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Czechoslovak Mathematical Journal, Vol. 63 (2013), No. 3, 763–775

Persistent URL: `http://dml.cz/dmlcz/143487`

Czechoslovak Mathematical Journal, 63 (138) (2013), 763–775

PRESSING DOWN LEMMA FOR λ-TREES AND ITS APPLICATIONS

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(Received May 14, 2012)

Abstract. For any ordinal λ of uncountable cofinality, a λ-tree is a tree T of height λ such that $|T_\alpha| < \mathrm{cf}(\lambda)$ for each $\alpha < \lambda$, where $T_\alpha = \{x \in T\colon \mathrm{ht}(x) = \alpha\}$. In this note we get a Pressing Down Lemma for λ-trees and discuss some of its applications. We show that if η is an uncountable ordinal and T is a Hausdorff tree of height η such that $|T_\alpha| \leqslant \omega$ for each $\alpha < \eta$, then the tree T is collectionwise Hausdorff if and only if for each antichain $C \subset T$ and for each limit ordinal $\alpha \leqslant \eta$ with $\mathrm{cf}(\alpha) > \omega$, $\{\mathrm{ht}(c)\colon c \in C\} \cap \alpha$ is not stationary in α. In the last part of this note, we investigate some properties of κ-trees, κ-Suslin trees and almost κ-Suslin trees, where κ is an uncountable regular cardinal.

Keywords: tree; D-space; λ-tree; property γ; collectionwise Hausdorff

MSC 2010: 54F05, 54F65

1. Introduction

Recall that a *tree* is a poset $T = (T, <)$ such that for every $x \in T$, the set $\hat{x} = \{y \in T\colon y < x\}$ is well-ordered by $<$. The order-type of $\hat{x}$ under $<$ is the height of x in T, which is denoted by $\mathrm{ht}(x)$. Given $A \subset T$, let $\widehat{A} = \bigcup\{\hat{a}\colon a \in A\}$. The αth level of T is the set $T_\alpha = \{x \in T\colon \mathrm{ht}(x) = \alpha\}$. We set $T \restriction \alpha = \bigcup\limits_{\beta<\alpha} T_\beta$. Define $T \restriction C = \bigcup\limits_{\beta\in C} T_\beta$. The height of T, $\mathrm{ht}(T)$, is the least α such that $T_\alpha = \emptyset$. An *antichain* of T is a pairwise incomparable subset of T. The *interval topology* on a tree T is the topology whose base is all sets of the form $(s,t] = \{x \in T\colon s < x \leqslant t\}$, together with all singletons $\{t\}$ such that t is a minimal member of T (see [2]). If a tree T with its interval topology is a Hausdorff topological space, then the tree T is called a *Hausdorff tree*. We know that if T is a Hausdorff tree, then for any elements $s, t \in T$ of a limit level of T, $t = s$ if and only if $\hat{t} = \hat{s}$.

Research supported by the National Natural Science Foundation of China (Grant No. 11271036), and by Doctoral Fund of Innovation of Beijing University of Technology.

An ω_1-*tree* is a tree T such that: (1) $\mathrm{ht}(T) = \omega_1$; (2) for each $\alpha < \omega_1$, $|T_\alpha| \leqslant \omega$; (3) for every $t \in T$ and for every α, $\mathrm{ht}(t) < \alpha < \omega_1$, t has at least two successors of height α; (4) if $\mathrm{ht}(t) = \mathrm{ht}(s)$ is a limit ordinal, $t = s$ if and only if $\hat{t} = \hat{s}$ (see [2]). In [8], Hart showed the Pressing-Down Lemma (PDL) for ω_1-trees. Some properties of ω_1-trees were investigated in [4] and [8].

For any uncountable regular cardinal κ, a κ-*tree* is a tree T such that $|T| = \kappa$ and $|T_\alpha| < \kappa$ for each $\alpha < \kappa$ (see [9]). For any ordinal λ of uncountable cofinality, a λ-*tree* is a tree T of height λ such that $|T_\alpha| < \mathrm{cf}(\lambda)$ for each $\alpha < \lambda$. In this note we get the following Pressing Down Lemma for λ-trees: Let T be a λ-tree, where λ is an ordinal of uncountable cofinality. If $A \subset T$ is a set which meets stationary (in λ) many levels and $f\colon A \to T$ is a function such that $f(x) < x$ for each $x \in A$, then there is $b \in T$ and there is a subset $A' \subset A$ which meets stationary (in λ) many levels such that $b \in (f(x), x]$ for each $x \in A'$. As a corollary, we get that if T is a λ-tree, where λ is an ordinal of uncountable cofinality, and a subtree $X \subset T$ is a subtree of T such that $\{\mathrm{ht}(x)\colon x \in X\}$ is stationary in λ, then X is not meta-Lindelöf. By this conclusion, we show that if T is a tree of height η such that $|T_\alpha| \leqslant \omega$ for each $\alpha \in \eta$ and a subtree $X \subset T$ is meta-Lindelöf, then X is a D-space.

Let T be a tree of height κ, where κ is an uncountable regular cardinal. A subset X of T is called *stationary* if and only if $\{\mathrm{ht}(x)\colon x \in X\}$ is stationary in κ. An ω_1-tree is an *almost ω_1-Suslin tree* if and only if it has no stationary antichain ([2]). It was proved in [2] that an ω_1-tree is an almost ω_1-Suslin tree if and only if its tree topology is a collectionwise Hausdorff topology. This conclusion is generalized in this note. We get the following conclusion. If T is a Hausdorff tree of height η, where η is an uncountable ordinal, and $|T_\alpha| \leqslant \omega$ for each $\alpha < \eta$, then the tree T is collectionwise Hausdorff if and only if for each antichain $C \subset T$ and for each limit ordinal $\alpha \leqslant \eta$ with $\mathrm{cf}(\alpha) > \omega$, $\{\mathrm{ht}(c)\colon c \in C\} \cap \alpha$ is not stationary in α.

