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THE CLASSIFICATION OF TWO STEP NILPOTENT COMPLEX LIE ALGEBRAS OF DIMENSION 8

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Abstract. A Lie algebra $\mathfrak{g}$ is called two step nilpotent if $\mathfrak{g}$ is not abelian and $[\mathfrak{g},\mathfrak{g}]$ lies in the center of $\mathfrak{g}$. Two step nilpotent Lie algebras are useful in the study of some geometric problems, such as commutative Riemannian manifolds, weakly symmetric Riemannian manifolds, homogeneous Einstein manifolds, etc. Moreover, the classification of two-step nilpotent Lie algebras has been an important problem in Lie theory. In this paper, we study two step nilpotent indecomposable Lie algebras of dimension 8 over the field of complex numbers. Based on the study of minimal systems of generators, we choose an appropriate basis and give a complete classification of two step nilpotent Lie algebras of dimension 8.

Keywords: two-step nilpotent Lie algebra; base; minimal system of generators; related sets; H-minimal system of generators

MSC 2010: 17B05, 17B30, 17B40

1. Introduction

The classification problem is one of the major problems in the theory of finite dimensional Lie algebras over an algebraically closed field of characteristic zero. Levis theorem and the classification of semisimple Lie algebras reduce this problem to the classification of solvable Lie algebras. The study of solvable Lie algebras can generally be reduced to the study of nilpotent Lie algebras. The first important research on the classification of nilpotent Lie algebras is due to Umlauf [15] in later 19th century. In his thesis, he presented the first nontrivial classification. Since then, several attempts have been made to develop some machinery whereby the classification problem could be reformulated. Seeley [14] and Gong [7] gave the classification of nilpotent Lie algebras of dimension 7 over $\mathbb{C}$ and $\mathbb{R}$. In dimension 8,

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there are only partial results, see [2], [5], [6], [10]. For the 2-step case, the problem can be reduced to the classification of 2-step nilpotent Lie algebras by considering 2, 3 and 4-dimensional center (see Section 3 for an interpretation). In 2011, Ren and Zhu [11] obtained a complete classification of 2-step nilpotent Lie algebras of dimension 8 with 2-dimensional center.

The purpose of this article is to give a complete classification of 2-step nilpotent Lie algebras of dimension 8 over the field of complex numbers. Namely, we will present the classification of 2-step nilpotent Lie algebras whose center is of dimension 3 or 4. As can be expected, the computation involved here is much more complicated.

The arrangement of this paper is as follows. In Section 2, we recall some definitions and fundamental results on nilpotent Lie algebras. In Section 3 we present the lists of two-step nilpotent Lie algebras of dimension 8 with 3 and 4 dimensional centers, respectively. In Section 4, we prove that each 8-dimensional two-step nilpotent Lie algebra with 3 or 4-dimensional center must be isomorphic to one of the Lie algebras in our lists in Section 3.

2. Preliminaries

In this section, we recall some elementary facts about nilpotent Lie algebras. All Lie algebras in this section are over an algebraic closed field of characteristic zero.

Lemma 2.1 ([13]). *If N is a nilpotent Lie algebra, then the following two assertions are equivalent:*

(1) *The set $\{x_1, x_2, \ldots, x_n\}$ is a minimal system of generators.*
(2) *The set $\{x_1 + N^2, x_2 + N^2, \ldots, x_n + N^2\}$ is a basis for the vector space N/N^2.*

If H is a maximal torus of N (i.e., a maximal abelian subalgebra of $\operatorname{Der} N$ consisting of semisimple linear transformations), then N can be decomposed into a direct sum of root spaces with respect to H:

$$N = \sum_{\alpha \in H^*} N_\alpha.$$

The scalar $\operatorname{mult}(\alpha) := \dim N_\alpha$ is called the multiplicity of the root α. We also denote $\dim[x] = \dim N_\alpha$ for nonzero $x \in N_\alpha$.

If the maximal torus of N is trivial, then N is characteristically nilpotent. By the results of R. Carles in [3], one easily deduces that the nilindex of N is greater than or equal to 3. Since the nilindex of a 2-step nilpotent Lie algebra is 2, the maximal torus of a 2-step nilpotent Lie algebra cannot be reduced to zero. In the following we fix a nonzero maximal torus of a 2-step nilpotent Lie algebra N.

Definition 2.1 ([13]). Let H be a maximal torus of N. A minimal system of generators consisting of root vectors for H is called an H-msg of N.

Definition 2.2 ([10]). Let $\{x_1, x_2, \dots, x_n\}$ be a minimal system of generators of a nilpotent Lie algebra N. The related set of x_i is defined to be the set $G(x_i) = \{x_j;\ [x_i, x_j] \neq 0\}$, and the number $p_i = |G(x_i)|$ is called the related number of x_i. The n-tuple of integers $(p_1, p_2, \dots, p_n)$ [4] is called the related sequence of $\{x_1, x_2, \dots, x_n\}$.

Definition 2.3 ([10]). A minimal system of generators is called a $(p_1, p_2, \dots, p_n)$-msg if its related sequence is $(p_1, p_2, \dots, p_n)$. It is called a $(p_1, p_2, \dots, p_n)$-H-msg if it is also an H-msg.

The study of nilpotent Lie algebras with generators was initiated by G. Favre [4], who obtained the classification of nilpotent Lie algebras with maximal rank.

Definition 2.4 ([9]). A nilpotent Lie algebra N is called quasi-cyclic if N has a subspace U such that $N = U \oplus U^2 \oplus \dots \oplus U^k$, where $U^i = [U, U^{i-1}]$.

If N is a quasi-cyclic nilpotent Lie algebra, it is clear that there exists a derivation I such that $I|_{U^s} = s \cdot \mathrm{id}$ (where id denotes the identity map). Hence there exists a maximal torus on N.

Lemma 2.2 ([9]). *If N is a 2-step nilpotent Lie algebra, then $C(N) = N^2$ if and only if $p_i > 0$ for any $(p_1, p_2, \dots, p_n)$-msg.*

Lemma 2.3 ([9]). *Let N be a 2-step nilpotent Lie algebra, and $\{x_1, x_2, \dots, x_n\}$ a minimal system of generators of N. If a linear transformation h of N satisfies*

$$h[x_i, x_j] = [h(x_i), x_j] + [x_i, h(x_j)],\ 1 \leqslant i, j \leqslant n$$

then $h \in \mathrm{Der}\, N$.

Lemma 2.4 ([9]). *Let N be a 2-step nilpotent Lie algebra, and $\{x_1, x_2, \dots, x_n\}$ a minimal system of generators of N. If there exists $h \in \mathrm{Der}\, N$ such that $h(x_i) = a_i x_i$ and $a_i \neq a_j$ $(i \neq j)$, then there exists a maximal torus H on N such that $\{x_1, x_2, \dots, x_n\}$ is an H-msg, and $\dim[x_i] = 1$ for any i.*

Lemma 2.5 ([9]). *Let N be a quasi-cyclic nilpotent Lie algebra, and $\{x_1, x_2, \ldots, x_n\}$ an H_1-msg of N, $\{y_1, y_2, \ldots, y_n\}$ an H_2-msg of N. Then there exists $\theta \in \mathrm{Aut}\,(N)$ such that*

$$(y_1, y_2, \ldots, y_n)^t = A(\theta(x_1), \theta(x_2), \ldots, \theta(x_n))^t$$

where $(y_1, y_2, \ldots, y_n)^t$ is the transpose of the matrix $(y_1, y_2, \ldots, y_n)$, and A is an $n \times n$ invertible matrix. In particular, if for any i, $\dim[x_i] = 1$, then A is a monomial matrix (i.e., each row or each column has exactly one nonzero entry).

