm{d}\tau$$

$$+ a_3 \int_{t_0}^{\xi} \frac{\xi-\tau}{p(\tau)} \int_{t_0}^{\tau} \left(f(s,u(s),u'(s),u''(s)) + q(s)u(s) \right)\,\mathrm{d}s\,\mathrm{d}\tau$$

$$+ a_4 \int_{t_0}^{\xi} \frac{1}{p(\tau)} \int_{t_0}^{\tau} \left(f(s,u(s),u'(s),u''(s)) + q(s)u(s) \right)\,\mathrm{d}s\,\mathrm{d}\tau + a_5 \left. \right),$$

$$K_{P,Q}(Nu)(t) = \int_{t_0}^t \frac{t-\tau}{p(\tau)} \int_{t_0}^\tau \big(Nu(s) - (QNu)(s)\big)\,\mathrm{d}s\,\mathrm{d}\tau$$

$$= \int_{t_0}^t \frac{t-\tau}{p(\tau)} \int_{t_0}^\tau \bigg(f\big(s, u(s), u'(s), u''(s)\big) + q(s)u(s)$$

$$- \frac{1}{\Lambda_1} \bigg(a_1 \int_{t_0}^T \frac{T-\tau}{p(\tau)} \int_{t_0}^\tau \big(f(s, u(s), u'(s), u''(s)) + q(s)u(s)\big)\,\mathrm{d}s\,\mathrm{d}\tau$$

$$+ a_2 \int_{t_0}^T \frac{1}{p(\tau)} \int_{t_0}^\tau \big(f(s, u(s), u'(s), u''(s)) + q(s)u(s)\big)\,\mathrm{d}s\,\mathrm{d}\tau$$

$$+ a_3 \int_{t_0}^\xi \frac{\xi-\tau}{p(\tau)} \int_{t_0}^\tau \big(f(s, u(s), u'(s), u''(s)) + q(s)u(s)\big)\,\mathrm{d}s\,\mathrm{d}\tau$$

$$+ a_4 \int_{t_0}^\xi \frac{1}{p(\tau)} \int_{t_0}^\tau \big(f(s, u(s), u'(s), u''(s)) + q(s)u(s)\big)\,\mathrm{d}s\,\mathrm{d}\tau + a_5 \bigg) \bigg).$$

Since the first and second-order derivatives are involved in the nonlinear term f explicitly and the nontrivial solutions u of the problem $(\mathcal{P})$ exitsts, let us prove now some a priori estimates which will be useful later.

Lemma 3.2. *Let $b_i > 0$ ($i = 1, 2, 3, 4$) and $c_i \neq 0$ ($i = 1, 2, 3$). A solution of the problem $(\mathcal{P})$ satisfies*

(3.7) $$\|u''\|_{C^1} \leqslant b_1 \|u\|_{C^1} + b_2, \quad \|u'\|_{C^1} \leqslant b_3 \|u\|_{C^1} + b_4,$$

where

$$b_1 = \frac{\zeta_3}{|c_1|} + \zeta_3 \int_{t_0}^T |q(s)|\,\mathrm{d}s, \qquad b_2 = \zeta_3 \int_{t_0}^T \alpha_r(s)\,\mathrm{d}s,$$

$$b_3 = \int_{t_0}^T \frac{1}{p(\tau)} \int_{t_0}^\tau |q(s)|\,\mathrm{d}s\,\mathrm{d}\tau + b_1 \bigg(|c_2|p(t_0) + p(t_0)\int_{t_0}^T \frac{1}{p(\tau)}\,\mathrm{d}\tau\bigg),$$

$$b_4 = b_2 \bigg(|c_2|p(t_0) + p(t_0)\int_{t_0}^T \frac{1}{p(\tau)}\,\mathrm{d}\tau\bigg) + \int_{t_0}^T \frac{1}{p(\tau)} \int_{t_0}^\tau \alpha_r(s)\,\mathrm{d}s\,\mathrm{d}\tau + |c_3|.$$

P r o o f. By virtue of $m(u(t_0), u''(t_0)) = 0$, there exists a nonzero real number c_1 such that

(3.8) $$u(t_0) = c_1 p(t_0) u''(t_0).$$

Also, the boundary conditions being $n(u(T), u'(T)) = 0$ or $l(u(\xi), u'(\xi), u''(\xi)) = 0$, there exist two nonzero real numbers c_2 and c_3 such that

(3.9) $$u'(t_0) = c_2 p(t_0) u''(t_0) + c_3.$$

For a.e. $t \in [t_0, T]$, by (3.3), (3.8), (3.9) and $\sup\limits_{t\in]t_0,T[} |u(t)| < r < \infty$, one has that

$$
|u''(t)| \leqslant p(t_0)|u''(t_0)| \sup_{t\in]t_0,T[} \frac{1}{p(t)}
$$

$$
+ \sup_{t\in]t_0,T[} \frac{1}{p(t)} \int_{t_0}^{T} |f(s,u(s),u'(s),u''(s)) + q(s)u(s)| \, ds \, d\tau
$$

$$
\leqslant \frac{|u(t_0)|}{|c_1|} \sup_{t\in]t_0,T[} \frac{1}{p(t)} + \sup_{t\in]t_0,T[} \frac{1}{p(t)} \int_{t_0}^{T} |f(s,u(s),u'(s),u''(s))| \, ds
$$

$$
+ \sup_{t\in]t_0,T[} \frac{1}{p(t)} \sup_{t\in]t_0,T[} |u(t)| \int_{t_0}^{T} |q(s)| \, ds,
$$

and so we have

$$
\sup_{t\in]t_0,T[} |u''(t)| \leqslant \frac{\zeta_3}{|c_1|} \sup_{t\in]t_0,T[} |u(t)| + \zeta_3 \int_{t_0}^{T} \alpha_r(s) \, ds + \sup_{t\in]t_0,T[} |u(t)|\zeta_3 \int_{t_0}^{T} |q(s)| \, ds.
$$