In the last part of this note, we investigate some properties of κ-trees, κ-Suslin trees, almost κ-Suslin trees, and ω_1'-trees. A κ-tree is an *almost κ-Suslin tree* if and only if it has no stationary antichain. We show that if there is a κ-tree with property γ, then there is a κ-tree with property γ which is not a κ-Suslin tree. We show that if there exists an almost κ-Suslin tree, then there exists an almost κ-Suslin tree which is not a κ-Suslin tree. The following are equivalent for a Hausdorff κ-tree T: T is normal and collectionwise Hausdorff; T has property γ; T is hereditarily collectionwise normal.

In this note, the set of all positive integers is denoted by $\mathbb{N}$ and ω is $\mathbb{N} \cup \{0\}$. In notation and terminology we will follow [3] and [9].

Lemma 2.1 ([6]). *Let α be an ordinal of uncountable cofinality. If $S \subset \alpha$ is stationary in α [i.e. $S \cap C \neq \emptyset$ for every closed unbounded (in short: cub) subset C of α] and $f\colon S \to \alpha$ is a regressive function on S [i.e. $f(\xi) < \xi$ whenever $\xi \in S \setminus \{0\}$], then there is a stationary subset $T \subset S$ and an ordinal $\varsigma \in \alpha$ with $f(\xi) \leqslant \varsigma$ for all $\xi \in T$. In particular, if α is an uncountable regular cardinal then T and ζ above may be chosen in such a way that $f(\xi) = \zeta$ for all $\xi \in T$.*

Definition 2.2 ([9]). For any uncountable regular cardinal κ, a *κ-tree* is a tree T of height κ such that $|T_\alpha| < \kappa$ for each $\alpha < \kappa$.

Definition 2.3. For any uncountable ordinal λ with $\mathrm{cf}(\lambda) \geqslant \omega_1$, a *$\lambda$-tree* is a tree T of height λ such that $|T_\alpha| < \mathrm{cf}(\lambda)$ for each $\alpha < \lambda$.

Theorem 2.4. *Let T be a λ-tree, where λ is an ordinal of uncountable cofinality, and let $A \subset T$ be a set which meets stationary (in λ) many levels. If $f\colon A \to T$ is a function such that $f(x) < x$ for each $x \in A$, then there is $b \in T$ and there is a subset $A' \subset A$ which meets stationary (in λ) many levels such that $b \in (f(x), x]$ for each $x \in A'$.*

P r o o f. If $A \subset T$ is a set which meets stationary (in λ) many levels, then $S = \{\mathrm{ht}(x)\colon x \in A\}$ is stationary in λ. For each $\alpha \in S$, we choose $x_\alpha \in A$ such that $\mathrm{ht}(x_\alpha) = \alpha$. Since $f(x_\alpha) < x_\alpha$, we have $\mathrm{ht}(f(x_\alpha)) < \mathrm{ht}(x_\alpha) = \alpha$ for each $\alpha \in S$. By Lemma 2.1, there is a stationary subset $S' \subset S$ and an ordinal $\delta < \lambda$ such that $\mathrm{ht}(f(x_\alpha)) < \delta$ for each $\alpha \in S'$. We can assume that $\mathrm{ht}(x_\alpha) > \delta$ for each $\alpha \in S'$. For each $x \in T_\delta$, denote $A_x = \{\alpha\colon \alpha \in S', x \in (f(x_\alpha), x_\alpha]\}$. So $S' = \bigcup\limits_{x \in T_\delta} A_x$. Suppose A_x is not stationary in λ for each $x \in T_\delta$. There is a cub set C_x of λ such that $C_x \cap A_x = \emptyset$ for each $x \in T_\delta$. Since $|T_\delta| < \mathrm{cf}(\lambda)$, we know that $\bigcap\limits_{x \in T_\delta} C_x$ is a cub set in λ. Thus $\Bigl(\bigcap\limits_{x \in T_\delta} C_x\Bigr) \cap S' = \emptyset$, a contradiction. So there is $b \in T_\delta$ such that $A_b = \{\alpha\colon \alpha \in S', b \in (f(x_\alpha), x_\alpha]\} \subset S'$ is a stationary set in λ. Thus $b \in (f(x_\alpha), x_\alpha]$ for each $\alpha \in A_b$. If $A' = \{x_\alpha\colon \alpha \in A_b\}$, then $A' \subset A$ is such that the set A' meets stationary (in λ) many levels and $b \in (f(x), x]$ for each $x \in A'$. $\square$

By the Pressing Down Lemma for an uncountable regular cardinal and a proof similar to that of Theorem 2.4, we can get the following corollary.

Corollary 2.5. *For any uncountable regular cardinal κ, let T be a κ-tree and let $A \subset T$ be a set which meets stationary (in κ) many levels. If $f\colon A \to T$ is a function such that $f(x) < x$ for each $x \in A$, then f is constant on a subset of A which meets stationary (in κ) many levels.*

In [8], using Lemma 2.1, K. P. Hart showed the following conclusion which is also a corollary of Corollary 2.5.

Corollary 2.6 ([8]) (Pressing Down Lemma for ω_1-trees)**.** *Let T be an ω_1-tree and let $A \subset T$ be a set which meets stationary (in ω_1) many levels. If $f\colon A \to T$ is a function such that $f(x) < x$ for each $x \in A$, then f is constant on a set which meets stationary (in ω_1) many levels.*

We can get the following proposition by Theorem 2.4.

