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ON THE LAPLACIAN, SIGNLESS LAPLACIAN AND NORMALIZED LAPLACIAN CHARACTERISTIC POLYNOMIALS OF A GRAPH

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Abstract. The Laplacian, signless Laplacian and normalized Laplacian characteristic polynomials of a graph are the characteristic polynomials of its Laplacian matrix, signless Laplacian matrix and normalized Laplacian matrix, respectively. In this paper, we mainly derive six reduction procedures on the Laplacian, signless Laplacian and normalized Laplacian characteristic polynomials of a graph which can be used to construct larger Laplacian, signless Laplacian and normalized Laplacian cospectral graphs, respectively.

Keywords: Laplacian matrix; signless Laplacian matrix; normalized Laplacian matrix; characteristic polynomial

MSC 2010: 05C50

1. Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Its *adjacency matrix* is defined to be the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if v_i is adjacent to v_j; and $a_{ij} = 0$, otherwise. The degree of a vertex v in a graph G is denoted by $d_G(v)$ or simply $d(v)$ if G is clear from the context. The *Laplacian matrix* $L(G) = D(G) - A(G)$ is the difference of $D(G) = \operatorname{diag}(d(v_1), d(v_2), \ldots, d(v_n))$, the diagonal matrix of vertex degrees, and the adjacency matrix. The *signless Laplacian matrix* and *normalized Laplacian matrix* are defined to be $Q(G) = D(G) + A(G)$ and $\mathcal{L}(G) = D(G)^{-1/2}(D(G) - A(G))D(G)^{-1/2}$ (with the convention that if the degree

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of v is 0 then $(d(v))^{-1/2} = 0$, see [3]), respectively. If $v \in V(G)$, let $L_v(G)$ ($Q_v(G)$, $\mathcal{L}_v(G)$) be the principal submatrix of $L(G)$ ($Q(G)$, $\mathcal{L}(G)$) formed by deleting the row and column corresponding to the vertex v. Similarly, if H is a subgraph of G, let $L_H(G)$ ($Q_H(G)$, $\mathcal{L}_H(G)$) be the principal submatrix of $L(G)$ ($Q(G)$, $\mathcal{L}(G)$) formed by deleting the rows and columns corresponding to all vertices of $V(H)$. In particular, if e is an edge of G, then $L_e(G)$ ($Q_e(G)$, $\mathcal{L}_e(G)$) is the principal submatrix of $L(G)$ ($Q(G)$, $\mathcal{L}(G)$) formed by deleting the rows and columns corresponding to the vertices of the edge e.

Throughout this paper, we shall denote by $\Phi(B) = \Phi(B;x) = \det(xI - B)$ the *characteristic polynomial* of the square matrix B. In particular, if $B = L(G)$, we write $\Phi(L(G))$ by $\Gamma(G;x)$ or simply by $\Gamma(G)$ and call $\Gamma(G)$ the *Laplacian characteristic polynomial* of G; if $B = Q(G)$, we write $\Phi(Q(G))$ by $\Psi(G;x)$ or simply by $\Psi(G)$ and call $\Psi(G)$ the *signless Laplacian characteristic polynomial* of G; if $B = \mathcal{L}(G)$, we write $\Phi(\mathcal{L}(G))$ by $\Theta(G;x)$ or simply by $\Theta(G)$ and call $\Theta(G)$ the *normalized Laplacian characteristic polynomial* of G. It is both convenient and consistent to define $\Phi(L_G(G)) = \Phi(Q_G(G)) = \Phi(\mathcal{L}_G(G)) = 1$. The Laplacian characteristic polynomial of a graph G plays an important role in investigating the eigenvalues of $L(G)$ (see [4]–[8], [10]–[12]).

For the characteristic polynomial of the adjacency matrix of a graph, Schwenk [9] obtained the following two results which display respectively the relations between the characteristic polynomial of $A(G)$ and the corresponding polynomials of $A(G-v)$ or $A(G-e)$, where $v \in V(G)$ and $e \in E(G)$.