The above argument yields

$$
\sup_{t\in]t_0,T[} |u''(t)| \leqslant \left(\frac{\zeta_3}{|c_1|} + \zeta_3 \int_{t_0}^{T} |q(s)| \, ds \right) \sup_{t\in]t_0,T[} |u(t)| + \zeta_3 \int_{t_0}^{T} \alpha_r(s) \, ds
$$

$$
= b_1 \sup_{t\in]t_0,T[} |u(t)| + b_2.
$$

Similarly to the above argument, we have

$$
|u'(t)| \leqslant |u'(t_0)| + p(t_0)|u''(t_0)| \int_{t_0}^{T} \frac{1}{p(\tau)} \, d\tau + \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} \alpha_r(s) \, ds \, d\tau
$$

$$
+ \sup_{t\in]t_0,T[} |u(t)| \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} |q(s)| \, ds \, d\tau
$$

$$
\leqslant \left(|c_2|p(t_0) + p(t_0) \int_{t_0}^{T} \frac{1}{p(\tau)} \, d\tau \right) |u''(t_0)| + \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} \alpha_r(s) \, ds \, d\tau
$$

$$
+ \sup_{t\in]t_0,T[} |u(t)| \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} |q(s)| \, ds \, d\tau + |c_3|
$$

$$\leqslant \left(|c_2| p(t_0) + p(t_0) \int_{t_0}^{T} \frac{1}{p(\tau)} \, d\tau \right) \sup_{t \in]t_0, T[} |u''(t)| + \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} \alpha_r(s) \, ds \, d\tau$$

$$+ \sup_{t \in]t_0, T[} |u(t)| \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} |q(s)| \, ds \, d\tau + |c_3|$$

$$\leqslant \left(|c_2| p(t_0) + p(t_0) \int_{t_0}^{T} \frac{1}{p(\tau)} \, d\tau \right) \left(b_1 \sup_{t \in]t_0, T[} |u(t)| + b_2 \right)$$

$$+ \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} \alpha_r(s) \, ds \, d\tau + \sup_{t \in]t_0, T[} |u(t)| \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} |q(s)| \, ds \, d\tau + |c_3|$$

$$= \left(\int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} |q(s)| \, ds \, d\tau + b_1 \left(|c_2| p(t_0) + p(t_0) \int_{t_0}^{T} \frac{1}{p(\tau)} \, d\tau \right) \right) \sup_{t \in]t_0, T[} |u(t)|$$

$$+ b_2 \left(|c_2| p(t_0) + p(t_0) \int_{t_0}^{T} \frac{1}{p(\tau)} \, d\tau \right) + \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} \alpha_r(s) \, ds \, d\tau + |c_3|,$$

therefore,

$$\sup_{t \in]t_0, T[} |u'(t)| \leqslant \left(\int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} |q(s)| \, ds \, d\tau + b_1 \left(|c_2| p(t_0) + p(t_0) \int_{t_0}^{T} \frac{1}{p(\tau)} \, d\tau \right) \right)$$

$$\times \sup_{t \in]t_0, T[} |u(t)| + b_2 \left(|c_2| p(t_0) + p(t_0) \int_{t_0}^{T} \frac{1}{p(\tau)} \, d\tau \right)$$

$$+ \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} \alpha_r(s) \, ds \, d\tau + |c_3| = b_3 \sup_{t \in]t_0, T[} |u(t)| + b_4.$$

This implies that the desired conclusion. $\qquad \square$

Lemma 3.3. *The mapping N is L-completely continuous.*

P r o o f. Assume that $u_n, u_0 \in E$ satisfy $\|u_n - u_0\| \to 0$ $(n \to \infty)$, thus there exists $R > 0$ such that $\|u_n\| \leqslant R$ for any $n \geqslant 1$. One has that

$$\|Nu_n - Nu_0\|_{C^1} = \sup_{t \in [t_0, T]} |Nu_n(t) - Nu_0(t)|$$

$$\leqslant \sup_{t \in [t_0, T]} \left(|f(t, u_n(t), u_n'(t), u_n''(t)) - f(t, u_0(t), u_0'(t), u_0''(t))| \right.$$

$$\left. + |q(t)||u_n(t) - u_0(t)| \right).$$

In view of the fact that f satisfies the Carathéodory conditions, we can obtain that for a.e. $t \in [t_0, T]$,

$$\|Nu_n - Nu_0\|_{C^1} \to 0 \quad (n \to \infty).$$

This means that the operator $N \colon E \to Z$ is continuous. By the definitions of QN and $K_{P,Q}N$, we can obtain that $QN \colon E \to Z$ and $K_{P,Q}N \colon E \to X$ are continuous.

Let $E \subseteq X$ be a bounded set and $r = \sup\{\|u\| \colon u \in E\} < \infty$. Then for a.e. $t \in [t_0, T]$ we have

$$|Nu_n(t)| \leqslant |f(t, u_n(t), u_n'(t), u_n''(t))| + |q(t)||u_n| \leqslant |\alpha_r(t)| + r|q(t)| := \psi(t),$$

$$|Q(Nu_n)| \leqslant \frac{1}{|\Lambda_1|}\left(|a_1| \int_{t_0}^T \frac{T-\tau}{p(\tau)} \int_{t_0}^\tau |Nu_n(s)| \,\mathrm{d}s\,\mathrm{d}\tau + |a_2| \int_{t_0}^T \frac{1}{p(\tau)} \int_{t_0}^\tau |Nu_n(s)| \,\mathrm{d}s\,\mathrm{d}\tau\right.$$

$$\left. + |a_3| \int_{t_0}^\xi \frac{\xi-\tau}{p(\tau)} \int_{t_0}^\tau |Nu_n(s)| \,\mathrm{d}s\,\mathrm{d}\tau + |a_4| \int_{t_0}^\xi \frac{1}{p(\tau)} \int_{t_0}^\tau |Nu_n(s)| \,\mathrm{d}s\,\mathrm{d}\tau + |a_5|\right)$$

$$\leqslant \frac{1}{|\Lambda_1|}\left(|a_1| \int_{t_0}^T \frac{T-\tau}{p(\tau)} \int_{t_0}^\tau \psi(s) \,\mathrm{d}s\,\mathrm{d}\tau + |a_2| \int_{t_0}^T \frac{1}{p(\tau)} \int_{t_0}^\tau \psi(s) \,\mathrm{d}s\,\mathrm{d}\tau\right.$$

$$\left. + |a_3| \int_{t_0}^\xi \frac{\xi-\tau}{p(\tau)} \int_{t_0}^\tau \psi(s) \,\mathrm{d}s\,\mathrm{d}\tau + |a_4| \int_{t_0}^\xi \frac{1}{p(\tau)} \int_{t_0}^\tau \psi(s) \,\mathrm{d}s\,\mathrm{d}\tau + |a_5|\right),$$

$$|K_{P,Q}(Nu_n(t))| \leqslant \int_{t_0}^t \frac{t-\tau}{p(\tau)} \int_{t_0}^\tau |Nu_n(s)| \,\mathrm{d}s\,\mathrm{d}\tau + \int_{t_0}^t \frac{t-\tau}{p(\tau)} \int_{t_0}^\tau |Q(Nu_n)(s)| \,\mathrm{d}s\,\mathrm{d}\tau$$