Proposition 2.7. *Let T be a λ-tree, where λ is an ordinal of uncountable cofinality. If a subtree $X \subset T$ and $\{\mathrm{ht}(x)\colon x \in X\}$ is stationary in λ, then X is not meta-Lindelöf.*

Proof. Let $\mathcal{W} = \{\hat{x} \cup \{x\}\colon x \in T\}$. If $\mathcal{U} = \{W \cap X\colon W \in \mathcal{W}\}$, then $\mathcal{U}$ is an open cover of X. Let $\mathcal{V}$ be any open (in X) refinement of $\mathcal{U}$. Thus $\mathcal{V}$ is also an open cover of X. For each $x \in X$, there is $V_x \in \mathcal{V}$ such that $x \in V_x$. If $x \in X$ and $\mathrm{ht}(x)$ is a limit ordinal, then there is $f(x) < x$ such that $(f(x), x] \cap X \subset V_x$. Denote $X_l = \{x\colon x \in X \text{ and } \mathrm{ht}(x) \text{ is a limit ordinal}\}$. Since $\{\mathrm{ht}(x)\colon x \in X\}$ is stationary in λ, the set $\{\mathrm{ht}(x)\colon x \in X_l\}$ is stationary in λ. Thus there is a subset $X' \subset X_l$ which meets stationary (in λ) many levels and $z \in X$ such that $z \in (f(x), x]$ for each $x \in X'$ by Theorem 2.4. Thus $[z, x] \cap X \subset (f(x), x] \cap X \subset V_x$ for each $x \in X'$, where the set X' meets stationary (in λ) many levels. Therefore the point z is contained in uncountably many elements of $\mathcal{V}$. So $\mathcal{V}$ is not point-countable. Therefore X is not meta-Lindelöf. □

By Proposition 2.7, we can get the following corollaries.

Corollary 2.8. *If T is a λ-tree, where λ is an ordinal of uncountable cofinality, then T is not meta-Lindelöf.*

Corollary 2.9. *If T is a κ-tree, where κ is an uncountable regular cardinal, then T is not meta-Lindelöf.*

Corollary 2.10 ([8])**.** *No ω_1-tree is meta-Lindelöf.*

The notion of a D-space was introduced by E. K. van Douwen and W. F. Pfeffer in [12]. A *neighborhood assignment* for a space X is a function φ from X to the topology of the space X such that $x \in \varphi(x)$ for any $x \in X$. A space X is called a *D-space*, if for any neighborhood assignment φ for X there exists a closed discrete subspace D of X such that $X = \bigcup\{\varphi(d)\colon\ d \in D\}$ (see [12]). It is an open problem as to whether every paracompact Hausdorff space is a D-space. Recall that a space X is a *generalized ordered space* (abbreviated GO space) if it is embeddable in a linearly ordered topological space. In [11] (E. K. van Douwen and J. Lutzer, 1997) and [5] (W. G. Fleissner and A. M. Stanley, 2001), it was proved that every GO space X is a D-space if and only if X is a paracompact space. We consider the D-property in a tree and get Theorem 2.11.

Let us recall some facts on D-spaces. The D-property is hereditary with respect to closed subsets. A countable union of closed D-subspaces in a space X is a D-space ([1]).

Theorem 2.11. *Let T be a tree of height η and $|T_\alpha| \leqslant \omega$ for each $\alpha \in \eta$. If a subtree $X \subset T$ and X is meta-Lindelöf, then X is a D-space.*

Proof. The proof is by induction. The statement is true if $\eta = 1$. Let η be an ordinal. Suppose that the statement is true for each ordinal $\xi < \eta$. Let $\varphi = \{\varphi(x)\colon\ x \in X\}$ be any neighborhood assignment for X. If η is a successor ordinal, then there is an ordinal β such that $\eta = \beta+1$. The set $X^{(\beta)} = \{x\colon\ \mathrm{ht}(x) = \beta, x \in X\}$ is a closed discrete subset of X. If $F = X \setminus \bigcup\{\varphi(d)\colon\ d \in X^{(\beta)}\}$, then F is a closed subset of X and $F \subset X \setminus X^{(\beta)}$. Since $X \setminus X^{(\beta)}$ is a D-space by induction and the D-property is hereditary with respect to closed subsets, the set F is a D-space. Thus F contains a closed discrete subspace D_1 such that $F \subset \bigcup\{\varphi(x)\colon\ x \in D_1\}$. The set $D_1 \cup X^{(\beta)}$ is a closed discrete subspace of X and $X = \bigcup\{\varphi(x)\colon\ x \in D_1 \cup X^{(\beta)}\}$. Thus X is a D-space. Now we assume that η is a limit ordinal. We consider two cases:

(1) If $\mathrm{cf}(\eta) = \omega$, then let $\{\alpha_n\colon\ n \in \omega\}$ be an increasing sequence of ordinals unbounded in η. For each $n \in \omega$, let $X_n = X \cap T \restriction (\alpha_n + 1)$. Then X_n is closed in X and thus meta-Lindelöf. Further, since $\mathrm{ht}(X_n) < \eta$, X_n is a D-space for each $n \in \omega$. Thus $X = \bigcup\limits_{n\in\omega} X_n$ is a D-space.

(2) Now we assume that $\mathrm{cf}(\eta) > \omega$. Since $X \subset T$ is meta-Lindelöf, $\{\mathrm{ht}(x)\colon\ x \in X\}$ is not stationary in η by Proposition 2.7. Let C be cub in η so that $C \cap \{\mathrm{ht}(x)\colon\ x \in X\} = \emptyset$. We can assume that the order type of C is η. As in case (1), for each $\alpha \in \eta$ let $X_\alpha = X \cap T \restriction (\alpha + 1)$. Then X_α is closed in X and thus meta-Lindelöf and a D-space. Further, $\bigcup\limits_{\beta\in\alpha} X_\beta$ is closed in X for each $\alpha \in C$. Thus by Guo and Junnila ([7]), X is a D-space. $\square$

Recall that a topological space X is a *collectionwise Hausdorff* space if and only if whenever Y is a discrete subspace of the space X, there is a disjoint collection $\{U_x\colon\ x \in Y\}$ of open sets of X such that $x \in U_x$ for each $x \in Y$ (such a collection being called a *separation* of Y).