Proposition 1.1. *Let v be a vertex of a graph G, let $\varphi(v)$ be the collection of cycles containing v, and let $V(Z)$ be the set of all vertices in the cycle Z. Then the characteristic polynomial $\Phi(A(G))$ satisfies*

$$\Phi(A(G)) = x\Phi(A(G-v)) - \sum_{w} \Phi(A(G-v-w)) - 2\sum_{Z\in\varphi(v)} \Phi(A(G-V(Z))),$$

where the first summation extends over those vertices w adjacent to v, and the second summation extends over all $Z \in \varphi(v)$.

Proposition 1.2. *Let $e = uv$ be an edge of G, and let $\mathscr{C}(e)$ be the set of all cycles containing e. Then $\Phi(A(G))$ satisfies*

$$\Phi(A(G)) = \Phi(A(G-e)) - \Phi(A(G-u-v)) - 2\sum_{Z\in\mathscr{C}(e)} \Phi(A(G-V(Z))),$$

where the summation extends over all $Z \in \mathscr{C}(e)$.

In this paper, we further investigate the Laplacian, signless Laplacian and normalized Laplacian characteristic polynomials of a graph, and obtain six reduction procedures which can be used to construct larger Laplacian, signless Laplacian and normalized Laplacian cospectral graphs, respectively.

2. The Laplacian and signless Laplacian characteristic polynomials of a graph

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set. A one-to-one mapping of S onto itself is called a *permutation* of S. Let $a_1, a_2, \ldots, a_m$ be m distinct elements of a set S. If a permutation σ of S is such that it sends

$$a_1 \to a_2 \to a_3 \to \ldots \to a_m \to a_1$$

and keeps the remaining elements of S, if any fixed, then σ is called a *cycle* of length m and is denoted by $(a_1, a_2, \ldots, a_m)$. A *transposition* is a cycle of length 2. Since permutations are mappings, we define the product of two permutations as the composition of two mappings.

Lemma 2.1 ([1]). *Every permutation either is a transposition or can be expressed as a product of transpositions. In particular, every cycle of length $m \geqslant 2$ can be expressed as a product of $m - 1$ transpositions.*

If a permutation σ is a product of an even number of transpositions, then it is called an *even permutation*. If σ is a product of an odd number of transpositions, then it is called an *odd permutation*.

The following result displays the relations between the characteristic polynomial of $L(G)$ and the polynomial of $L_v(G)$.

Theorem 2.1. *Let v be a vertex of a graph G, let $\varphi(v)$ be the collection of cycles containing v. Then the Laplacian characteristic polynomial $\Gamma(G)$ satisfies*

$$\Gamma(G) = (x - d(v))\Phi(L_v(G)) - \sum_{w} \Phi(L_{vw}(G)) - 2 \sum_{Z \in \varphi(v)} (-1)^{|Z|}\Phi(L_Z(G)),$$

where the first summation extends over those vertices w adjacent to v, and the second summation extends over all $Z \in \varphi(v)$, and $|Z|$ denotes the length of Z.

P r o o f. Let $B = (b_{ij}) = xI - D + A$. Without loss of generality, we can assume that the first row and column of $D - A$ correspond to the vertex v, that is $v = v_1$. Then it is easy to see that $b_{ii} = x - d(v_i)$ $(i = 1, 2, \ldots, n)$, and $b_{ij} = 1$ if

$v_iv_j \in E(G)$; $b_{ij} = 0$, otherwise. In order to get the result, the basic technique in use is to take the computation of the determinant as a sum over permutations and then place permutations into several groups depending on whether or not a particular entry appears. According to the definition of the determinant, we have

$$\Gamma(G) = \det(xI - D + A) = \sum_P \varepsilon_P b_{1i_1} b_{2i_2} \dots b_{ni_n}, \tag{2.1}$$

where the summation runs over all permutations

$$\begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}, \quad \text{and} \quad \varepsilon_P = \begin{cases} 1, & \text{if } P \text{ is even;} \\ -1, & \text{if } P \text{ is odd.} \end{cases}$$

A term in equation (2.1), $S_P = \varepsilon_P b_{1i_1} b_{2i_2} \dots b_{ni_n} \neq 0$ if and only if for $j \neq i_j$, $v_j v_{i_j} \in E(G)$ $(j = 1, 2, \dots, n)$. Note that P may be represented as a product, say $P = \sigma_1 \sigma_2 \dots \sigma_k$, of disjoint cycles.