$$\leqslant \int_{t_0}^t \frac{t-\tau}{p(\tau)} \int_{t_0}^\tau \psi(s) \,\mathrm{d}s\,\mathrm{d}\tau + \frac{1}{|\Lambda_1|} \int_{t_0}^t \frac{t-\tau}{p(\tau)} \int_{t_0}^\tau \left(|a_1| \int_{t_0}^T \frac{T-\tau}{p(\tau)} \int_{t_0}^\tau \psi(s) \,\mathrm{d}s\,\mathrm{d}\tau\right.$$

$$+ |a_2| \int_{t_0}^T \frac{1}{p(\tau)} \int_{t_0}^\tau \psi(s) \,\mathrm{d}s\,\mathrm{d}\tau + |a_3| \int_{t_0}^\xi \frac{\xi-\tau}{p(\tau)} \int_{t_0}^\tau \psi(s) \,\mathrm{d}s\,\mathrm{d}\tau$$

$$\left. + |a_4| \int_{t_0}^\xi \frac{1}{p(\tau)} \int_{t_0}^\tau \psi(s) \,\mathrm{d}s\,\mathrm{d}\tau + |a_5|\right) \mathrm{d}s\,\mathrm{d}\tau$$

$$= \int_{t_0}^t \frac{t-\tau}{p(\tau)} \int_{t_0}^\tau \psi(s) \,\mathrm{d}s\,\mathrm{d}\tau + \frac{1}{|\Lambda_1|} \int_{t_0}^t \frac{(t-\tau)(\tau-t_0)}{p(\tau)} \left(|a_1| \int_{t_0}^T \frac{T-\tau}{p(\tau)} \int_{t_0}^\tau \psi(s) \,\mathrm{d}s\,\mathrm{d}\tau\right.$$

$$+ |a_2| \int_{t_0}^T \frac{1}{p(\tau)} \int_{t_0}^\tau \psi(s) \,\mathrm{d}s\,\mathrm{d}\tau + |a_3| \int_{t_0}^\xi \frac{\xi-\tau}{p(\tau)} \int_{t_0}^\tau \psi(s) \,\mathrm{d}s\,\mathrm{d}\tau$$

$$\left. + |a_4| \int_{t_0}^\xi \frac{1}{p(\tau)} \int_{t_0}^\tau \psi(s) \,\mathrm{d}s\,\mathrm{d}\tau + |a_5|\right) \mathrm{d}\tau.$$

Since the functions $\alpha_r, q \in L^1([t_0, T], \mathbb{R})$, we get that $\psi \in L^1([t_0, T], \mathbb{R})$. Further,

$$\|Nu_n\|_{L^1} \leqslant \int_0^T |\psi(t)| \,\mathrm{d}t := \chi < \infty.$$

It follows that $Q(N(E))$ and $K_{P,Q}(N(E))$ are bounded.

It is easy to see that $\{Q(Nu_n)\}_{n=1}^\infty$ is equicontinuous at a.e. $t \in [t_0, T]$, so we only show that $\{K_{P,Q}(Nu_n)\}_{n=1}^\infty$ is equicontinuous at a.e. $t \in [t_0, T]$. For any $t_1, t_2 \in$

$[t_0, T]$ with $t_1 < t_2$ one has

$$(3.10) \qquad |K_{P,Q}(Nu_n(t_1)) - K_{P,Q}(Nu_n(t_2))| \leqslant \int_{t_1}^{t_2} |(K_{P,Q}(Nu_n))'(t)| \, \mathrm{d}t$$

$$\leqslant \int_{t_1}^{t_2} \left| \int_{t_0}^{t} \frac{1}{p(\tau)} \int_{t_0}^{\tau} Nu_n(s) - (QNu_n)(s) \, \mathrm{d}s \, \mathrm{d}\tau \right| \mathrm{d}t$$

$$\leqslant \int_{t_1}^{t_2} \int_{t_0}^{t} \frac{1}{p(\tau)} \int_{t_0}^{\tau} |Nu_n(s)| \, \mathrm{d}s \, \mathrm{d}\tau \, \mathrm{d}t$$

$$+ \int_{t_1}^{t_2} \int_{t_0}^{t} \frac{1}{p(\tau)} \int_{t_0}^{\tau} |(QNu_n)(s)| \, \mathrm{d}s \, \mathrm{d}\tau \, \mathrm{d}t$$

$$\leqslant \int_{t_1}^{t_2} \int_{t_0}^{t} \frac{1}{p(\tau)} \int_{t_0}^{\tau} \psi(s) \, \mathrm{d}s \, \mathrm{d}\tau$$

$$+ \frac{1}{|\Lambda_1|} \int_{t_1}^{t_2} \int_{t_0}^{t} \frac{\tau - t_0}{p(\tau)} \left(|a_1| \int_{t_0}^{T} \frac{T - \tau}{p(\tau)} \int_{t_0}^{\tau} \psi(s) \, \mathrm{d}s \, \mathrm{d}\tau \right.$$

$$+ |a_2| \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} \psi(s) \, \mathrm{d}s \, \mathrm{d}\tau + |a_3| \int_{t_0}^{\xi} \frac{\xi - \tau}{p(\tau)} \int_{t_0}^{\tau} \psi(s) \, \mathrm{d}s \, \mathrm{d}\tau$$

$$\left. + |a_4| \int_{t_0}^{\xi} \frac{1}{p(\tau)} \int_{t_0}^{\tau} \psi(s)| \, \mathrm{d}s \, \mathrm{d}\tau + |a_5| \right) \mathrm{d}\tau \, \mathrm{d}t.$$

Since $\psi \in L^1([t_0, T], \mathbb{R})$, thus (3.10) shows that $\{K_{P,Q}(Nu_n)\}_{n=1}^{\infty}$ is equicontinuous at a.e. $t \in [t_0, T]$. On the other hand,

$$(3.11) \quad |K_{P,Q}(Nu_n'(t_1)) - K_{P,Q}(Nu_n'(t_2))|$$

$$\leqslant \int_{t_1}^{t_2} \frac{1}{p(\tau)} \int_{t_0}^{\tau} |Nu_n(s) - (QNu_n)(s)| \, \mathrm{d}s \, \mathrm{d}\tau$$

$$\leqslant \int_{t_1}^{t_2} \frac{1}{p(\tau)} \int_{t_0}^{\tau} \left(\psi(s) + \left| \frac{a_1}{\Lambda_1} \right| \int_{t_0}^{T} \frac{T - \tau}{p(\tau)} \int_{t_0}^{\tau} \psi(s) \, \mathrm{d}s \, \mathrm{d}\tau \right.$$

$$+ \left| \frac{a_2}{\Lambda_1} \right| \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} \psi(s) \, \mathrm{d}s \, \mathrm{d}\tau + \left| \frac{a_3}{\Lambda_1} \right| \int_{t_0}^{\xi} \frac{\xi - \tau}{p(\tau)} \int_{t_0}^{\tau} \psi(s) \, \mathrm{d}s \, \mathrm{d}\tau$$

$$\left. + \left| \frac{a_4}{\Lambda_1} \right| \int_{t_0}^{\xi} \frac{1}{p(\tau)} \int_{t_0}^{\tau} \psi(s) \, \mathrm{d}s \, \mathrm{d}\tau + \left| \frac{a_5}{\Lambda_1} \right| \right) \mathrm{d}s \, \mathrm{d}\tau.$$