Definition 2.12 ([9]). For any uncountable regular cardinal κ, a κ-*Suslin tree* is a tree T such that $|T| = \kappa$ and every chain and every antichain of T have cardinality $< \kappa$.

Definition 2.13. For any uncountable regular cardinal κ, a κ-tree is an *almost κ-Suslin tree* if and only if it has no stationary antichain.

The notion of an almost Suslin tree which appears in [2] will be called an almost ω_1-Suslin tree in this note. The notion of an ω_1-Suslin tree is called a Suslin tree in [2].

Lemma 2.14 ([2]). *Let T be an ω_1-tree. T is an almost ω_1-Suslin tree if and only if its tree topology is collectionwise Hausdorff.*

In getting an almost ω_1-Suslin tree is collectionwise Hausdorff ([2]), the item (3) which appears in the definition of an ω_1-tree is not needed. This is proved in Theorem 2.15. In proving the following theorem, some basic facts will be used. For example, every Hausdorff tree is regular; every countable discrete subspace Y in a regular space X can be separated by disjoint open sets of X (i.e. there is an open neighborhood V_x of x for each $x \in Y$ such that $V_x \cap V_y = \emptyset$ if x, y are distinct points of Y). Since these facts are well known, we omit the proofs. We generalize Lemma 2.14 and get the following theorem.

Theorem 2.15. *Let T be a Hausdorff tree of height η such that $|T_\alpha| \leqslant \omega$ for each $\alpha < \eta$, where η is an uncountable ordinal. The tree T is collectionwise Hausdorff if and only if for each antichain $C \subset T$ and for each limit ordinal $\alpha \leqslant \eta$ with $\operatorname{cf}(\alpha) > \omega$, $\{\operatorname{ht}(c)\colon\ c \in C\} \cap \alpha$ is not stationary in α.*

Proof. "$\Rightarrow$" Suppose that there is a limit ordinal $\alpha \leqslant \eta$ with $\operatorname{cf}(\alpha) > \omega$ and there is an antichain $C \subset T$ such that $\{\operatorname{ht}(c)\colon\ c \in C\} \cap \alpha$ is stationary in α. If $E = C \cap \Big(\bigcup\limits_{\beta<\alpha} T_\beta\Big)$, then E is an antichain of T. Thus the set E is a discrete subspace of T. The tree T is collectionwise Hausdorff, hence the set E can be separated by disjoint open sets of the form $(f(x), x]$, where $f(x) < x$ for each $x \in E$. By Theorem 2.4, there is $E_1 \subset E$ which meets stationary (in α) many levels of T and $z \in T$ such that $z \in (f(x), x]$ for each $x \in E_1$. If $a, b \in E_1$ and $a \neq b$, then $z \in (f(a), a] \cap (f(b), b]$. This is a contradiction with $(f(a), a] \cap (f(b), b] = \emptyset$.

“⇐“ Let X be any discrete subspace of T. Since T is Hausdorff, the tree T is regular. If $\alpha < \omega_1$, then the set $T \restriction \alpha$ is an open countable regular subspace of T. Thus every discrete subspace of $T \restriction \alpha$ can be separated by disjoint open sets of $T \restriction \alpha$. So $X \cap (T \restriction \alpha)$ can be separated by disjoint open sets of T.

We first prove a claim.

Claim. If the ordinal η has an uncountable cofinality, then the set $H = \{\mathrm{ht}(x)\colon x \in X\}$ is not stationary in η.

Proof of Claim. Suppose that the claim is not true. Then the set H is stationary in η. For each $x \in X$, there is an open set U_x disjoint from $X \setminus \{x\}$. Thus we can pick $f(x) < x$ such that $(f(x), x] \cap X = \{x\}$. There is $X' \subset X$, which meets stationary (in η) many levels and $z \in T$ such that $z \in (f(x), x]$ for each $x \in X'$ by Theorem 2.4. If $H' = \{\mathrm{ht}(x)\colon x \in X'\}$, then the set H' is stationary in η. For any $x \in X'$ we have $z < x$ and $(z, x) \cap X' = \emptyset$.

In what follows, we show that for any two distinct points $x_1, x_2 \in X'$, the points x_1 and x_2 are incomparable. Suppose that the points x_1 and x_2 are comparable, we can assume $x_1 < x_2$. Thus $(z, x_1] \subset (z, x_2)$. So $x_1 \in (z, x_2] \cap X'$ which is a contradiction with $(z, x_2) \cap X' = \emptyset$. Thus the points x_1 and x_2 are incomparable. So the set X' is an antichain of the tree T. Thus the set H' is not stationary in η by the known conditions. This contradicts the fact that the set H' is stationary in η. Thus we have proved the claim.

Now we continue to prove the sufficiency of the condition. The proof is by induction.

We first prove the case of $\eta = \omega_1$. By the claim the set $\{\mathrm{ht}(x)\colon x \in X\}$ is not stationary in ω_1. So there is a cub set $F \subset \omega_1$ such that $F \cap \{\mathrm{ht}(x)\colon x \in X\} = \emptyset$. Hence $X \cap (T \restriction F) = \emptyset$. Since F is a cub set of ω_1, we know that $T \restriction F$ is closed in T. If $Y = T \setminus (T \restriction F)$, then the set Y is an open subspace of T and $X \subset Y$. Let $\{\alpha_v\colon v \in \omega_1\}$ be the monotone enumeration of F such that, if $v \in \omega_1$ is a limit ordinal then $\alpha_v = \sup\{\alpha_t\colon t < v\}$; the ordinal α_{v+1} is a successor ordinal for each $v \in \omega_1$.