Consider the first cycle with length l, say $\sigma_1 = (12 \dots l)$, of P. It is easy to see that if $S_P \neq 0$ and $l \geqslant 3$, then this corresponds to a cycle containing a vertex v of G. Conversely, to a cycle containing a vertex v, say $v(= v_1)v_2v_3 \dots v_l v$, of G, the corresponding term σ_1 of P may be $(123 \dots l)$ or $(1l \dots 32)$; if $S_P \neq 0$ and $l = 2$, then there exists a one-to-one mapping between the first cycle with length 2 of P and the edge containing the vertex v of G; if $S_P \neq 0$ and $l = 1$, then $\sigma_1 = (11)$, corresponds to the vertex v of G. Let M be the set of permutations of $S = \{1, 2, \dots, n\}$ such that for each $S_P \neq 0$, $P \in M$, let $M_1 = \{P;\ P \in M$, the first term of P is $b_{11}\}$, $M_{uv} = \{P;\ P \in M$, the first cycle of P corresponds to the edge uv of $G\}$ and $M_Z = \{P;\ P \in M$, the first cycle of P corresponds to a cycle Z with length l, containing vertex v of $G\}$, $(|Z| = l \geqslant 3)$. Then $M = M_1 \cup \Big(\bigcup_{uv \in E(G)} M_{uv}\Big) \cup \Big(\bigcup_{Z \in \varphi(v)} M_Z\Big)$. Thus, we have

$$\begin{aligned} \Gamma(G) = \sum_P \varepsilon_P b_{1i_1} b_{2i_2} \dots b_{ni_n} &= \sum_{P \in M_1} \varepsilon_P b_{1i_1} b_{2i_2} \dots b_{ni_n} \\ &+ \sum_{uv \in E(G)} \sum_{P \in M_{uv}} \varepsilon_P b_{1i_1} b_{2i_2} \dots b_{ni_n} + \sum_{Z \in \varphi(v)} \sum_{P \in M_Z} \varepsilon_P b_{1i_1} b_{2i_2} \dots b_{ni_n}. \end{aligned} \tag{2.2}$$

From Lemma 2.1 and applying the definition of the determinant again, we have

$$\begin{aligned} \sum_{P \in M_1} \varepsilon_P b_{1i_1} b_{2i_2} \dots b_{ni_n} &= \sum_{P_1} \varepsilon_{P_1} b_{11} b_{2i_2} \dots b_{ni_n} \\ &= (x - d(v)) \sum_{P_1} \varepsilon_{P_1} b_{2i_2} \dots b_{ni_n} = (x - d(v)) \Phi(L_v(G)), \end{aligned} \tag{2.3}$$

where the second summation runs over all permutations

$$\begin{pmatrix} 2 & \dots & n \\ i_2 & \dots & i_n \end{pmatrix}.$$

Similarly, if $v(=v_1)v_2 \in E(G)$, then

$$\begin{aligned} \sum_{P \in M_{v_1 v_2}} \varepsilon_P b_{1i_1} b_{2i_2} \dots b_{ni_n} &= -\sum_{P_2} \varepsilon_{P_2} b_{12} b_{21} b_{3i_3} \dots b_{ni_n} \\ &= -\sum_{P_2} \varepsilon_{P_2} b_{3i_3} \dots b_{ni_n} = -\Phi(L_{v_1 v_2}(G)), \end{aligned}$$

where the second summation runs over all permutations

$$\begin{pmatrix} 3 & \dots & n \\ i_3 & \dots & i_n \end{pmatrix}.$$

Thus, we have

$$\sum_{uv \in E(G)} \sum_{P \in M_{uv}} \varepsilon_P b_{1i_1} b_{2i_2} \dots b_{ni_n} = -\sum_{w} \Phi(L_{vw}(G)). \tag{2.4}$$