(3.11) shows that $\{K_{P,Q}(Nu_n)'\}_{n=1}^{\infty}$ is also equicontinuous at a.e. $t \in [t_0, T]$. Hence, by the Arzelà-Ascoli theorem, $\{QNu_n\}_{n=1}^{\infty}$ and $\{K_{P,Q}(Nu_n)\}$ are compact on an arbitrary bounded $E \subseteq X$, and the mapping $N \colon X \to Z$ is L-completely continuous. $\square$

Now we are ready to apply the coincidence degree theorem of Mawhin to give sufficient conditions for the existence of at least one nontrivial solution to the problem ($\mathcal{P}$).

Theorem 3.1. Let $(|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})\|q\|_{L^1} < 1$, let $f:$ $[t_0, T] \times \mathbb{R}^3 \to \mathbb{R}$ satisfy the Carathéodory conditions, and let us assume:

(H$_1$) There exist functions $\alpha, \beta, \gamma, \theta \in L^1[t_0, T]$ and constants $\varepsilon_i \in \,]0, 1[\; (i = 1, 2, 3)$ such that for any $(\mu, \nu, \vartheta) \in \mathbb{R}^3$ and a.e. $t \in [t_0, T]$, one of the following three conditions is fulfilled:

(3.12) $\quad |f(t, \mu, \nu, \vartheta)| \leqslant \alpha(t)|\mu| + \beta(t)|\mu|^{\varepsilon_1} + \gamma(t)|\nu|^{\varepsilon_2} + \delta(t)|\vartheta|^{\varepsilon_3} + \theta(t),$

(3.13) $\quad |f(t, \mu, \nu, \vartheta)| \leqslant \alpha(t)|\mu|^{\varepsilon_1} + \beta(t)|\nu| + \gamma(t)|\nu|^{\varepsilon_2} + \delta(t)|\vartheta|^{\varepsilon_3} + \theta(t),$

(3.14) $\quad |f(t, \mu, \nu, \vartheta)| \leqslant \alpha(t)|\mu|^{\varepsilon_1} + \beta(t)|\nu|^{\varepsilon_2} + \gamma(t)|\vartheta| + \delta(t)|\vartheta|^{\varepsilon_3} + \theta(t).$

(H$_2$) There exists a constant $M > 0$ such that for any $u \in \mathrm{Dom}\, L$, if $|p(t)u''(t)| > M$ for a.e. $t \in [t_0, T]$, then

$$
a_1 \int_{t_0}^{T} \frac{T - \tau}{p(\tau)} \int_{t_0}^{\tau} \big(f(s, u(s), u'(s), u''(s)) + q(s)u(s)\big)\, \mathrm{d}s\, \mathrm{d}\tau
$$
$$
+ a_2 \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} \big(f(s, u(s), u'(s), u''(s)) + q(s)u(s)\big)\, \mathrm{d}s\, \mathrm{d}\tau
$$
$$
+ a_3 \int_{t_0}^{\xi} \frac{\xi - \tau}{p(\tau)} \int_{t_0}^{\tau} \big(f(s, u(s), u'(s), u''(s)) + q(s)u(s)\big)\, \mathrm{d}s\, \mathrm{d}\tau
$$
$$
+ a_4 \int_{t_0}^{\xi} \frac{1}{p(\tau)} \int_{t_0}^{\tau} \big(f(s, u(s), u'(s), u''(s)) + q(s)u(s)\big)\, \mathrm{d}s\, \mathrm{d}\tau + a_5 \neq 0.
$$

(H$_3$) There exists a constant $\overline{M} > 0$ such that for any $\omega \in \mathbb{R}$, if $|\omega| < \overline{M}$, then we have either

$$
(3.15) \qquad \frac{\omega}{\Lambda_1}\bigg(a_1 \int_{t_0}^{T} \frac{T - \tau}{p(\tau)} \int_{t_0}^{\tau} (f(s, \omega, 0, 0) + \omega q(s))\, \mathrm{d}s\, \mathrm{d}\tau
$$
$$
+ a_2 \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} (f(s, \omega, 0, 0) + \omega q(s))\, \mathrm{d}s\, \mathrm{d}\tau
$$
$$
+ a_3 \int_{t_0}^{\xi} \frac{\xi - \tau}{p(\tau)} \int_{t_0}^{\tau} (f(s, \omega, 0, 0) + \omega q(s))\, \mathrm{d}s\, \mathrm{d}\tau
$$
$$
+ a_4 \int_{t_0}^{\xi} \frac{1}{p(\tau)} \int_{t_0}^{\tau} (f(s, \omega, 0, 0) + \omega q(s))\, \mathrm{d}s\, \mathrm{d}\tau + a_5\bigg) < 0
$$

or

$$
(3.16) \qquad \frac{\omega}{\Lambda_1}\bigg(a_1 \int_{t_0}^{T} \frac{T - \tau}{p(\tau)} \int_{t_0}^{\tau} (f(s, \omega, 0, 0) + \omega q(s))\, \mathrm{d}s\, \mathrm{d}\tau
$$
$$
+ a_2 \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} (f(s, \omega, 0, 0) + \omega q(s))\, \mathrm{d}s\, \mathrm{d}\tau
$$

$$+ a_3 \int_{t_0}^{\xi} \frac{\xi - \tau}{p(\tau)} \int_{t_0}^{\tau} (f(s, \omega, 0, 0) + \omega q(s)) \, ds \, d\tau$$

$$+ a_4 \int_{t_0}^{\xi} \frac{1}{p(\tau)} \int_{t_0}^{\tau} (f(s, \omega, 0, 0) + \omega q(s)) \, ds \, d\tau + a_5 \bigg) > 0.$$