Remark 2.1. When N is quasi-cyclic, then by Lemma 2.5 it is easily seen that the related sequence $(p_1, p_2, \ldots, p_n)$ $(p_i \geqslant p_{i+1})$ of $\{x_1, x_2, \ldots, x_n\}$ is an invariant of N if $\{x_1, x_2, \ldots, x_n\}$ is an H-msg, and $\dim[x_i] = 1$ for each x_i.

It is important to find new invariants in the classification of Lie algebras. Goze and Ancochea obtained the classification of complex nilpotent Lie algebras of dimension 7 (see [1]) by introducing a new invariant which are called the characteristic sequence (see [8]). Later, using the same method, they also obtained the classification of complex filiform Lie algebras of dimension 8 (see [2]). Unfortunately, the characteristic sequences of any 2-step nilpotent Lie algebras of dimension 8 with 2 (or 3, 4)-dimensional center are all of the type $(2, 2, 1, 1, 1, 1)$ (resp. $(2, 2, 1, 1, 1, 1)$, $(2, 1, 1, 1, 1, 1, 1)$). Therefore this method cannot be used to obtain a complete classification of all 2-step nilpotent Lie algebras of dimension 8.

It is worth mentioning that, some other invariants of 2-step Lie algebras were used by Revoy [12] to study 2-step nilpotent Lie algebras. Revoy determined the classes of 2-step nilpotent Lie algebras with small generating sets.

In this paper, we shall apply Lemma 2.5 to determine whether two 2-step nilpotent Lie algebras of dimension 8 are isomorphic. The key here is to find an H-msg of N using our method. By Lemma 2.4, in order to find an H-msg of N, we need only to find a semisimple derivation h whose eigenvalues are distinct (i.e., each eigenvalue has multiplicity one).

3. Lists of the Lie algebras

We now start the classification of 2-step nilpotent Lie algebras of dimension 8. Since 2-step nilpotent Lie algebras of dimension less or equal to 7 have been completely classified, we can assume that the nilpotent Lie algebras are indecomposable (i.e., cannot be decomposed into direct sum of two or more ideals). If N is an indecomposable 2-step nilpotent Lie algebra of dimension 8, then the number k of generators cannot be less than 4. In fact, if $k \leqslant 3$, then by the results in [8] we have $\dim N \leqslant k + C_k^2 \leqslant 3 + 3 = 6$, which is a contradiction. The above assertion amounts to saying that the center of N has dimension 2, 3 or 4, since N is not the Heisenberg algebra.

Obviously, the assumption that N is indecomposable implies that $C(N) = N^2$. Thus $p_i > 0$ for any $(p_1, p_2, \ldots, p_n)$-msg. Let $\{x_1, x_2, \ldots, x_8\}$ be a basis of N.

The case that N has 2-dimensional center has been settled by Ren and Lin in [11]. We now recall the classification.

Theorem 3.1 ([11]). *If* $\dim N = 8$ *and* $\dim N^2 = 2$, *then* N *is isomorphic to one of the following Lie algebras:*

$N_1^{8,2}$: *There exists a* $(1,1,1,1,1,1)$-msg $\{x_1, x_2, \ldots, x_6\}$ *such that*

$$[x_1, x_2] = x_7, [x_3, x_4] = x_8, [x_5, x_6] = x_7 + x_8.$$

$N_2^{8,2}$: *There exists a* $(2,1,1,2,1,1)$-msg $\{x_1, x_2, \ldots, x_6\}$ *such that*

$$[x_1, x_2] = [x_4, x_5] = x_7, [x_1, x_3] = [x_4, x_6] = x_8.$$

$N_3^{8,2}$: *There exists a* $(1,1,1,2,2,1)$-msg $\{x_1, x_2, \ldots, x_6\}$ *such that*

$$[x_1, x_2] = [x_4, x_5] = x_7, [x_3, x_4] = [x_5, x_6] = x_8.$$

$N_4^{8,2}$: *There exists a* $(1,1,1,2,2,1)$-msg $\{x_1, x_2, \ldots, x_6\}$ *such that*

$$[x_1, x_2] = [x_3, x_4] = [x_5, x_6] = x_7, [x_4, x_5] = x_8.$$

$N_5^{8,2}$: *There exists a* $(1,2,2,2,2,1)$-msg $\{x_1, x_2, \ldots, x_6\}$ *such that*

$$[x_1, x_2] = [x_3, x_4] = [x_5, x_6] = x_7, [x_2, x_3] = [x_4, x_5] = x_8.$$

In particular, in each $N_i^{8,2}$, $\{x_1, x_2, \ldots, x_6\}$ *is an* H_i-msg. *Moreover, for any* i, $\dim[x_i] = 1$ *for* $i = 1, 2, 3, 4$, $\dim H_i = 4$, *and* $\dim H_5 = 3$.

The above theorem gives a complete classification of the case that the center of N has dimension 2. To accomplish a classification of all 2-step nilpotent Lie algebras of dimension 8, we need to give the classification for the cases that the center has dimension 3 or 4. We first consider the 3-dimensional case.

Theorem 3.2. *The following* 2*-step nilpotent Lie algebras of dimension* 8 *with* 3*-dimensional center are mutually nonisomorphic:*

$N_1^{8,3}$: *There exists a* $(2,2,2,1,1)$-msg $\{x_1, x_2, \ldots, x_5\}$ *such that*

$$[x_1, x_2] = [x_4, x_5] = x_6, [x_2, x_3] = x_7, [x_1, x_3] = x_8.$$

$N_2^{8,3}$: *There exists a* $(1,2,2,2,1)$-msg $\{x_1, x_2, \ldots, x_5\}$ *such that*

$$[x_1, x_2] = [x_4, x_5] = x_6, [x_2, x_3] = x_7, [x_3, x_4] = x_8.$$

$N_3^{8,3}$: *There exists a* $(1,2,2,2,1)$-msg $\{x_1, x_2, \ldots, x_5\}$ *such that*

$$[x_1, x_2] = x_6, [x_2, x_3] = [x_4, x_5] = x_7, [x_3, x_4] = x_8.$$

$N_4^{8,3}$: *There exists a* $(2,2,2,2,2)$-msg $\{x_1, x_2, \ldots, x_5\}$ *such that*

$$[x_1, x_2] = x_6, [x_2, x_3] = [x_4, x_5] = x_7, [x_3, x_4] = [x_5, x_1] = x_8.$$