Then $X = \big(\bigcup\{X \cap (T \restriction \alpha_{v+1} \setminus T \restriction (\alpha_v + 1))\colon v \in \omega_1\}\big) \cup (X \cap (T \restriction (\alpha_0 + 1)))$. The set $T \restriction (\alpha_0 + 1)$ is an open subspace of T. For each $v \in \omega_1$, the set $T \restriction \alpha_{v+1} \setminus T \restriction (\alpha_v + 1)$ is an open subspace of T and $X \cap (T \restriction \alpha_{v+1} \setminus T \restriction (\alpha_v + 1))$ is countable. Since $\alpha_{v+1} < \omega_1$ for each $v \in \omega_1$, we know that $T \restriction \alpha_{v+1}$ is collectionwise Hausdorff. Thus $X \cap (T \restriction \alpha_{v+1} \setminus T \restriction (\alpha_v + 1))$ can be separated by disjoint open sets of $T \restriction \alpha_{v+1} \setminus T \restriction (\alpha_v + 1)$ for each $v \in \omega_1$. Similarly, we know that $X \cap (T \restriction (\alpha_0 + 1))$ can be separated by disjoint open sets of $T \restriction (\alpha_0 + 1)$. $\{T \restriction \alpha_{v+1} \setminus T \restriction (\alpha_v + 1)\colon v \in \mathrm{cf}(\eta)\} \cup \{T \restriction (\alpha_0 + 1)\}$ is the family that is a disjoint open cover of Y. Thus the set X can be separated by disjoint open sets of T.

Suppose that the statement is true for each ordinal $\omega_1 \leqslant \beta < \eta$, that is to say, if T_1 is a tree of height β such that for any antichain $C \subset T_1$, $\{\mathrm{ht}(c)\colon c \in C\} \cap \alpha$ is not stationary in α for each limit ordinal $\alpha \leqslant \beta$ with $\mathrm{cf}(\alpha) > \omega$, then the tree T_1 is collectionwise Hausdorff. In what follows, we show the case that $\mathrm{ht}(T) = \eta$. Let X be any discrete subspace of T; we consider two cases:

(1) The ordinal η is a successor ordinal. So there is an ordinal β such that $\eta = \beta+1$.

(a) If $\beta = \gamma + 1$, then for each $x \in X \cap T_\beta$ there is an open set $\{x\}$ disjoint from $T \restriction \beta$. Since the clopen subspace $T \restriction \beta$ is collectionwise Hausdorff by induction, X can be separated by disjoint open sets of T.

(b) Let β be a limit ordinal. Let $T_\beta = \{x_n\colon n \in \omega\}$. We will define $\{f(x_n)\colon n \in \omega\}$ so that for each $n \in \omega$, $f(x_n) < x_n$ and $\{[f(x_n), x_n]\colon n \in \omega\}$ is a pairwise disjoint locally finite family of clopen sets. Note that if T_β is finite this is an elementary exercise.

First suppose $\mathrm{cf}(\beta) = \omega$. Since T is Hausdorff it is routine to choose $(f(x_n))_{n\in\omega}$ such that $[f(x_i), x_i] \cap [f(x_j), x_j] = \emptyset$ for $i \neq j$, $\mathrm{ht}(f(x_i)) > \mathrm{ht}(f(x_j))$ for $j < i$ and $\sup\{\mathrm{ht}(f(x_n))\colon n \in \omega\} = \beta$. Let $x \in T \restriction \beta$. So there exists $j \in \omega$ such that $\mathrm{ht}(f(x_j)) > \mathrm{ht}(x)$. So $\hat{x} \cap [f(x_i), x_i] = \emptyset$ for each $i > j$.

Now suppose $\mathrm{cf}(\beta) > \omega$. Since T is Hausdorff it is routine to choose $(g(x_n))_{n\in\omega}$ such that $\{[g(x_n), x_n]\colon n \in \omega\}$ is a pairwise disjoint family of clopen sets. Let $\alpha = \sup\{\mathrm{ht}(g(x_n))\colon n \in \omega\}$. For each $n \in \omega$, let $f(x_n) \in [g(x_n), x_n]$ be such that $\mathrm{ht}(f(x_n)) = \alpha + 1$. Let $x \in T \restriction \beta$. If $\mathrm{ht}(x) \leqslant \alpha$, then there is nothing to show. Suppose $\mathrm{ht}(x) > \alpha$. Consider $(\{x\} \cup \hat{x}) \cap T_{\alpha+1} = a$. If $a \neq f(x_n)$ for any n, then we are done. If $a = f(x_n)$, then $(\{x\} \cup \hat{x}) \cap [f(x_i), x_i] = \emptyset$ for all $i \neq n$. Thus, $\{[f(x_n), x_n]\colon n \in \omega\}$ is a clopen, locally finite family. Therefore, $\{[f(x_n), x_n]\colon n \in \omega\}$ is locally finite and so $T \setminus \bigcup_{n\in\omega} [f(x_n), x_n]$ is open and contains $X \setminus T_\beta$. By the inductive hypothesis T_β is collectionwise Hausdorff.

(2) The ordinal η is a limit ordinal.

(a) If $\mathrm{cf}(\eta) = \omega$, then let $\{\alpha_n\colon n \in \omega\}$ be a sequence of ordinals which is cofinal in η such that $\alpha_n < \alpha_{n+1}$ for each $n \in \omega$. We can assume that α_n is a successor ordinal for each $n \in \omega$.