Then for every $q(t) \in L^1[t_0, T]$, the problem $(\mathcal{P})$ when

$$l\bigg(\widetilde{m}((p(t_0))^{-1}) + \tilde{n}((p(t_0))^{-1})(\xi - t_0) + c \int_{t_0}^{\xi} \frac{\xi - \tau}{p(\tau)} \, d\tau,$$

$$\tilde{n}((p(t_0))^{-1}) + c \int_{t_0}^{\xi} \frac{d\tau}{p(\tau)}, \frac{c}{p(\xi)} \bigg) = 0$$

admits at least one nontrivial solution at resonance provided that

$$\|\alpha\|_{L^1} + \|\beta\|_{L^1} + (2b_3)^{\varepsilon_2} \|\gamma\|_{L^1} + (2b_1)^{\varepsilon_3} \|\delta\|_{L^1}$$
$$< \frac{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\}) \|q\|_{L^1}}{|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\}}.$$

P r o o f. The mapping L and N are defined by (3.1) and (3.2), respectively. We note that L is a Fredholm mapping of index zero and N is L-completely continuous by Lemma 3.1 and Lemma 3.3, respectively. Let

$$\Omega_1 = \{u \in \operatorname{Dom} L \setminus \operatorname{Ker} L \colon Lu = \lambda Nu \text{ for some } \lambda \in \,]0, 1[\}.$$

For $u \in \Omega_1$, we have $u \notin \operatorname{Ker} L$ and $Nu \in \operatorname{Im} L = \operatorname{Ker} Q$, thus

$$a_1 \int_{t_0}^{T} \frac{T - \tau}{p(\tau)} \int_{t_0}^{\tau} (f(s, u(s), u'(s), u''(s)) + q(s)u(s)) \, ds \, d\tau$$

$$+ a_2 \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} (f(s, u(s), u'(s), u''(s)) + q(s)u(s)) \, ds \, d\tau$$

$$+ a_3 \int_{t_0}^{\xi} \frac{\xi - \tau}{p(\tau)} \int_{t_0}^{\tau} (f(s, u(s), u'(s), u''(s)) + q(s)u(s)) \, ds \, d\tau$$

$$+ a_4 \int_{t_0}^{\xi} \frac{1}{p(\tau)} \int_{t_0}^{\tau} (f(s, u(s), u'(s), u''(s)) + q(s)u(s)) \, ds \, d\tau + a_5 = 0.$$

It follows from condition (H_2) that there exists $\bar{t} \in [t_0, T]$ such that $|p(\bar{t})u''(\bar{t})| \leqslant M$. Taking account of $\int_{t_0}^{\bar{t}} (p(\tau)u''(\tau))' \, d\tau = p(\bar{t})u''(\bar{t}) - p(t_0)u''(t_0)$, one has

$$(3.17) \qquad p(t_0)|u''(t_0)| \leqslant |p(\bar{t})u''(\bar{t})| + \left| \int_{t_0}^{\bar{t}} (p(\tau)u''(\tau))' \, d\tau \right|$$

$$\leqslant M + \|(pu'')'\|_{L^1} = M + \|Lu\|_{L^1} < M + \|Nu\|_{L^1}.$$

Also, for $u \in \Omega_1$, observe that $(I - P)u \in \operatorname{Im} K_P = \operatorname{Dom} L \cap \operatorname{Ker} P$, hence by (3.17) we can obtain

$$(3.18) \qquad \|(I - P)u\| = \|K_P L(I - P)u\| = \max\{\zeta_1, \zeta_2, \zeta_3\} \|L(I - P)u\|_{L^1}$$
$$= \max\{\zeta_1, \zeta_2, \zeta_3\} \|Lu\|_{L^1} < \max\{\zeta_1, \zeta_2, \zeta_3\} \|Nu\|_{L^1}.$$

So combining (3.8), (3.9), (3.17) and (3.18), we have

$$\|u\| \leqslant \|Pu\| + \|(I - P)u\| \leqslant |u(t_0)| + |u'(t_0)|(T - t_0)$$
$$+ p(t_0)|u''(t_0)| \left| \int_{t_0}^{T} \frac{T - \tau}{p(\tau)} \, d\tau \right| + \max\{\zeta_1, \zeta_2, \zeta_3\} \|Nu\|_{L^1}$$
$$\leqslant |c_1| p(t_0)|u''(t_0)| + (|c_2 p(t_0)|u''(t_0)| + |c_3|)(T - t_0)$$
$$+ p(t_0)|u''(t_0)|\zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\} \|Nu\|_{L^1}$$
$$\leqslant (|c_1| + |c_2|(T - t_0) + \zeta_1) p(t_0)|u''(t_0)| + \max\{\zeta_1, \zeta_2, \zeta_3\} \|Nu\|_{L^1} + |c_3|(T - t_0)$$
$$\leqslant (|c_1| + |c_2|(T - t_0) + \zeta_1)(M + \|Nu\|_{L^1}) + \max\{\zeta_1, \zeta_2, \zeta_3\} \|Nu\|_{L^1} + |c_3|(T - t_0)$$
$$\leqslant (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\}) \|Nu\|_{L^1}$$
$$+ (|c_1| + |c_2|(T - t_0) + \zeta_1) M + |c_3|(T - t_0)$$
$$\leqslant (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})$$
$$\times \left(\int_{t_0}^{T} |f(s, u(s), u'(s), u''(s))| \, ds + \|q\|_{L^1} \|u\|_{C^1} \right)$$
$$+ (|c_1| + |c_2|(T - t_0) + \zeta_1) M + |c_3|(T - t_0)$$
$$\leqslant (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})$$
$$\times \left(\int_{t_0}^{T} |f(s, u(s), u'(s), u''(s))| \, ds + \|q\|_{L^1} \|u\| \right)$$
$$+ (|c_1| + |c_2|(T - t_0) + \zeta_1) M + |c_3|(T - t_0),$$

hence,

$$\|u\| \leqslant \frac{|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\}}{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\}) \|q\|_{L^1}}$$
$$\times \int_{t_0}^{T} |f(s, u(s), u'(s), u''(s))| \, ds$$
$$+ \frac{(|c_1| + |c_2|(T - t_0) + \zeta_1) M + |c_3|(T - t_0)}{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\}) \|q\|_{L^1}}.$$