$N_5^{8,3}$: *There exists a* $(3,2,1,1,1)$-msg $\{x_1, x_2, \ldots, x_5\}$ *such that*

$$[x_1, x_2] = x_6, [x_1, x_3] = x_7, [x_1, x_4] = x_8, [x_2, x_5] = x_7.$$

$N_6^{8,3}$: *There exists a* $(3,2,2,2,1)$-msg $\{x_1, x_2, \ldots, x_5\}$ *such that*

$$[x_1, x_2] = x_6, [x_1, x_3] = x_7, [x_1, x_4] = x_8, [x_2, x_3] = x_8, [x_4, x_5] = x_7.$$

$N_7^{8,3}$: *There exists a* $(3,2,2,2,1)$-msg $\{x_1, x_2, \ldots, x_5\}$ *such that*

$$[x_1, x_2] = x_6, [x_1, x_3] = x_7, [x_1, x_5] = x_8, [x_2, x_4] = x_8, [x_3, x_4] = x_6.$$

$N_8^{8,3}$: *There exists a* $(3,3,2,1,1)$-msg $\{x_1, x_2, \ldots, x_5\}$ *such that*

$$[x_1, x_2] = x_6, [x_1, x_3] = x_7, [x_2, x_3] = x_8, [x_1, x_4] = x_8, [x_2, x_5] = x_7.$$

$N_9^{8,3}$: *There exists a* $(3,3,2,2,2)$-msg $\{x_1, x_2, \ldots, x_5\}$ *such that*

$$[x_1, x_2] = x_6, [x_1, x_3] = x_7, [x_2, x_3] = x_8, [x_1, x_4] = x_8, [x_2, x_5] = x_7, [x_4, x_5] = x_6.$$

$N_{10}^{8,3}$: *There exists a* $(1,2,2,2,1)$-msg $\{x_1, x_2, \ldots, x_5\}$ *such that*

$$[x_1, x_2] = x_6, [x_2, x_3] = [x_3, x_4] = x_7, [x_4, x_5] = x_8.$$

$N_{11}^{8,3}$: *There exists a* $(2,2,2,2,2)$-msg $\{x_1, x_2, \ldots, x_5\}$ *such that*

$$[x_1, x_2] = x_6, [x_2, x_3] = x_7, [x_3, x_4] = x_8, [x_4, x_5] = [x_5, x_1] = x_7.$$

In particular, $\{x_1, x_2, \ldots, x_5\}$ *is an* H_i-msg *in each* $N_i^{8,3}$*, and* $\dim[x_i] = 1$ *for any* $1 \leqslant i \leqslant 9$.

P r o o f. First we consider $N_1^{8,3}$. By Lemma 2.3, there exists an $h_1 \in \operatorname{Der} N$ such that the matrix of h_1 relative to $\{x_1, x_2, \ldots, x_8\}$ is $\operatorname{diag}(1, 0, -1, 3, -2, 1, -1, 0)$. By Lemma 2.4, there exists a maximal torus H_1 on N such that $\{x_1, x_2, \ldots, x_5\}$ is an H_1-msg, and $\dim[x_i] = 1$ for each x_i.

For any $h \in H_1$, the matrix of h relative to $\{x_1, x_2, \ldots, x_8\}$ has the form

$$\mathrm{diag}(a_1, a_2, a_3, a_4, a_5, a_1 + a_2, a_2 + a_3, a_1 + a_3).$$

Since $[x_1, x_2] = [x_4, x_5]$, we have $a_1 + a_2 = a_4 + a_5$. It is easily seen that $\{t_1, t_2, t_3, t_4\}$ is a basis of H_1, where the matrices of t_1, t_2, t_3, t_4 relative to $\{x_1, x_2, \ldots, x_8\}$ are $\mathrm{diag}(1,0,0,0,1,1,0,1)$, $\mathrm{diag}(0,1,0,0,1,1,1,0)$, $\mathrm{diag}(0,0,1,0,0,0,0,1)$, $\mathrm{diag}(0,0,0,1,-1,0,0,0)$, respectively.

This argument can be applied to other cases to get a $(p_1, p_2, p_3, p_4, p_5)$-msg and a basis of a maximal torus relative to $\{x_1, x_2, \ldots, x_8\}$ for each Lie algebras $N_j^{(8,3)}$. We summarize the results in the following table:

	$(p_1, p_2, p_3, p_4, p_5)$-msg	a basis of maximal torus relative to $\{x_1, x_2, \ldots, x_8\}$	rank
$N_1^{8,3}$	$(2,2,2,1,1)$	$\mathrm{diag}(1,0,0,0,1,1,0,1)$, $\mathrm{diag}(0,1,0,0,1,1,1,0)$, $\mathrm{diag}(0,0,1,0,0,0,0,1)$, $\mathrm{diag}(0,0,0,1,-1,0,0,0)$.	4
$N_2^{8,3}$	$(1,2,2,2,1)$	$\mathrm{diag}(1,0,0,0,1,1,0,0)$, $\mathrm{diag}(0,1,0,0,1,1,1,0)$, $\mathrm{diag}(0,0,1,0,0,0,1,1)$, $\mathrm{diag}(0,0,0,1,0,0,0,1)$.	4
$N_3^{8,3}$	$(1,2,2,2,1)$	$\mathrm{diag}(1,0,0,0,0,1,0,0)$, $\mathrm{diag}(0,1,0,1,0,1,1,1)$, $\mathrm{diag}(0,0,1,1,0,0,1,2)$, $\mathrm{diag}(0,0,0,-1,1,0,0,-1)$.	4
$N_4^{8,3}$	$(2,2,2,2,2)$	$\mathrm{diag}(2,0,0,1,-1,2,0,1)$, $\mathrm{diag}(0,2,0,1,1,2,2,1)$, $\mathrm{diag}(0,0,1,0,1,0,1,1)$.	3
$N_5^{8,3}$	$(3,2,1,1,1)$	$\mathrm{diag}(1,0,0,0,1,1,1,1)$, $\mathrm{diag}(0,1,0,0,-1,1,0,0)$, $\mathrm{diag}(0,0,1,0,1,0,1,0)$, $\mathrm{diag}(0,0,0,1,0,0,0,1)$.	4
$N_6^{8,3}$	$(3,2,2,2,1)$	$\mathrm{diag}(1,0,0,-1,2,1,1,0)$, $\mathrm{diag}(0,1,0,1,-1,1,0,1)$, $\mathrm{diag}(0,0,1,1,0,0,1,1)$.	3
$N_7^{8,3}$	$(3,2,2,2,1)$	$\mathrm{diag}(1,0,0,1,0,1,1,1)$, $\mathrm{diag}(0,1,0,1,2,1,0,2)$, $\mathrm{diag}(0,0,1,-1,-1,0,1,-1)$.	3
$N_8^{8,3}$	$(3,3,2,1,1)$	$\mathrm{diag}(1,0,0,-1,1,1,1,0)$, $\mathrm{diag}(0,1,0,1,-1,1,0,1)$, $\mathrm{diag}(0,0,1,1,1,0,1,1)$.	3
$N_9^{8,3}$	$(3,3,2,2,2)$	$\mathrm{diag}(1,3,2,4,0,4,3,5)$, $\mathrm{diag}(3,1,2,0,4,4,5,3)$.	2
$N_{10}^{8,3}$	$(1,2,2,2,1)$	$\mathrm{diag}(1,0,0,0,0,1,0,0)$, $\mathrm{diag}(0,1,0,1,0,1,1,1)$, $\mathrm{diag}(0,0,1,0,0,0,1,0)$, $\mathrm{diag}(0,0,0,0,1,0,0,1)$.	4
$N_{11}^{8,3}$	$(2,2,2,2,2)$	$\mathrm{diag}(1,0,0,1,-1,1,0,1)$, $\mathrm{diag}(0,1,0,0,1,1,1,0)$, $\mathrm{diag}(0,0,1,0,1,0,1,1)$.	3

Combining this table with Lemma 2.5, we conclude that the above eleven algebras are mutually nonisomorphic. □

Now we turn to the case in which the center of the two-step nilpotent Lie algebra has dimension 4. We first give the list of such Lie algebras.