If $Z = v(=v_1)v_2 \dots v_l v$ is a cycle of G, then

$$\begin{aligned} \sum_{P \in M_Z} \varepsilon_P b_{1i_1} b_{2i_2} \dots b_{ni_n} &= -2(-1)^l \sum_{P_3} \varepsilon_{P_3} b_{12} b_{23} \dots b_{l1} b_{l+1 i_{l+1}} \dots b_{ni_n} \\ &= -2(-1)^l \sum_{P_3} \varepsilon_{P_3} b_{l+1 i_{l+1}} \dots b_{ni_n} \\ &= -2(-1)^l \Phi(L_Z(G)), \end{aligned}$$

where the second summation runs over all permutations

$$\begin{pmatrix} l+1 & \dots & n \\ i_{l+1} & \dots & i_n \end{pmatrix}.$$

Thus, we have

$$\sum_{Z \in \varphi(v)} \sum_{P \in M_Z} \varepsilon_P b_{1i_1} b_{2i_2} \dots b_{ni_n} = -2 \sum_{Z \in \varphi(v)} (-1)^{|Z|} \Phi(L_Z(G)). \tag{2.5}$$

Substituting equations (2.3)–(2.5) into equation (2.2), we have

$$\Gamma(G) = (x - d(v))\Phi(L_v(G)) - \sum_{w} \Phi(L_{vw}(G)) - 2 \sum_{Z \in \varphi(v)} (-1)^{|Z|} \Phi(L_Z(G)).$$

□

By reasoning similar to that of Theorem 2.1, we have

Corollary 2.1. *Let H be a proper subgraph of G, and let v be a vertex of G such that $v \notin V(H)$. Then we have*

$$\Phi(L_H(G)) = (x - d(v))\Phi(L_{H,v}(G)) - \sum_{\substack{uv\in E(G);\\ u\notin V(H)}} \Phi(L_{H,uv}(G))$$
$$- 2 \sum_{\substack{Z\in\varphi(v);\\ V(Z)\cap V(H)=\emptyset}} (-1)^{|Z|}\Phi(L_{H,Z}(G)).$$

Let us consider a special case of this theorem when v is a pendant vertex.

Corollary 2.2. *Let v be a pendant vertex of G, and u be the vertex adjacent to v. Then*

$$\Gamma(G) = (x - 1)\Gamma(G - v) - x\Phi(L_{uv}(G)).$$

P r o o f. From Theorem 2.1 we have

$$\Gamma(G) = (x - 1)\Phi(L_v(G)) - \Phi(L_{uv}(G)). \tag{2.6}$$

Note that

$$\Phi(L_v(G)) = \Gamma(G - v) - \Phi(L_u(G - v)) = \Gamma(G - v) - \Phi(L_{uv}(G)).$$

Substituting the above equation into equation (2.6), we have

$$\Gamma(G) = (x - 1)\Gamma(G - v) - x\Phi(L_{uv}(G)).$$

□

Let G, r and H, s be two disjoint rooted graphs with roots r and s, respectively. The coalescence of two rooted graphs G, r and H, s, denoted by $G \cdot H$, is the graph formed by identifying the two roots r and s. Suppose that the new vertex is w. Then we have

Corollary 2.3. *If G and H are two rooted graphs with roots r and s, respectively, then the Laplacian characteristic polynomial of the coalescence $G \cdot H$ is*

$$\Gamma(G \cdot H) = \Gamma(G)\Phi(L_s(H)) + \Gamma(H)\Phi(L_r(G)) - x\Phi(L_r(G))\Phi(L_s(H)).$$

P r o o f. We apply Theorem 2.1 to $G\cdot H$ with the coalesced vertex w as the vertex v to get

$$\begin{aligned}(2.7)\qquad \Gamma(G\cdot H) &= (x-d_{G\cdot H}(w))\Phi(L_w(G\cdot H)) - \sum_{wu\in E(G\cdot H)}\Phi(L_{wu}(G\cdot H))\\ &\quad - 2\sum_{Z\in\varphi_{G\cdot H}(w)}(-1)^{|Z|}\Phi(L_Z(G\cdot H))\\ &= (x-d_G(r)-d_H(s))\Phi(L_r(G))\Phi(L_s(H))\\ &\quad - \Phi(L_s(H))\Big[\sum_{ru\in E(G)}\Phi(L_{ru}(G)) + 2\sum_{Z\in\varphi_G(r)}(-1)^{|Z|}\Phi(L_Z(G))\Big]\\ &\quad - \Phi(L_r(G))\Big[\sum_{sv\in E(H)}\Phi(L_{sv}(H)) + 2\sum_{Z\in\varphi_H(s)}(-1)^{|Z|}\Phi(L_Z(H))\Big],\end{aligned}$$

where $\varphi_G(r)$ denotes the collection of all cycles containing the vertex r of G.