If (3.12) holds, by (3.7), then

$$
\begin{aligned}
\text{(3.19)} \ \|u\| &\leqslant \frac{|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\}}{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})\|q\|_{L^1}} \\
&\quad \times \left(\|\alpha\|_{L^1}\|u\|_{C^1} + \|\beta\|_{L^1}\|u\|_{C^1}^{\varepsilon_1} + \|\gamma\|_{L^1}\|u'\|_{C^1}^{\varepsilon_2} + \|\delta\|_{L^1}\|u''\|_{C^1}^{\varepsilon_3} + \|\theta\|_{L^1} \right) \\
&\quad + \frac{(|c_1| + |c_2|(T - t_0) + \zeta_1)M + |c_3|(T - t_0)}{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})\|q\|_{L^1}} \\
&\leqslant \frac{|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\}}{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})\|q\|_{L^1}} \\
&\quad \times \left(\|\alpha\|_{L^1}\|u\|_{C^1} + \|\beta\|_{L^1}\|u\|_{C^1}^{\varepsilon_1} + \|\gamma\|_{L^1}(b_3\|u\|_{C^1} + b_4)^{\varepsilon_2} \right. \\
&\quad \left. + \|\delta\|_{L^1}(b_1\|u\|_{C^1} + b_2)^{\varepsilon_3} + \|\theta\|_{L^1} \right) \\
&\quad + \frac{(|c_1| + |c_2|(T - t_0) + \zeta_1)M + |c_3|(T - t_0)}{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})\|q\|_{L^1}} \\
&\leqslant \frac{|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\}}{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})\|q\|_{L^1}} (\|\alpha\|_{L^1}\|u\|_{C^1} \\
&\quad + \|\beta\|_{L^1}\|u\|_{C^1}^{\varepsilon_1} + (2b_3)^{\varepsilon_2}\|\gamma\|_{L^1}\|u\|_{C^1}^{\varepsilon_2} + (2b_1)^{\varepsilon_3}\|\delta\|_{L^1}\|u\|_{C^1}^{\varepsilon_3} \\
&\quad + (2b_4)^{\varepsilon_2}\|\gamma\|_{L^1} + (2b_2)^{\varepsilon_3}\|\delta\|_{L^1} + \|\theta\|_{L^1}) \\
&\quad + \frac{(|c_1| + |c_2|(T - t_0) + \zeta_1)M + |c_3|(T - t_0)}{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})\|q\|_{L^1}}.
\end{aligned}
$$

Since $\varepsilon_i \in \,]0, 1[\,$ $(i = 1, 2, 3)$, $u \notin \operatorname{Ker} L$, the rest of the proof is divided in two cases.

Case 1. $\|u\|_{C^1}^{\varepsilon_1}, \|u\|_{C^1}^{\varepsilon_2}, \|u\|_{C^1}^{\varepsilon_3} \in \,]1, \|u\|_{C^1}[\,$.

Case 2. $\|u\|_{C^1}^{\varepsilon_1}, \|u\|_{C^1}^{\varepsilon_2}, \|u\|_{C^1}^{\varepsilon_3} \in \,]\|u\|_{C^1}, 1]$.

For Case 1, in view of $\|u\|_{C^1}^{\varepsilon_1}, \|u\|_{C^1}^{\varepsilon_2}, \|u\|_{C^1}^{\varepsilon_3} \leqslant \|u\|_{C^1}$, (3.19) yields

$$
\begin{aligned}
\|u\| &\leqslant \frac{|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\}}{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})\|q\|_{L^1}} \\
&\quad \times ((\|\alpha\|_{L^1} + \|\beta\|_{L^1} + (2b_3)^{\varepsilon_2}\|\gamma\|_{L^1} + (2b_1)^{\varepsilon_3}\|\delta\|_{L^1})\|u\|_{C^1} \\
&\quad + (2b_4)^{\varepsilon_2}\|\gamma\|_{L^1} + (2b_2)^{\varepsilon_3}\|\delta\|_{L^1} + \|\theta\|_{L^1}) \\
&\quad + \frac{(|c_1| + |c_2|(T - t_0) + \zeta_1)M + |c_3|(T - t_0)}{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})\|q\|_{L^1}}.
\end{aligned}
$$

Noting that $\|u\|_{C^1} \leqslant \|u\|$, we have

$$
\begin{aligned}
\|u\| &\leqslant \frac{\mathcal{C}_1((2b_4)^{\varepsilon_2}\|\gamma\|_{L^1} + (2b_2)^{\varepsilon_3}\|\delta\|_{L^1} + \|\theta\|_{L^1})}{1 - \mathcal{C}_1(\|\alpha\|_{L^1} + \|\beta\|_{L^1} + (2b_3)^{\varepsilon_2}\|\gamma\|_{L^1} + (2b_1)^{\varepsilon_3}\|\delta\|_{L^1})} \\
&\quad + \frac{\mathcal{C}_2}{1 - \mathcal{C}_1(\|\alpha\|_{L^1} + \|\beta\|_{L^1} + (2b_3)^{\varepsilon_2}\|\gamma\|_{L^1} + (2b_1)^{\varepsilon_3}\|\delta\|_{L^1})},
\end{aligned}
$$

where

$$\mathcal{C}_1 := \frac{|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\}}{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})\|q\|_{L^1}},$$

$$\mathcal{C}_2 := \frac{(|c_1| + |c_2|(T - t_0) + \zeta_1)M + |c_3|(T - t_0)}{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})\|q\|_{L^1}}.$$

For Case 2, we get that the right hand side of (3.19) equals a constant, that is

$$\|u\| \leqslant \frac{|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\}}{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})\|q\|_{L^1}}$$
$$\times \left(\|\alpha\|_{L^1} + \|\beta\|_{L^1} + ((2b_3)^{\varepsilon_2} + (2b_4)^{\varepsilon_2})\|\gamma\|_{L^1} + ((2b_1)^{\varepsilon_3} + (2b_2)^{\varepsilon_3})\|\delta\|_{L^1} + \|\theta\|_{L^1}\right)$$
$$+ \frac{(|c_1| + |c_2|(T - t_0) + \zeta_1)M + |c_3|(T - t_0)}{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})\|q\|_{L^1}}.$$

Thus for $\varepsilon_i \in\ ]0, 1[\ (i = 1, 2, 3)$ and $u \in \Omega_1$ there exists $\widetilde{M} > 0$ such that $\|u\| \leqslant \widetilde{M}$ when

$$\|\alpha\|_{L^1} + \|\beta\|_{L^1} + (2b_3)^{\varepsilon_2}\|\gamma\|_{L^1} + (2b_1)^{\varepsilon_3}\|\delta\|_{L^1}$$
$$< \frac{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})\|q\|_{L^1}}{|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\}},$$

that is, Ω_1 is bounded.

If (3.13) or (3.14) holds, we can argue in an analogous manner and derive the desired conclusion.