Theorem 3.3. *The following two-step nilpotent Lie algebras of dimension* 8 *with* 4 *dimensional center are mutually nonisomorphic:*

$N_1^{8,4}$: *There exists a* $(2,2,2,2)$-msg $\{x_1, x_2, x_3, x_4\}$ *such that*

$$[x_1, x_2] = x_5, [x_2, x_3] = x_6, [x_3, x_4] = x_7, [x_4, x_1] = x_8.$$

$N_2^{8,4}$: *There exists a* $(3,2,2,1)$-msg $\{x_1, x_2, x_3, x_4\}$ *such that*

$$[x_1, x_2] = x_5, [x_1, x_3] = x_6, [x_2, x_3] = x_7, [x_1, x_4] = x_8.$$

$N_3^{8,4}$: *There exists a* $(3,3,2,2)$-msg $\{x_1, x_2, x_3, x_4\}$ *such that*

$$[x_1, x_2] = x_5, [x_1, x_3] = x_6, [x_2, x_4] = x_6, [x_2, x_3] = x_7, [x_1, x_4] = x_8.$$

Proof. Similarly to the proof of Theorem 3.2, one can deduce the following results.

First consider $N_1^{8,4}$. It is easily seen that there exists an $h_1 \in \operatorname{Der} N$ such that the matrix of h_1 relative to $\{x_1, x_2, \ldots, x_8\}$ is $\operatorname{diag}(1,2,3,4,3,5,7,5)$. Moreover, there exists a maximal torus H_1 on N such that $\{x_1, x_2, x_3, x_4\}$ is an H_1-msg, and $\dim[x_i] = 1$ for each x_i. It is easy to show that $\{t_1, t_2, t_3, t_4\}$ is a basis of H_1, where the matrices of t_1, t_2, t_3, t_4 relative to $\{x_1, x_2, \ldots, x_8\}$ are $\operatorname{diag}(1,0,0,0,1,0,0,1)$, $\operatorname{diag}(0,1,0,0,1,1,0,1)$, $\operatorname{diag}(0,0,1,0,0,1,1,0)$, $\operatorname{diag}(0,0,0,1,0,0,1,1)$, respectively.

Next consider $N_2^{8,4}$. Similarly to the above case, there exists an $h_2 \in \operatorname{Der} N$ such that the matrix of h_2 relative to $\{x_1, x_2, \ldots, x_8\}$ is $\operatorname{diag}(1,2,3,4,3,4,5,5)$. Moreover, there exists a maximal torus H_2 on N such that $\{x_1, x_2, x_3, x_4\}$ is an H_2-msg, and $\dim[x_i] = 1$ for each x_i. It is easy to show that $\{t_1, t_2, t_3, t_4\}$ is a basis of H_2, where the matrices of t_1, t_2, t_3, t_4 relative to $\{x_1, x_2, \ldots, x_8\}$ are $\operatorname{diag}(1,0,0,0,1,1,0,1)$, $\operatorname{diag}(0,1,0,0,1,0,1,0)$, $\operatorname{diag}(0,0,1,0,0,1,1,0)$, $\operatorname{diag}(0,0,0,1,0,0,0,1)$, respectively.

Finally, we consider $N_3^{8,4}$. In this case, there exists an $h_3 \in \operatorname{Der} N$ such that the matrix of h_3 relative to $\{x_1, x_2, \ldots, x_8\}$ is $\operatorname{diag}(1,2,4,3,3,5,6,4)$. Further, there exists a maximal torus H_3 on N such that $\{x_1, x_2, x_3, x_4\}$ is an H_3-msg, and $\dim[x_i] = 1$ for each x_i. On the other hand, it is easy to show that $\{t_1, t_2, t_3\}$ is a basis of H_3, where the matrices of t_1, t_2, t_3 relative to $\{x_1, x_2, \ldots, x_8\}$ are $\operatorname{diag}(1,0,0,1,1,1,0,2)$, $\operatorname{diag}(0,1,0,-1,1,0,1,-1)$, $\operatorname{diag}(0,0,1,1,0,1,1,1)$, respectively.

Combining the above observations with Lemma 2.5, we conclude that the above three Lie algebras are mutually nonisomorphic. □

4. The classification

In this section we shall prove that the lists in Theorems 3.1 and 3.2 give a complete classification of 8-dimensional two-step nilpotent Lie algebras whose center has dimension 3 and 4, respectively. Precisely, we have the following two theorems:

Theorem 4.1. *If* $\dim N = 8$ *and* $\dim N^2 = 3$*, then* N *is isomorphic to one of the Lie algebras* $N_i^{8,3}$, $1 \leqslant i \leqslant 11$.

Theorem 4.2. *If* $\dim N = 8$, $\dim N^2 = 4$*, then* N *is isomorphic to one of the Lie algebras* $N_i^{8,4}$, $1 \leqslant i \leqslant 3$.

In order to prove Theorems 4.1 and 4.2, we need the following notation and lemmas.

Definition 4.1. The classical lexicographic order on the set of n-tuples is defined as follows: $(p_1, p_2, \ldots, p_n) \geqslant (q_1, q_2, \ldots, q_n)$ if there exists i such that $p_k = q_k, 1 \leqslant k \leqslant i-1$ and $p_i > q_i$. A minimal system of generators is called a minimal-$(p_1, p_2, \ldots, p_n)$-msg if its related sequence $(p_1, p_2, \ldots, p_n)$ is minimal with respect to this order (i.e., $(q_1, q_2, \ldots, q_n) \geqslant (p_1, p_2, \ldots, p_n)$, for any related sequence $(q_1, q_2, \ldots, q_n)$).

Remark 4.1. The minimal sequence $(p_1, p_2, \ldots, p_n)$ of $\{x_1, x_2, \ldots, x_n\}$ is an invariant of N. We call $(p_1, p_2, \ldots, p_n)$ the minimal order sequence of N.

Lemma 4.1. *If* $\dim N = 8$ *and* $\dim N^2 = 3$*, then* N *can only have minimal order sequences* $(2,2,2,1,1)$, $(2,2,2,2,2)$, $(3,2,1,1,1)$, $(3,2,2,2,1)$ *or* $(3,3,2,1,1)$.

P r o o f. Let $(p_1, p_2, \ldots, p_5)$ be the minimal order sequence of N, and $\{x_1, x_2, \ldots, x_5\}$ its related minimal system of generators.