Applying Theorem 2.1, respectively, to G and H, we have

$$\Gamma(G) = (x-d_G(r))\Phi(L_r(G)) - \sum_{ru\in E(G)}\Phi(L_{ru}(G)) - 2\sum_{Z\in\varphi_G(r)}(-1)^{|Z|}\Phi(L_Z(G)).$$

Then

$$(2.8)\qquad \sum_{ru\in E(G)}\Phi(L_{ru}(G)) + 2\sum_{Z\in\varphi_G(r)}(-1)^{|Z|}\Phi(L_Z(G)) = (x-d_G(r))\Phi(L_r(G)) - \Gamma(G).$$

Similarly,

$$(2.9)\qquad \sum_{sv\in E(H)}\Phi(L_{sv}(H)) + 2\sum_{Z\in\varphi_H(s)}(-1)^{|Z|}\Phi(L_Z(H)) = (x-d_H(s))\Phi(L_s(H)) - \Gamma(H).$$

Substituting equations (2.8) and (2.9) into equation (2.7), we have

$$\Gamma(G\cdot H) = \Gamma(G)\Phi(L_s(H)) + \Gamma(H)\Phi(L_r(G)) - x\Phi(L_r(G))\Phi(L_s(H)).$$

□

The rooted graphs G_1, r_1 and G_2, r_2 are called Laplacian cospectrally rooted if not only $\Gamma(G_1) = \Gamma(G_2)$, that is G_1 and G_2 are Laplacian cospectral, but also $\Phi(L_{r_1}(G_1)) = \Phi(L_{r_2}(G_2))$. From Corollary 2.3 we have

Corollary 2.4. *If G_1, r_1 and G_2, r_2 are Laplacian cospectrally rooted and H is any rooted graph, then $\Gamma(G_1\cdot H) = \Gamma(G_2\cdot H)$.*

Remark 2.1. Corollary 2.4 asserts that a Laplacian cospectrally rooted pair can be used to build larger Laplacian cospectral graphs. Figure 1 displays two Laplacian cospectrally rooted graphs.

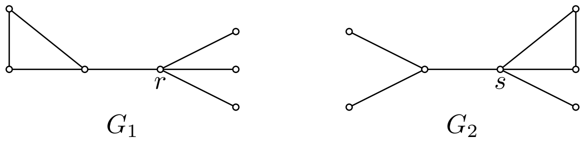

Figure 1. Laplacian cospectrally rooted graphs G_1 and G_2 with roots r and s, respectively.

The next theorem displays the relation between the Laplacian characteristic polynomial of G and the polynomial of $G-e$, where $e \in E(G)$.

Theorem 2.2. *Let $e = uv$ be an edge of G, and let $\mathscr{C}_G(e)$ be the set of all cycles containing e in G. Then the Laplacian characteristic polynomial of G satisfies*

$$\Gamma(G) = \Gamma(G-e) - \Phi(L_u(G-e)) - \Phi(L_v(G-e)) - 2\sum_Z (-1)^{|Z|}\Phi(L_Z(G)),$$

where the summation extends over all $Z \in \mathscr{C}_G(e)$.

Proof. Applying Theorem 2.1 to G and $G-e$, respectively, we have

$$\Gamma(G) = (x - d_G(u))\Phi(L_u(G)) - \sum_{uw\in E(G)} \Phi(L_{uw}(G)) - 2\sum_{Z\in\varphi_G(u)} (-1)^{|Z|}\Phi(L_Z(G))$$

and

$$\Gamma(G-e) = (x - d_G(u)+1)\Phi(L_u(G-e)) - \sum_{uw\in E(G-e)} \Phi(L_{uw}(G-e)) - 2\sum_{Z\in\varphi_{G-e}(u)} (-1)^{|Z|}\Phi(L_Z(G-e)),$$

where $\varphi_G(v)$ is the collection of all cycles containing the vertex v of G. Then

$$\begin{aligned}\Gamma(G) - \Gamma(G-e) &= (x - d_G(u))\Phi(L_u(G)) - (x - d_G(u)+1)\Phi(L_u(G-e)) \\ &\quad - \Big[\sum_{uw\in E(G)} \Phi(L_{uw}(G)) - \sum_{uw\in E(G-e)} \Phi(L_{uw}(G-e))\Big] \\ &\quad - 2\Big[\sum_{Z\in\varphi_G(u)} (-1)^{|Z|}\Phi(L_Z(G)) - \sum_{Z\in\varphi_{G-e}(u)} (-1)^{|Z|}\Phi(L_Z(G-e))\Big].\end{aligned} \tag{2.10}$$