Let

$$\Omega_2 = \{u \in \operatorname{Ker} L: \ Nu \in \operatorname{Im} L\}$$

for $u \in \Omega_2$. Taking $u(t) \equiv \omega\ (\omega \in \mathbb{R})$, a.e. $t \in [t_0, T]$, $Nu \in \operatorname{Im} L = \operatorname{Ker} Q$, we have

$$a_1 \int_{t_0}^{T} \frac{T - \tau}{p(\tau)} \int_{t_0}^{\tau} (f(s, \omega, 0, 0) + \omega q(s))\, \mathrm{d}s\, \mathrm{d}\tau$$
$$+ a_2 \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} (f(s, \omega, 0, 0) + \omega q(s))\, \mathrm{d}s\, \mathrm{d}\tau$$
$$+ a_3 \int_{t_0}^{\xi} \frac{\xi - \tau}{p(\tau)} \int_{t_0}^{\tau} (f(s, \omega, 0, 0) + \omega q(s))\, \mathrm{d}s\, \mathrm{d}\tau$$
$$+ a_4 \int_{t_0}^{\xi} \frac{1}{p(\tau)} \int_{t_0}^{\tau} (f(s, \omega, 0, 0) + \omega q(s))\, \mathrm{d}s\, \mathrm{d}\tau + a_5 = 0,$$

since $QNu = 0$. From (H_3) we know that $\|u\| = |e| \leqslant \overline{M}$, thus Ω_2 is bounded.

378

If (3.15) holds, then let

$$\Omega_3 = \{u \in \operatorname{Ker} L \colon -\lambda Ju + (1-\lambda)(QNu) = 0, \quad \lambda \in [0,1]\},$$

where $J\colon \operatorname{Ker} L \to \operatorname{Im} Q$ is a linear isomorphism given by $J(k) = k$ for any $k \in \mathbb{R}$. Since $u(t) = k$, thus

$$\lambda k = (1-\lambda)(QNk) = \frac{(1-\lambda)}{\Lambda_1}\left(a_1 \int_{t_0}^{T} \frac{T-\tau}{p(\tau)} \int_{t_0}^{\tau} (f(s,k,0,0) + kq(s))\,\mathrm{d}s\,\mathrm{d}\tau \right.$$

$$+ a_2 \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} (f(s,k,0,0) + kq(s))\,\mathrm{d}s\,\mathrm{d}\tau$$

$$+ a_3 \int_{t_0}^{\xi} \frac{\xi-\tau}{p(\tau)} \int_{t_0}^{\tau} (f(s,k,0,0) + kq(s))\,\mathrm{d}s\,\mathrm{d}\tau$$

$$\left. + a_4 \int_{t_0}^{\xi} \frac{1}{p(\tau)} \int_{t_0}^{\tau} (f(s,k,0,0) + kq(s))\,\mathrm{d}s\,\mathrm{d}\tau + a_5 \right).$$

If $\lambda = 1$, then $k = 0$ and in the case $\lambda \in [0,1[$, if $|k| < \overline{M}$, we have

$$\lambda k^2 = \frac{k(1-\lambda)}{\Lambda_1}\left(a_1 \int_{t_0}^{T} \frac{T-\tau}{p(\tau)} \int_{t_0}^{\tau} (f(s,k,0,0) + kq(s))\,\mathrm{d}s\,\mathrm{d}\tau \right.$$

$$+ a_2 \int_{t_0}^{T} \frac{1}{p(\tau)} \int_{t_0}^{\tau} (f(s,k,0,0) + kq(s))\,\mathrm{d}s\,\mathrm{d}\tau$$

$$+ a_3 \int_{t_0}^{\xi} \frac{\xi-\tau}{p(\tau)} \int_{t_0}^{\tau} (f(s,k,0,0) + kq(s))\,\mathrm{d}s\,\mathrm{d}\tau$$

$$\left. + a_4 \int_{t_0}^{\xi} \frac{1}{p(\tau)} \int_{t_0}^{\tau} (f(s,k,0,0) + kq(s))\,\mathrm{d}s\,\mathrm{d}\tau + a_5 \right) < 0,$$

which is a contradiction. Again, if (3.16) holds, then let

$$\Omega_3 = \{u \in \operatorname{Ker} L \colon \lambda Ju + (1-\lambda)(QNu) = 0, \ \lambda \in [0,1]\},$$

where J is as above, similarly to the above argument. Thus in either case $\|u\| = |k| \leqslant \overline{M}$ for any $u \in \Omega_3$, that is, Ω_3 is bounded.

Let Ω be a bounded open subset of X such that $\bigcup_{i=1}^{3} \Omega_i \subset \Omega$. By Lemma 3.3, we can check that $K_P(I-Q)N\colon \overline{\Omega} \to X$ is compact, thus N is L-compact on $\overline{\Omega}$.

Finally, we verify that the condition (3) of Theorem 2.1 is fulfilled. We define a homotopy

$$H(u,\lambda) = \pm\lambda Ju + (1-\lambda)(QNu).$$

According to the above argument, we have

$$H(u, \lambda) \neq 0, \quad u \in \partial\Omega \cap \operatorname{Ker} L,$$

thus, by the degree property of homotopy invariance, we obtain

$$\deg(QN_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) = \deg(H(\cdot, 0), \Omega \cap \operatorname{Ker} L, 0)$$
$$= \deg(H(\cdot, 1), \Omega \cap \operatorname{Ker} L, 0) = \deg(\pm J, \Omega \cap \operatorname{Ker} L, 0) \neq 0.$$

Thus, the conditions of Theorem 2.1 are satisfied, that is, the operator equation $Lu = Nu$ admits at least one nontrivial solution in $\operatorname{Dom} L \cap \overline{\Omega}$. Therefore, the problem $(\mathcal{P})$ has at least one solution in $C^1[t_0, T]$. $\qquad\square$

4. APPLICATION

Consider the problem

$$(4.1) \qquad \begin{cases} (p(t)u''(t))' - q(t)u(t) + \frac{1}{2}(\Theta + 1)u''(t)u(t) + \Xi(1 - u'(t)^2) = 0, \\ m(u(0), u''(0)) = 0, \ n(u(1), u'(1)) = 0, \ l(u(\frac{3}{4}), u'(\frac{3}{4}), u''(\frac{3}{4})) = 0. \end{cases}$$

The equation (4.1) is the well known Falkner-Skan equation [6], [11] when $p(t) = 1$ and $q(t) = 0$, $0 < t < 1$, which describes a nonlinear one-dimensional third-order boundary value problem, whose solutions are the similarity solutions of the two-dimensional incompressible laminar boundary layer equations. When $\Theta = 1$, it arises in the study of two-dimensional incompressible viscous flow past a thin semiinfinite flat plate Ξ, $0 \leqslant \Xi \leqslant 1$. The special case $\Xi = 0$ is Blasius's equation, in which the wedge reduces to a flat plate. The special case $\Xi = 1/2$ is called Homann's equation, in which the wedge reduces to a flat plate. The special case $\Xi = 1$ is called Hiemenz's equation. $\Xi > 0$ corresponds to a flow toward the wedge, otherwise, to a flow away from the wedge. When $\Theta \neq 1$, taking $\Theta = \Xi$, we have a Blasius flow over a flat plate with a sharp edge as $\Xi = 0$; a flow over a wedge with half angle $\theta_{\frac{1}{2}} = \Xi/(\Xi + 1)$, $0 < \theta_{\frac{1}{2}} < \pi/2$ as $0 < \Xi < 1$; a Hiemenz flow toward a plane stagnation point as $\Xi = 1$; a flow into a corner with $\theta_{\frac{1}{2}} > \pi/2$ as $1 < \Xi < 2$; no corresponding simple ideal flow as $2 < \Xi$.