We first assert that there do not exist x_{i_1}, x_{i_2} such that $\{x_{i_3}, x_{i_4}, x_{i_5}\} \subset G(x_{i_1}) \cap G(x_{i_2})$, where $\{i_1, i_2, i_3, i_4, i_5\} = \{1,2,3,4,5\}$. In fact, otherwise, we could deduce that either $[x_{i_1}, x_{i_3}]$, $[x_{i_1}, x_{i_4}]$, $[x_{i_1}, x_{i_5}]$ are linearly dependent, or $[x_{i_1}, x_{i_3}]$, $[x_{i_1}, x_{i_4}]$, $[x_{i_1}, x_{i_5}]$ and $[x_{i_2}, x_{i_3}]$, $[x_{i_2}, x_{i_4}]$, $[x_{i_2}, x_{i_5}]$ are linearly independent, since $\dim C(N) = 3$. We now show that each of the above assertions leads to a contradiction.

(1) $[x_{i_1}, x_{i_3}]$, $[x_{i_1}, x_{i_4}]$, $[x_{i_1}, x_{i_5}]$ are linearly dependent. We may assume $p_{i_3} \geqslant p_{i_4} \geqslant p_{i_5}$, $a[x_{i_1}, x_{i_3}] + b[x_{i_1}, x_{i_4}] + c[x_{i_1}, x_{i_5}] = 0$. Suppose $x'_{i_3} = ax_{i_3} + bx_{i_4} + cx_{i_5}$, $x'_{i_4} = bx_{i_4} + cx_{i_5}$. Then either $(p_{i_1}, p_{i_2}, p'_{i_3}, p_{i_4}, p_{i_5})$ is less than $(p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}, p_{i_5})$ $(a \neq 0)$, or $(p_{i_1}, p_{i_2}, p_{i_3}, p'_{i_4}, p_{i_5})$ is less than $(p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}, p_{i_5})$ $(a = 0)$. This is a contradiction.

(2) Both $[x_{i_1}, x_{i_3}]$, $[x_{i_1}, x_{i_4}]$, $[x_{i_1}, x_{i_5}]$ and $[x_{i_2}, x_{i_3}]$, $[x_{i_2}, x_{i_4}]$, $[x_{i_2}, x_{i_5}]$ are linearly independent. It is easy to show that there exist a, b, c and $k \neq 0$ such that

$$a[x_{i_1}, x_{i_3}] + b[x_{i_1}, x_{i_4}] + c[x_{i_1}, x_{i_5}] = k(a[x_{i_2}, x_{i_3}] + b[x_{i_2}, x_{i_4}] + c[x_{i_2}, x_{i_5}]).$$

We may assume $p_{i_3} \geqslant p_{i_4} \geqslant p_{i_5}$. Suppose $x'_{i_3} = ax_{i_3} + bx_{i_4} + cx_{i_5}$, $x'_{i_4} = bx_{i_4} + cx_{i_5}$, and $x'_{i_1} = x_{i_1} - kx_{i_2}$. Then either $(p'_{i_1}, p_{i_2}, p'_{i_3}, p_{i_4}, p_{i_5})$ is less than $(p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}, p_{i_5})$ $(a \neq 0)$, or $(p'_{i_1}, p_{i_2}, p_{i_3}, p'_{i_4}, p_{i_5})$ is less than $(p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}, p_{i_5})$ $(a = 0,\ b \neq 0)$, or $(p'_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}, p_{i_5})$ is less than $(p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}, p_{i_5})$ $(a = 0,\ b = 0)$. This is also a contradiction. Thus $p_2 \neq 4$.

Secondly, it is easy to show that none of $(4,1,1,1,1)$, $(4,2,2,1,1)$, $(4,2,2,2,2)$, $(4,3,1,1,1)$, $(4,3,2,2,1)$, $(4,3,3,1,1)$, $(4,3,3,2,2)$, $(4,3,3,3,1)$, $(4,3,3,3,3)$ can be the minimal order sequence of N. Thus $p_1 < 4$.

Finally, if $(p_1, p_2, p_3, p_4, p_5) = (3,3,3,3,2)$, we may suppose $\{x_1, x_2\} = G(x_5)$. Then we have $\{x_3, x_4, x_5\} = G(x_1) \cap G(x_2)$. By the above assertion we conclude that $(3,3,3,3,2)$ cannot be the minimal order sequence of N.

If $(p_1, p_2, p_3, p_4, p_5) = (3,3,3,2,1)$, we may assume $\{x_1\} = G(x_5)$. Then it is easy to show that $[x_1, x_2]$, $[x_1, x_3]$, $[x_1, x_5]$ are linearly independent, and $[x_2, x_3] = a[x_1, x_2] + b[x_1, x_3] + c[x_1, x_5]$ $(a \neq 0)$. Let $x'_2 = ax_2 + bx_3 + cx_5$, $x'_3 = x_1 + a^{-1}x_3$. Then the related sequence $(p_1, p'_2, p'_3, p_4, p_5)$ of $\{x_1, x'_2, x'_3, x_4, x_5\}$ is less than $(3,3,3,2,1)$. This is a contradiction.

If $(p_1, p_2, p_3, p_4, p_5) = (3,3,2,2,2)$, then from the first result we know that $[x_1, x_2] \neq 0$. Thus we may assume $\{x_1, x_2\} = G(x_3)$, and $\{x_1, x_5\} = G(x_4)$, $\{x_2, x_4\} = G(x_5)$. It is easy to show that $[x_1, x_2]$, $[x_1, x_3]$, $[x_2, x_3]$ are linearly independent. Let $[x_1, x_2] = x_6$, $[x_1, x_3] = x_7$, $[x_2, x_3] = x_8$. Then we have $[x_1, x_4]$, $[x_2, x_5]$, $[x_4, x_5] \in \{ax_7 + bx_8; a, b \in C\}$. Thus there exist a and b such that $x'_1 = ax_1 + bx_5$, $[x'_1, x_4] = [x'_1, x_3]$. Let $x'_3 = x_3 - x_4$. Then we have $\{x_2, x_4\} = G(x'_1)$, $\{x'_1, x'_3, x_5\} = G(x_2)$, $\{x_2, x_5\} = G(x'_3)$, $\{x'_1, x_5\} = G(x_4)$, and $\{x_2, x'_3, x_4\} = G(x_5)$. But this contradicts the assumption that $[x_2, x'_3]$, $[x'_3, x_5]$, $[x_2, x_5]$ are linearly dependent. So $(3,3,2,2,2)$ cannot be the minimal order sequence of N.

Finally, using the assumption that N is indecomposable, one can easily show that $(2,1,1,1,1)$ cannot be the minimal order sequence of N.

Combining the above results with Theorem 3.2, we conclude that $(2,2,2,1,1)$, $(2,2,2,2,2)$, $(3,2,1,1,1)$, $(3,2,2,2,1)$ and $(3,3,2,1,1)$ are the only possible minimal order sequences of N. □

Lemma 4.2. *Suppose* $\dim N = 8$ *and* $\dim N^2 = 3$. *If* $\{x_1, x_2, x_3, x_4, x_5\}$ *is a minimal system of generators of* N, *and there exists* $h \in \operatorname{Der} N$ *such that* $h(x_i) = a_i x_i$, *and* $a_i \neq a_j (i \neq j)$, *then* N *is isomorphic to one of the Lie algebras* $N_i^{8,3}$, $1 \leqslant i \leqslant 9$.