Since $\omega_1 < \eta$, we can assume that $\omega_1 < \alpha_n$ for each $n \in \omega$. Therefore $T = \big(\bigcup\{T \restriction \alpha_{n+1} \setminus T \restriction \alpha_n\colon n \in \omega\}\big) \cup (T \restriction \alpha_0)$. The set $T \restriction \alpha_0$ is clopen in T. For each $n \in \omega$ the set $T \restriction \alpha_{n+1} \setminus T \restriction \alpha_n$ is also a clopen set in T. By induction, the set $X \cap (T \restriction \alpha_{n+1} \setminus T \restriction \alpha_n)$ can be separated by disjoint open sets of $T \restriction \alpha_{n+1} \setminus T \restriction \alpha_n$ for each $n \in \omega$. The set $X \cap (T \restriction \alpha_0)$ can also be separated by disjoint open sets of $T \restriction \alpha_0$ by induction. Thus X can be separated by disjoint open sets of T.

(b) Now we assume $\mathrm{cf}(\eta) \geqslant \omega_1$. In this case, for each antichain $C \subset T$, the set $\{\mathrm{ht}(c)\colon c \in C\} \cap \alpha$ is not stationary in α if $\alpha \leqslant \eta$ is a limit ordinal and $\mathrm{cf}(\alpha) > \omega$.

By the claim we get that $H = \{\mathrm{ht}(x)\colon\ x \in X\}$ is not stationary in η. Thus there is a cub set $C \subset \eta$ such that $X \cap (T \restriction C) = \emptyset$. If $Y = T \setminus (T \restriction C)$, then Y is an open subspace of T and $X \subset Y$. Let $C_1 = \{a_\alpha\colon\ \alpha \in \mathrm{cf}(\eta)\}$ be such that C_1 is homeomorphic to $\mathrm{cf}(\eta)$ and C_1 is unbounded in η. Thus C_1 is a closed unbounded set of η. So $C \cap C_1$ is a closed unbounded set of η. Therefore $H \cap (C \cap C_1) = \emptyset$. The set $C \cap C_1$ is also closed unbounded in C_1. So we assume $C \cap C_1 = \{\alpha_v\colon\ v \in \mathrm{cf}(\eta)\}$ such that $\alpha_{v_1} < \alpha_{v_2}$ if $v_1 < v_2$ and $v_1, v_2 \in \mathrm{cf}(\eta)$. The set $\{\alpha_v\colon\ v \in \mathrm{cf}(\eta)\}$ also satisfies that the ordinal α_{v+1} is a successor ordinal for each $v \in \mathrm{cf}(\eta)$, and if $v \in \mathrm{cf}(\eta)$ is a limit ordinal then $\alpha_v = \sup\{\alpha_t\colon\ t < v\}$. Thus $X \subset Y \subset T \setminus (T \restriction C \cap C_1)$. Denote $Y_1 = T \setminus (T \restriction C \cap C_1)$.

For each $v \in \mathrm{cf}(\eta)$ the set $T \restriction \alpha_{v+1} \setminus T \restriction (\alpha_v + 1)$ is an open subspace of T. Thus the family $\{T \restriction \alpha_{v+1} \setminus T \restriction (\alpha_v + 1)\colon\ v \in \mathrm{cf}(\eta)\} \cup \{T \restriction (\alpha_0 + 1)\}$ is a disjoint open cover of Y_1. If $v \in \mathrm{cf}(\eta)$, then the set $X \cap (T \restriction \alpha_{v+1} \setminus T \restriction (\alpha_v + 1))$ is a discrete subspace of a tree $T \restriction \alpha_{v+1}$ and $\mathrm{ht}(T \restriction \alpha_{v+1}) < \eta$. If $\mathrm{ht}(T \restriction \alpha_{v+1}) < \omega_1$, then we know that the space $T \restriction \alpha_{v+1}$ is collectionwise Hausdorff. If $\mathrm{ht}(T \restriction \alpha_{v+1}) \geqslant \omega_1$, then the tree $T \restriction \alpha_{v+1}$ is collectionwise Hausdorff by induction. Therefore the set $X \cap (T \restriction \alpha_{v+1} \setminus T \restriction (\alpha_v + 1))$ can be separated by disjoint open sets of the space $T \restriction \alpha_{v+1} \setminus T \restriction (\alpha_v + 1)$. The set $T \restriction \alpha_0$ is an open subspace of T. By a similar argument, the discrete set $X \cap (T \restriction \alpha_0)$ can also be separated by disjoint open sets of $T \restriction \alpha_0$. Thus X can be separated by disjoint open sets of T.

So the tree T is collectionwise Hausdorff. □

It was proved in [2] that if there is an almost ω_1-Suslin tree, then there exists an almost ω_1-Suslin tree which is not an ω_1-Suslin tree. In what follows, we denote any uncountable regular cardinal by κ. Clearly, every κ-Suslin tree is an almost κ-Suslin tree for any uncountable regular cardinal κ. But as the following theorem shows, the two concepts are not identical.

Theorem 2.16. *If there exists an almost κ-Suslin tree, then there exists an almost κ-Suslin tree which is not a κ-Suslin tree.*

P r o o f. Let T be an almost κ-Suslin tree. If T is not a κ-Suslin tree, then we are done. Suppose T is a κ-Suslin tree. For each $\alpha < \kappa$, pick $x_\alpha \in T_\alpha$. Let

$$T^* = T \cup \{(x_\alpha, 1)\colon\ \alpha < \kappa\}$$

and define a partial ordering $<^*$ on T^* by

$$s, t \in T \to [s <^* t \Leftrightarrow s < t];$$
$$x \leqslant x_\alpha \to x <^* (x_\alpha, 1).$$

In all other cases there is no ordering between elements. For each $\alpha < \kappa$, the height of the point $(x_\alpha, 1)$ is $\alpha + 1$ in T^*. The collection $A = \{(x_\alpha, 1) \colon \alpha < \kappa\}$ is clearly an antichain of T^*. Hence T^* is not a κ-Suslin tree. We show that T^* is an almost κ-Suslin tree.