Set $c_1 = c_2 = 1/50$, $p(t) = 10 - x$, $q(t) = 1/12$. It is easy to calculate that $\zeta_1 \approx 0.051$, $\zeta_2 \approx 0.105$, $\zeta_3 = 1/9$, $b_1 \approx 5.565$, $b_3 \approx 6.426$. Let $r = \sup\{\|\mu\| \colon \mu \in E\} < 1$,

so $\mu, \nu, \vartheta \leqslant 1$, and by the Young inequality ($p - q$ inequality), we have

$$
\begin{aligned}
|f(t, \mu, \nu, \vartheta)| &= \left| \frac{1}{2}(\Theta + 1)\mu\vartheta + \Xi(1 - \nu^2) \right| \\
&= \frac{1}{2}(\Theta + 1)|\mu\vartheta| + \Xi(\nu^2 + 1) \\
&\leqslant \frac{1}{3}(\Theta + 1)|\mu|^{3/2} + \frac{1}{6}(\Theta + 1)|\vartheta|^3 + \frac{2}{3}\Xi|\nu|^{3/2} + \frac{1}{3}\Xi|\nu|^3 + \Xi \\
&\leqslant \frac{1}{3}(\Theta + 1)|\mu|^{3/4} + \frac{1}{3}\Xi|\nu| + \frac{2}{3}\Xi|\nu|^{3/7} + \frac{1}{6}(\Theta + 1)|\vartheta|^{3/5} + \Xi,
\end{aligned}
$$

where $f(t, \mu, \nu, \vartheta)$ satisfies (3.13). Taking $\Theta = 1$, $\Xi = 1/2$, we consider the well-known Homann's equation, in which the wedge reduces to a flat plate. Further, we have

$$
\begin{aligned}
\|\alpha\|_{L^1} &+ \|\beta\|_{L^1} + (2b_3)^{\varepsilon_2}\|\gamma\|_{L^1} + (2b_1)^{\varepsilon_3}\|\delta\|_{L^1} \\
&= \frac{2}{3} + \frac{1}{6} + \frac{1}{3}(12.853)^{3/7} + \frac{1}{3}(11.129)^{3/5} \approx \frac{17}{6} \\
&< \frac{1 - (|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\})\|q\|_{L^1}}{|c_1| + |c_2|(T - t_0) + \zeta_1 + \max\{\zeta_1, \zeta_2, \zeta_3\}} \\
&= \frac{1 - (\frac{1}{25} + 0.051 + \frac{1}{9})\frac{1}{12}}{\frac{1}{25} + 0.051 + \frac{1}{9}} \approx 4.855.
\end{aligned}
$$

Then all hypotheses of Theorem 3.1 hold. Hence, the problem (4.1) when

$$
l\left(\widetilde{m}\left(\frac{1}{10} \right) + \frac{3}{4}\tilde{n}\left(\frac{1}{10} \right) + c\left(\frac{3}{4} - \frac{37}{4}\ln\frac{40}{37} \right), \; \tilde{n}\left(\frac{1}{10} \right) + c\ln\frac{40}{37}, \frac{4}{37}c \right) = 0
$$

admits at least one nontrivial solution at resonance.

References

[1] *R. P. Agarwal*: Focal Boundary Value Problems for Differential and Difference Equations. Mathematics and its Applications 436, Kluwer Academic Publishers, Dordrecht, 1998.

[2] *D. R. Anderson*: Discrete third-order three-point right-focal boundary value problems. Comput. Math. Appl. *45* (2003), 861–871.

[3] *D. R. Anderson, J. M. Davis*: Multiple solutions and eigenvalues for third-order right focal boundary value problems. J. Math. Anal. Appl. *267* (2002), 135–157.

[4] *C. P. Gupta*: Solvability of multi-point boundary value problem at resonance. Result. Math. *28* (1995), 270–276.

[5] *N. Kosmatov*: Multi-point boundary value problems on an unbounded domain at resonance. Nonlinear Anal., Theory Methods Appl. *68* (2008), 2158–2171.

[6] *S. J. Liao*: A second-order approximate analytical solution of a simple pendulum by the process analysis method. J. Appl. Mech. *59* (1992), 970–975.

[7] *B. Liu, J. S. Yu*: Solvability of multi-point boundary value problem at resonance III. Appl. Math. Comput. *129* (2002), 119–143.

[8] *Z. Liu, L. Debnath, S. Kang*: Existence of monotone positive solutions to a third order two-point generalized right focal boundary value problem. Comput. Math. Appl. *55* (2008), 356–367.

[9] *J. Mawhin*: Topological Degree Methods in Nonlinear Boundary Value Problems. Regional Conference Series in Mathematics 40. AMS, Providence, RI, 1979.

[10] *J. Mawhin, D. Ruiz*: A strongly nonlinear Neumann problem at resonance with restrictions on the nonlinearity just in one direction. Topol. Methods Nonlinear Anal. *20* (2002), 1–14.

[11] *H. Nasr, I. A. Hassanien, H. M. El-Hawary*: Chebyshev solution of laminar boundary layer flow. Int. J. Comput. Math. *33* (1990), 127–132.

[12] *Juan J. Nieto*: Existence of solutions in a cone for nonlinear alternative problems. Proc. Am. Math. Soc. *94* (1985), 433–436.

[13] *I. Rachůnková, S. Staněk*: Topological degree method in functional boundary value problems at resonance. Nonlinear Anal., Theory Methods Appl. *27* (1996), 271–285.

[14] *J. Santanilla*: Some coincidence theorem in wedges, cones, and convex sets. J. Math. Anal. Appl. *105* (1985), 357–371.

[15] *P. J. Y. Wong*: Multiple fixed-sign solutions for a system of generalized right focal problems with deviating arguments. J. Math. Anal. Appl. *323* (2006), 100–118.

Author's address: *Youwei Zhang*, Department of Mathematics, Hexi University, Zhangye, Gansu 734000, China, e-mail: `ywzhang0288@163.com`.