P r o o f. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a $(p_1, p_2, p_3, p_4, p_5)$-$H$-msg, with $p_i \geqslant p_{i+1}$. Suppose there exists $h \in \operatorname{Der} N$ such that $h(x_i) = a_i x_i$, with $a_i \neq a_j (i \neq j)$. Then $[x_i, x_j]$, $[x_i, x_k]$ ($[x_i, x_j], [x_i, x_k] \neq 0$) are linearly independent, since they lie in different eigenspaces of the linear transformation h. Therefore we have $(p_1, p_2, p_3, p_4, p_5) \leqslant (3, 3, 2, 2, 2)$. A direct computation shows:

If $(p_1, p_2, p_3, p_4, p_5) = (2, 2, 2, 1, 1)$, then N is isomorphic to $N_1^{8,3}$, $N_2^{8,3}$ or $N_3^{8,3}$.
If $(p_1, p_2, p_3, p_4, p_5) = (2, 2, 2, 2, 2)$, then N is isomorphic to $N_4^{8,3}$.
If $(p_1, p_2, p_3, p_4, p_5) = (3, 2, 1, 1, 1)$, then N is isomorphic to $N_5^{8,3}$.
If $(p_1, p_2, p_3, p_4, p_5) = (3, 2, 2, 2, 1)$, then N is isomorphic to $N_6^{8,3}$ or $N_7^{8,3}$.
If $(p_1, p_2, p_3, p_4, p_5) = (3, 3, 2, 1, 1)$, then N is isomorphic to $N_8^{8,3}$.
If $(p_1, p_2, p_3, p_4, p_5) = (3, 3, 2, 2, 2)$, then N is isomorphic to $N_9^{8,3}$.
This completes the proof of the lemma. □

Lemma 4.3. *If N has a minimal-$(2, 2, 2, 1, 1)$-msg, then N is isomorphic to $N_1^{8,3}$, $N_2^{8,3}$, $N_3^{8,3}$, or $N_{10}^{8,3}$.*

P r o o f. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a minimal-$(2, 2, 2, 1, 1)$-msg. By the assumption that N is indecomposable, one easily sees that, if $[x_4, x_5] \neq 0$, then N is isomorphic to $N_1^{8,3}$. Otherwise we may assume that neither of the elements $[x_4, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$ or $[x_3, x_5]$ is 0. In this case, one easily deduces that, if $[x_4, x_1]$, $[x_1, x_2]$, $[x_3, x_5]$ are linearly independent, then N is isomorphic to $N_{10}^{8,3}$; if $[x_4, x_1]$, $[x_1, x_2]$, $[x_3, x_5]$ are linearly dependent, then N is isomorphic to $N_2^{8,3}$ or $N_3^{8,3}$. □

Lemma 4.4. *If N has a minimal-$(3, 2, 1, 1, 1)$-msg, then N is isomorphic to $N_5^{8,3}$.*

P r o o f. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a minimal-$(3, 2, 1, 1, 1)$-msg. We may assume $[x_2, x_3] \neq 0$. Then it is easy to show that $[x_1, x_4]$, $[x_1, x_5]$, $[x_1, x_2]$ are linearly independent, and $[x_2, x_3] \in \{a[x_1, x_4] + b[x_1, x_5];\ a, b \in C\}$. Therefore N is isomorphic to $N_5^{8,3}$. □

Lemma 4.5. *If N has a minimal-$(3, 2, 2, 2, 1)$-msg, then N is isomorphic to $N_7^{8,3}$.*

P r o o f. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a minimal-$(3, 2, 2, 2, 1)$-msg. If $G(x_1) = \{x_2, x_3, x_4\}$, $G(x_2) = \{x_1, x_3\}$, $G(x_3) = \{x_1, x_2\}$, $G(x_4) = \{x_1, x_5\}$ and $G(x_5) = \{x_4\}$, then $[x_1, x_2]$, $[x_1, x_3]$, $[x_1, x_4]$ are linearly independent. Further, $[x_1, x_2]$, $[x_1, x_3]$, $[x_2, x_3]$ are also linearly independent. Thus we may assume $[x_1, x_4] = a[x_2, x_3]$ and $[x_5, x_4] = b[x_1, x_2] + c[x_1, x_3]$ $(a, b, c \neq 0)$. Suppose $x_1' = x_1 + x_5 - (a/b) \cdot x_3$, $x_2' = bx_2 + cx_3 - x_4$. Then the related sequence of $\{x_1', x_2', x_3, x_4, x_5\}$ is $(2, 2, 2, 2, 2)$. So we have $G(x_1) = \{x_2, x_3, x_5\}$, $G(x_2) = \{x_1, x_4\}$, $G(x_3) = \{x_1, x_3\}$, $G(x_4) = \{x_2, x_3\}$ and $G(x_5) = \{x_1\}$. It is easy to show that the following sets of vectors are all linearly independent:

(1) $[x_1, x_2]$, $[x_1, x_3]$, $[x_1, x_5]$,
(2) $[x_2, x_4]$, $[x_3, x_4]$,
(3) $[x_2, x_4]$, $[x_2, x_1]$,
(4) $[x_3, x_1]$, $[x_3, x_4]$.

From this we see that one can choose a new basis of N satisfying $[x_1, x_5] = [x_3, x_4]$, $[x_1, x_3] = [x_2, x_4]$, and keeping other brackets invariant. Thus N is isomorphic to $N_7^{8,3}$. □

Lemma 4.6. *If N has a minimal-$(3,3,2,1,1)$-msg, then N is isomorphic to $N_8^{8,3}$.*

Proof. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a minimal-$(3,3,2,1,1)$-msg. Without losing generality, we may assume that $[x_1, x_2]$, $[x_1, x_3]$, $[x_2, x_3]$ are linearly independent, that $[x_1, x_4]$, $[x_2, x_5]$ are linearly independent, and that $[x_1, x_4], [x_2, x_5] \in \{a[x_1, x_3] + b[x_2, x_3] \mid a, b \in C\}$. Therefore there exist a, b, k and t such that $[x_1, x_3'] = k[x_2, x_5]$, $[x_2, x_3'] = t[x_1, x_4]$, and $(x_3' = ax_4 + bx_5)$. Thus N is isomorphic to $N_8^{8,3}$. □

Finally, we prove

Lemma 4.7. *If N has a minimal-$(2,2,2,2,2)$-msg $\{x_1, x_2, x_3, x_4, x_5\}$, then either N is isomorphic to $N_{11}^{8,3}$, or N has a derivation $h \in \operatorname{Der} N$ such that $h(y_i) = a_i y_i$, with $a_i \neq a_j$ $(i \neq j)$, and $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle y_1, y_2, y_3, y_4, y_5 \rangle$.*

Proof. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a minimal-$(2,2,2,2,2)$-msg. We may assume $[x_1, x_2] = x_6$, $[x_2, x_3] = x_7$, $[x_3, x_4] = x_8$, $[x_4, x_5] = \alpha_1 x_6 + \alpha_2 x_7 + \alpha_3 x_8$, and $[x_5, x_1] = \beta_1 x_6 + \beta_2 x_7 + \beta_3 x_8$. Let $\alpha = \{\alpha_i \neq 0\}$ and $\beta = \{\beta_i \neq 0\}$. It is easy to check that $|\alpha| > 0$, $|\beta| > 0$. By symmetry, we may assume $|\alpha| \geqslant |\beta|$. Now let $V = \langle x_1, x_2, x_3, x_4, x_5 \rangle$ be the vector space spanned by $\{x_1, x_2, x_3, x_4, x_5\}$. We have the following cases:

Case 2–1: $\alpha_1 \neq 0$, $\beta_1 \neq 0$, $\alpha_2 = \alpha_3 = \beta_2 = \beta_3 = 0$. Obviously, this case does not occur.