Note that

$$\Phi(L_u(G)) = \Phi(L_u(G-e)) - \Phi(L_{uv}(G-e)) = \Phi(L_u(G-e)) - \Phi(L_{uv}(G)), \tag{2.11}$$

$$\sum_{\substack{uw\in E(G)\\ w\neq v}} \Phi(L_{uw}(G)) = \sum_{uw\in E(G-e)} \Phi(L_{uw}(G-e)) - \sum_{\substack{uw\in E(G)\\ w\neq v}} \Phi(L_{uw,v}(G)), \tag{2.12}$$

and

$$\sum_{\substack{Z\in\varphi_{G-e}(u)\\ v\notin V(Z)}} (-1)^{|Z|}\Phi(L_Z(G)) = \sum_{\substack{Z\in\varphi_{G-e}(u)\\ v\notin V(Z)}} (-1)^{|Z|}\Phi(L_Z(G-e)) - \sum_{\substack{Z\in\varphi_{G-e}(u)\\ v\notin V(Z)}} (-1)^{|Z|}\Phi(L_{Z,v}(G-e)). \tag{2.13}$$

Substituting equations (2.11)–(2.13) into equation (2.10), we have

$$\begin{aligned}\Gamma(G) - \Gamma(G-e) = & -\Phi(L_u(G-e)) - (x - d_G(u)+1)\Phi(L_{uv}(G)) \\ & + \sum_{\substack{uw\in E(G)\\ w\neq v}} \Phi(L_{uw,v}(G)) - 2\sum_{Z\in\mathscr{C}_G(e)} (-1)^{|Z|}\Phi(L_Z(G)) \\ & + 2\sum_{\substack{Z\in\varphi_{G-e}(u)\\ v\notin V(Z)}} (-1)^{|Z|}\Phi(L_{Z,v}(G-e)).\end{aligned} \tag{2.14}$$

From Corollary 2.1, we have

$$\begin{aligned}\Phi(L_v(G-e)) &= (x-d_G(u)+1)\Phi(L_{uv}(G-e)) - \sum_{\substack{uw\in E(G-e)\\ w\neq v}} \Phi(L_{uw,v}(G-e)) \\ &\quad - 2\sum_{\substack{Z\in\varphi_{G-e}(u)\\ v\notin V(Z)}} (-1)^{|Z|}\Phi(L_{Z,v}(G-e)) \\ &= (x-d_G(u)+1)\Phi(L_{uv}(G)) - \sum_{\substack{uw\in E(G)\\ w\neq v}} \Phi(L_{uw,v}(G)) \\ &\quad - 2\sum_{\substack{Z\in\varphi_{G-e}(u)\\ v\notin V(Z)}} (-1)^{|Z|}\Phi(L_{Z,v}(G-e)).\end{aligned} \tag{2.15}$$

Substituting equation (2.15) into equation (2.14), we have

$$\Gamma(G) - \Gamma(G-e) = -\Phi(L_u(G-e)) - \Phi(L_v(G-e)) - 2\sum_{Z\in\mathscr{C}_G(e)} (-1)^{|Z|}\Phi(L_Z(G)).$$

Thus, we have

$$\Gamma(G) = \Gamma(G-e) - \Phi(L_u(G-e)) - \Phi(L_v(G-e)) - 2\sum_{Z\in\mathscr{C}_G(e)} (-1)^{|Z|}\Phi(L_Z(G)).$$

□

From Corollary 2.1, by reasoning similar to that of Theorem 2.2, we have

Corollary 2.5. *Let H be a proper subgraph of G, and let $e = uv$ be an edge of G such that $u, v \notin V(H)$. Let $\mathscr{C}(e)$ be the set of all cycles containing e. Then we have*

$$\begin{aligned}\Phi(L_H(G)) = \Phi(L_H(G-e)) - \Phi(L_{H,u}(G-e)) - \Phi(L_{H,v}(G-e))\\ - 2\sum_{\substack{Z\in\mathscr{C}(e);\\ V(Z)\cap V(H)=\emptyset}} (-1)^{|Z|}\Phi(L_{H,Z}(G)).\end{aligned}$$

From Theorem 2.2, the following known result is immediate.