Let $E \subset T^*$ be an antichain. Let $E^* = E \setminus T$. For each $x \in E^*$, $\mathrm{ht}(x)$ is a successor so $\{\mathrm{ht}(x) \colon x \in E^*\}$ is non-stationary. Notice that $E = E^* \cup (E \cap T)$. Further, $E \cap T$ is an antichain in T and thus $\{\mathrm{ht}(x) \colon x \in E \cap T\}$ is non-stationary. Therefore, E is non-stationary in T^*. □

An ω_1-tree T is said to have *property* γ if for any antichain $A \subset T$ there is a cub set $C \subset \omega_1$ such that $T \setminus (T \restriction C)$ contains a closed neighborhood of A (see [2]).

Definition 2.17. A κ-tree T is said to have *property* γ if for any antichain $A \subset T$ there is a cub set $C \subset \kappa$ such that $T \setminus (T \restriction C)$ contains a closed neighborhood of A.

If T is a κ-Suslin tree and A is an antichain of T, then there is $\alpha \in \kappa$ such that $A \subset T \restriction \alpha$, so $T \setminus (T \restriction C)$ is a closed neighborhood of A, where $C = \kappa \setminus (\alpha + 1)$. So it is clear that each κ-Suslin tree has property γ. However, the two concepts are not identical. It was proved in [2] that if there is an ω_1-tree with property γ, then there is an ω_1-tree with property γ which is not an ω_1-Suslin tree. We get that it also holds for κ-trees.

Theorem 2.18. *If there is a κ-tree with property γ, then there is a κ-tree with property γ which is not a κ-Suslin tree.*

Proof. Let T be a κ-tree with property γ. If T is not a κ-Suslin tree, then we have finished. Suppose T is a κ-Suslin tree. Let x_α be any element of T_α for each non-zero $\alpha < \kappa$, and obtain a tree T^* from T as in Theorem 2.16. If $B = \{(x_\alpha, 1) \colon \alpha \in \kappa\}$, then B is clearly an antichain of T^*. Hence T^* is not a κ-Suslin tree.

Let A be any antichain of T^*. The tree T is a κ-Suslin tree, so every antichain of T has cardinality $< \kappa$. Put $b = \sup\{\mathrm{ht}(a) \colon a \in A \cap T\}$. Let $C = \{\alpha \in \kappa \colon \alpha$ is a limit ordinal and $\alpha > b + 1\}$. Thus $A \cap T \subset T \setminus (T \restriction C)$ and C is closed and unbounded in κ.

Let $U = (T \restriction (b+1)) \cup (A \setminus T)$. We only need to show that $\overline{U} \cap (T^* \restriction C) = \emptyset$. Let $t \in T^* \restriction C$. Then $\mathrm{ht}(t) > b + 1$. Thus, $\hat{t} \setminus T \restriction (b+1) \neq \emptyset$ and $(\{t\} \cup \hat{t}) \setminus T \restriction (b+1)$ is an open neighborhood of t. Further, $\hat{t} \cap T^* \setminus T = \emptyset$. So $\hat{t} \cap A \setminus T = \emptyset$. Thus, $\overline{U} \cap (T^* \restriction C) = \emptyset$. □

The trees in [10] are Hausdorff trees. Thus we let the tree in Lemma 2.19 be a Hausdorff tree.

Lemma 2.19 ([10]). *Let S be a subspace of a Hausdorff tree. The following are equivalent:*

(1) *S is normal and collectionwise Hausdorff.*
(2) *S is strong collectionwise Hausdorff.*
(3) *S is hereditarily collectionwise normal.*

Lemma 2.20. *Let T be a κ-tree. If T is collectionwise Hausdorff, then T is an almost κ-Suslin tree.*

P r o o f. Suppose that T is not an almost κ-Suslin tree, then there is an antichain C of T such that $A^* = \{\mathrm{ht}(a)\colon\ a \in C\}$ is stationary in κ. Being an antichain of T, the set C is a discrete subspace of the tree T. The tree T is collectionwise Hausdorff, hence there is a disjoint collection $\{V_a\colon\ a \in C\}$ of open sets of T such that $a \in V_a$ for each $a \in C$. For each $a \in C$ there is an $f(a) < a$ such that $(f(a), a] \subset V_a$. Therefore there is $C_1 \subset C$ which meets stationary (in κ) many levels of T and $z \in T$ such that $z \in (f(a), a]$ for each $a \in C_1$ by Theorem 2.4. For any distinct points $a, b \in C_1$, we have $z \in (f(a), a] \cap (f(b), b]$. Thus $V_a \cap V_b \neq \emptyset$. This is a contradiction with $V_a \cap V_b = \emptyset$. Thus T is an almost κ-Suslin tree. $\square$

In [8], Hart showed that if T is an ω_1-tree and T has property γ, then T is hereditarily collectionwise normal. By the proof of this result, we can get a similar result for a κ-tree. Thus we have the following theorem.

Theorem 2.21. *The following are equivalent for a Hausdorff κ-tree T:*

(1) *T is normal and collectionwise Hausdorff.*
(2) *T has property γ.*
(3) *T is hereditarily collectionwise normal.*

P r o o f. (1) and (3) are equivalent by Lemma 2.19, and we can get (2)$\Rightarrow$(3) by a proof which is similar to the proof of Theorem 2.1 in [8]. To complete the proof, we only need to show (1)$\Rightarrow$(2).