Case 2–2: $\alpha_1 \neq 0$, $\beta_2 \neq 0$, $\alpha_2 = \alpha_3 = \beta_1 = \beta_3 = 0$. We can change the basis to a new one satisfying $[x_1, x_2] = x_6$, $[x_2, x_3] = x_7$, $[x_3, x_4] = x_8$, $[x_4, x_5] = x_6$, and $[x_5, x_1] = x_7$. Then one can check that this algebra is isomorphic to $N_4^{8,3}$.

Case 2–3: $\alpha_1 \neq 0$, $\beta_3 \neq 0$, $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = 0$. This case can be reduced to Case 2–2.

Case 2–4: $\alpha_2 \neq 0$, $\beta_1 \neq 0$, $\alpha_1 = \alpha_3 = \beta_2 = \beta_3 = 0$. This case does not occur.

Case 2–5: $\alpha_2 \neq 0$, $\beta_2 \neq 0$, $\alpha_1 = \alpha_3 = \beta_1 = \beta_3 = 0$. Obviously, in this case N is isomorphic to $N_{11}^{8,3}$.

Case 2–6: $\alpha_2 \neq 0$, $\beta_3 \neq 0$, $\alpha_1 = \alpha_3 = \beta_1 = \beta_2 = 0$. This case can be reduced to Case 2–2.

Case 2–7: $\alpha_3 \neq 0$, $\beta_1 \neq 0$, $\alpha_1 = \alpha_2 = \beta_2 = \beta_3 = 0$. This case does not occur.

Case 2–8: $\alpha_3 \neq 0$, $\beta_2 \neq 0$, $\alpha_1 = \alpha_2 = \beta_1 = \beta_3 = 0$. This case can be reduced to Case 2–4.

Case 2–9: $\alpha_3 \neq 0$, $\beta_3 \neq 0$, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$. This case can be reduced to Case 2–1.

Case 3–1: $\alpha_2 \neq 0$, $\alpha_3 \neq 0$, $\beta_3 \neq 0$, $\alpha_1 = \beta_1 = \beta_2 = 0$. In this case we can change the basis to a new one satisfying $[x_1, x_2] = x_6$, $[x_2, x_3] = x_7$, $[x_3, x_4] = x_8$, $[x_4, x_5] = x_7 + x_8$, and $[x_5, x_1] = x_8$. Let h be an element of $\mathfrak{gl}(N)$ whose matrix relative to $\{x_1, x_2, \ldots, x_8\}$ is

$$\begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 \end{pmatrix}.$$

Then it is easy to show that $h \in \operatorname{Der} N$, that h is semisimple on V and that it has distinct eigenvalues 3, -1, 1, 2, 0.

Case 3–2: $\alpha_1 \neq 0$, $\alpha_3 \neq 0$, $\beta_3 \neq 0$, $\alpha_2 = \beta_1 = \beta_2 = 0$. This case can be reduced to Case 3–1.

Case 3–3: $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\beta_3 \neq 0$, $\alpha_3 = \beta_1 = \beta_2 = 0$. This case can be reduced to Case 2–3.

Case 3–4: $\alpha_2 \neq 0$, $\alpha_3 \neq 0$, $\beta_1 \neq 0$, $\alpha_1 = \beta_2 = \beta_3 = 0$. This case can be reduced to Case 2–7.

Case 3–5: $\alpha_1 \neq 0$, $\alpha_3 \neq 0$, $\beta_1 \neq 0$, $\alpha_2 = \beta_2 = \beta_3 = 0$. This case can be reduced to Case 2–5.

Case 3–6: $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\beta_1 \neq 0$, $\alpha_3 = \beta_2 = \beta_3 = 0$. In this case we can change the basis to a new one satisfying $[x_1, x_2] = x_6$, $[x_2, x_3] = x_7$, $[x_3, x_4] = x_8$, $[x_4, x_5] = x_6 + x_7$, and $[x_5, x_1] = x_6$. Then it is easily seen that N is isomorphic to $N_6^{8,3}$.

Case 3–7: $\alpha_2 \neq 0$, $\alpha_3 \neq 0$, $\beta_2 \neq 0$, $\alpha_1 = \beta_1 = \beta_3 = 0$. This case can be reduced to Case 3–5.

Case 3–8: $\alpha_1 \neq 0$, $\alpha_3 \neq 0$, $\beta_2 \neq 0$, $\alpha_2 = \beta_1 = \beta_3 = 0$. This case can be reduced to Case 3–3.

Case 3–9: $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\beta_2 \neq 0$, $\alpha_3 = \beta_1 = \beta_3 = 0$. This case can be reduced to Case 3–1.

Case 4–1: $|\alpha| = 3$, $\beta_3 \neq 0$, $\beta_1 = \beta_2 = 0$. This case can be reduced to Case 3–1.

Case 4–2: $|\alpha| = 3$, $\beta_2 \neq 0$, $\beta_1 = \beta_3 = 0$. This case can be reduced to Case 4–1.

Case 4–3: $|\alpha| = 3$, $\beta_1 \neq 0$, $\beta_2 = \beta_3 = 0$. The algebra is isomorphic to $N_{11}^{8,3}$.

Case 4–4: $|\alpha| = 2$, $|\beta| = 2$, $\alpha_1 = \beta_1 = 0$. In this case we can change the basis to a new one satisfying $[x_1, x_2] = x_6$, $[x_2, x_3] = x_7$, $[x_3, x_4] = x_8$, $[x_4, x_5] = x_7 + x_8$, $[x_5, x_1] = x_7 + \lambda x_8$, and $\lambda \neq 0, 1, 9/8$ (when λ is replaced by $1 - \lambda$, the resulting algebra is isomorphic to the original one). Let h be an element of $\mathfrak{gl}(N)$ whose matrix relative to $\{x_1, x_2, \ldots, x_8\}$ is

$$\begin{pmatrix} 2\lambda - 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 - 2\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\lambda - 1 & 0 & 1 - \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 2\lambda \end{pmatrix}.$$

Then it is easy to check that $h \in \operatorname{Der} N$, that h is semisimple on V and that the restriction of h to V has distinct eigenvalues $2\sqrt{\lambda^2 - \lambda}$, $-2\sqrt{\lambda^2 - \lambda}$, 0, $\lambda + \sqrt{\lambda^2 - \lambda}$, and $\lambda - \sqrt{\lambda^2 - \lambda}$.

Case 4–5: $|\alpha| = 2$, $|\beta| = 2$, $\alpha_3 = \beta_1 = 0$. In this case we can change the basis to a new one satisfying $[x_1, x_2] = x_6$, $[x_2, x_3] = x_7$, $[x_3, x_4] = x_8$, $[x_4, x_5] = x_6 + x_7$ and $[x_5, x_1] = x_7 + x_8$. Let h be an element in $\mathfrak{gl}(N)$ whose matrix relative to $\{x_1, x_2, \ldots, x_8\}$ is

$$\begin{pmatrix} 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

Then one can check that $h \in \operatorname{Der} N$, that h is semisimple on V, and that the restriction of h to V has distinct eigenvalues 0, -1, 2, -2, 1.

Case 4–6: $|\alpha| = 2$, $|\beta| = 2$, $\alpha_2 = \beta_1 = 0$. This case can be reduced to Case 4–5.

Case 4–7: $|\alpha| = 2$, $|\beta| = 2$, $\alpha_1 = \beta_2 = 0$. This case can be reduced to Case 3–1.

Case 4–8: $|\alpha| = 2$, $|\beta| = 2$, $\alpha_2 = \beta_2 = 0$. This case can be reduced to Case 4–4.

Case 4–9: $|\alpha| = 2$, $|\beta| = 2$, $\alpha_3 = \beta_2 = 0$. This case can be reduced to Case 4–6.

Case 4–10: $|\alpha| = 2$, $|\beta| = 2$, $\alpha_1 = \beta_3 = 0$. This case can be reduced to Case 2–5.

Case 4–11: $|\alpha| = 2$, $|\beta| = 2$, $\alpha_2 = \beta_3 = 0$. This case can be reduced to Case 4–7.

Case 4–12: $|\alpha| = 2$, $|\beta| = 2$, $\alpha_3 = \beta_3 = 0$. This case can be reduced to Case 4–4.

Case 5–1: $|\alpha| = 3$, $|\beta| = 2$, $\beta_1 = 0$. In this case we can change the basis to a new one satisfying $[x_1, x_2] = x_6$, $[x_2, x_3] = x_7$, $[x_3, x_4] = x_8$, $[x_4, x_5] = x_6 + x_7 + x_8$ and $[x_5, x_1] = x_7 + \lambda x_8$, where $\lambda \neq 0$. Let h be an element of $\mathfrak{gl}(N)$ whose matrix of h relative to $\{x_1, x_2, \ldots, x_8\}$ is

$$\begin{pmatrix} -1 & -1 & 0 & -1 & -\lambda & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \lambda - 1 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda & \lambda & -2 \end{pmatrix}.$$

Then it is easy to check that $h \in \operatorname{Der} N$, that h is semisimple on V and that the restriction of h to V has distinct eigenvalues $0, -1, 1, -2, -3$.

Case 5–2: $|\alpha| = 3$, $|\beta| = 2$, $\beta_2 = 0$. In this case we can change the basis to a new one satisfying $[x_1, x_2] = x_6$, $[x_2, x_3] = x_7$, $[x_3, x_4] = x_8$, $[x_4, x_5] = x_6 + x_7 + x_8$ and $[x_5, x_1] = x_6 + \lambda x_8$, where $\lambda \neq 0$. Let h be the element in $\mathfrak{gl}(N)$ whose matrix relative to $\{x_1, x_2, \ldots, x_8\}$ is

$$\begin{pmatrix} 3\lambda & 1 & \lambda & 0 & \lambda - 1 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 2\lambda & 0 & -\lambda & 0 & 0 & 0 \\ 0 & \lambda & \lambda & \lambda & \lambda(\lambda - 1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\lambda & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 3\lambda \end{pmatrix}.$$

Then it is easy to check that $h \in \operatorname{Der} N$, that h is semisimple on V and that the restriction of h to V has distinct eigenvalues $0, -\lambda, \lambda, 2\lambda, 3\lambda$.

Case 5–3: $|\alpha| = 3$, $|\beta| = 2$, $\beta_3 = 0$. This case can be reduced to Case 4–12.

Case 6–1: $|\alpha| = 3$, $|\beta| = 3$. In this case we can change the basis to a new one satisfying $[x_1, x_2] = x_6$, $[x_2, x_3] = x_7$, $[x_3, x_4] = x_8$, $[x_4, x_5] = x_6 + x_7 + x_8$, and $[x_5, x_1] = x_6 + \lambda x_7 + \mu x_8$, where $\lambda \neq 0$ and $\mu \neq 0, 1$. Let h be the element in $\mathfrak{gl}(N)$

whose matrix relative to $\{x_1, x_2, \ldots, x_8\}$ is

$$\begin{pmatrix} \mu & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -\lambda & 1-2\mu & -1 & 1-\lambda & 0 & 0 & 0 & 0 \\ \mu-\lambda & 1 & 2\mu-1 & 1 & 0 & 0 & 0 & 0 \\ \mu & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\ -\lambda & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -\mu & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & -\mu & 0 & \mu \end{pmatrix}.$$

Then it is easy to check that $h \in \operatorname{Der} N$, that h is semisimple on V and that the restriction of h to V has distinct eigenvalues 0, $-2\sqrt{\mu^2-\mu}$, $-\sqrt{\mu^2-\mu}$, $\sqrt{\mu^2-\mu}$, $2\sqrt{\mu^2-\mu}$.

Combining all the results in the above cases we complete the proof of the lemma. □

Now we are ready to prove Theorems 4.1 and 4.2.

P r o o f of Theorem 4.1. This follows directly from Lemmas 4.1–4.7. □

P r o o f of Theorem 4.2. It is easy to know that the minimal order sequence of N can only be $(2,2,2,2)$, $(3,2,2,1)$ or $(3,3,2,2)$. Now we prove the theorem case by case, using an argument similar (but simpler) to that in Lemma 4.7.

(1) If the minimal order sequence of N is $(2,2,2,2)$, then N is isomorphic to $N_1^{8,4}$.

(2) If the minimal order sequence of N is $(3,2,2,1)$, then N is isomorphic to $N_2^{8,4}$.

(3) Let $\{x_1, x_2, x_3, x_4\}$ be a minimal-$(3,3,2,2)$-msg of N. Obviously, $[x_1, x_2] \neq 0$, and the vectors $[x_1, x_3]$, $[x_1, x_4]$, $[x_2, x_3]$ and $[x_2, x_4]$ are linearly dependent. From this one easily deduces that N is isomorphic to $N_3^{8,4}$. □

The method used in this paper can be easily applied to greater dimensions, but the computation involved will be much more complicated.

In the classification of 7 dimensional Lie algebras, there is a one parameter family of nonisomorphic Lie algebras. This phenomenon does not appear for 2-step nilpotent Lie algebras in dimension 7. By the results of this paper, it does not appear in dimension 8 either. In dimension 9 or greater, it was proved by Gauger [6] that there are infinitely many isomorphism classes of 2-step nilpotent Lie algebras. So the number 8 is the largest dimension for which the isomorphism classes of 2-step nilpotent Lie algebras are finite.

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