Let A be any antichain of T. Since the tree T is collectionwise Hausdorff, T is an almost κ-Suslin tree by Lemma 2.20. Hence $A^* = \{\mathrm{ht}(a)\colon\ a \in A\}$ is not stationary. So there is a cub set C of κ such that $C \cap A^* = \emptyset$, thus A and $T \restriction C$ are two disjoint closed sets of T. The tree T is normal, so there are two disjoint open sets U, V of T such that $A \subset U$ and $T \restriction C \subset V$. Thus $A \subset U \subset \overline{U} \subset T \setminus V \subset T \setminus (T \restriction C)$. Therefore $T \setminus (T \restriction C)$ contains a closed neighborhood of A. So T has property γ. $\square$

In [4] and [8], some properties of ω_1-trees were investigated. In what follows, we consider a tree T such that the item (3) which appears in the definition of an ω_1-tree is not required. We call such a tree T an ω_1'*-tree*. An *Aronszajn tree* is an ω_1-tree with no uncountable branch. It follows from the item (3) which appears in the definition of an ω_1-tree that every ω_1-Suslin tree is an Aronszajn tree. The ordinal ω_1 is an ω_1'-tree with no uncountable antichain, but it has an uncountable branch. The following conclusion appears in [2]. Let T be an ω_1-tree. T is an ω_1-Suslin tree if and only if whenever A, B are disjoint closed subsets of the space T, $\widehat{A} \cap \widehat{B}$ is countable. For an ω_1'-tree, we have the following result.

Theorem 2.22. *Let T be an ω_1'-tree. If whenever A and B are disjoint closed subsets of the space T, $\widehat{A} \cap \widehat{B}$ is countable, then T has no uncountable antichain.*

P r o o f. Suppose that the statement is not true. There is a maximal uncountable antichain C of T. Thus the set C is a closed discrete subspace of the space T. For any $a \in C$, put $\hat{a} = \{x\colon\ x \in T, x < a\}$. Since the set C is uncountable and $|T_0| \leqslant \omega$, there are $x_0 \in T_0$ and $C_0 \subset C$ such that $|C_0| = \omega_1$ and $\hat{a} \cap T_0 = \{x_0\}$ for each $a \in C_0$. Since $|T_\alpha| \leqslant \omega$ for each $\alpha \in \omega_1$, the set $\{\mathrm{ht}(x)\colon\ x \in F\}$ is unbounded in ω_1 if F is an uncountable subset of C.

Let $\alpha \in \omega_1$. Assume that C_β is defined for each $\beta < \alpha$ satisfying $|C_\beta| = \omega_1$ and there is $x_\beta \in T_\beta$ such that $\hat{a} \cap T_\beta = \{x_\beta\}$ for each $a \in C_\beta$. The family $\{C_\beta\colon\ \beta < \alpha\}$ also satisfies that $C_{\beta+1} \subset C_\beta$ if $\beta + 1 < \alpha$.

If $\alpha = \beta + 1$ for an ordinal β, then $|C_\beta| = \omega_1$. Since C_β is uncountable and $|T_\alpha| \leqslant \omega$, there are $x_\alpha \in T_\alpha$ and $C_\alpha \subset C_\beta$ such that $|C_\alpha| = \omega_1$ and $\hat{a} \cap T_\alpha = \{x_\alpha\}$ for each $a \in C_\alpha$.

Now we assume that α is a limit ordinal. Since C is uncountable and $|T_\alpha| \leqslant \omega$, there are $x_\alpha \in T_\alpha$ and $C_\alpha \subset C$ such that $|C_\alpha| = \omega_1$ and $\hat{a} \cap T_\alpha = \{x_\alpha\}$ for each $a \in C_\alpha$.

Thus we can get a set $C_\alpha \subset C$ and a point $x_\alpha \in T_\alpha$ for each $\alpha \in \omega_1$ such that $|C_\alpha| = \omega_1$ and $\hat{a} \cap T_\alpha = \{x_\alpha\}$ for each $a \in C_\alpha$. The family $\{C_\alpha\colon\ \alpha \in \omega_1\}$ satisfies that $C_{\alpha+1} \subset C_\alpha$ for each $\alpha \in \omega_1$. So $x_\alpha < x_{\alpha+1}$ for each $\alpha \in \omega_1$.

Let $y_1 \in C_1$ and $y_2 \in C_2 \setminus \{y_1\}$. Let $\alpha \in \omega_1$. Assume that we have a set $\{y_{2\beta+1}, y_{2\beta+2}\colon\ \beta < \alpha\}$ of distinct points of T. Pick $y_{2\alpha+1} \in C_{2\alpha+1} \setminus \{y_{2\beta+1}, y_{2\beta+2}\colon$ $\beta < \alpha\}$ and $y_{2\alpha+2} \in C_{2\alpha+2} \setminus (\{y_{2\beta+1}, y_{2\beta+2}\colon\ \beta < \alpha\} \cup \{y_{2\alpha+1}\})$.

If $A = \{y_{2\alpha+1}\colon\ \alpha \in \omega_1\}$ and $B = \{y_{2\alpha+2}\colon\ \alpha \in \omega_1\}$, then A and B are disjoint closed subsets of T. Since $C_{2\alpha+2} \subset C_{2\alpha+1}$ for each $\alpha \in \omega_1$, we have $\widehat{y_{2\alpha+1}} \cap T_{2\alpha+1} = \widehat{y_{2\alpha+2}} \cap T_{2\alpha+1}$. Thus $\widehat{A} \cap \widehat{B}$ is uncountable. This is a contradiction with the fact that $\widehat{A} \cap \widehat{B}$ is countable. Thus the tree T has no uncountable antichain. □

Acknowledgement. The authors would like to thank the referee for many valuable remarks, corrections, and suggestions which greatly improved the paper.

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