Corollary 2.6 ([5])**.** *Let G_1 and G_2 be two vertex disjoint graphs, and let $G = G_1u : vG_2$ be the graph obtained by joining the vertex u of G_1 and the vertex v of G_2 by an edge. Then*

$$\Gamma(G) = \Gamma(G_1)\Gamma(G_2) - \Gamma(G_1)\Phi(L_v(G_2)) - \Phi(L_u(G_1))\Gamma(G_2).$$

By reasoning similar to that of Theorems 2.1 and 2.2, we can obtain similar results for the characteristic polynomials of the adjacency matrix (see Propositions 1.1 and 1.2) and the signless Laplacian matrix of a graph, respectively.

Theorem 2.3. *Let v be a vertex of a graph G, let $\varphi(v)$ be the set of all cycles containing v. Then the signless Laplacian characteristic polynomial $\Psi(G)$ satisfies*

$$\Psi(G) = (x - d(v))\Phi(Q_v(G)) - \sum_w \Phi(Q_{vw}(G)) - 2\sum_{Z\in\varphi(v)} \Phi(Q_Z(G)),$$

where the first summation extends over those vertices w adjacent to v, and the second summation extends over all $Z \in \varphi(v)$.

Corollary 2.7. *Let H be a proper subgraph of G, and let v be a vertex of G such that $v \notin V(H)$. Then we have*

$$\Phi(Q_H(G)) = (x-d(v))\Phi(Q_{H,v}(G)) - \sum_{\substack{uv\in E(G);\\ u\notin V(H)}} \Phi(Q_{H,uv}(G)) - 2\sum_{\substack{Z\in\varphi(v);\\ V(Z)\cap V(H)=\emptyset}} \Phi(Q_{H,Z}(G)).$$

Corollary 2.8. *If G and H are two rooted graphs with roots r and s, respectively, then the signless Laplacian characteristic polynomial of the coalescence $G \cdot H$ satisfies*

$$\Psi(G \cdot H) = \Psi(G)\Phi(Q_s(H)) + \Psi(H)\Phi(Q_r(G)) - x\Phi(Q_r(G))\Phi(Q_s(H)).$$

Similarly, rooted graphs G_1, r_1 and G_2, r_2 are called signless Laplacian cospectrally rooted if not only $\Psi(G_1) = \Psi(G_2)$, but also $\Phi(Q_{r_1}(G_1)) = \Phi(Q_{r_2}(G_2))$. From Corollary 2.8, we have

Corollary 2.9. *If G_1, r_1 and G_2, r_2 are signless Laplacian cospectrally rooted and H is any rooted graph, then $\Psi(G_1 \cdot H) = \Psi(G_2 \cdot H)$.*

Remark 2.2. Corollary 2.9 asserts that a signless Laplacian cospectrally rooted pair can be used to build larger signless Laplacian cospectral graphs. Figure 2 displays two signless Laplacian cospectrally rooted graphs.

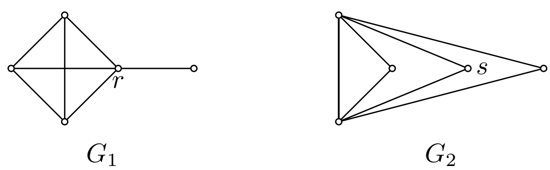

Figure 2. Signless Laplacian cospectrally rooted graphs with roots r and s, respectively.

Theorem 2.4. *Let $e = uv$ be an edge of G, and let $\mathscr{C}(e)$ be the set of all cycles containing e. Then the signless Laplacian characteristic polynomial of G satisfies*

$$\Psi(G) = \Psi(G-e) - \Phi(Q_u(G-e)) - \Phi(Q_v(G-e)) - 2\sum_{Z} \Phi(Q_Z(G)),$$

where the summation extends over all $Z \in \mathscr{C}(e)$.

Corollary 2.10. *Let H be a proper subgraph of G, and let $e = uv$ be an edge of G such that $u, v \notin V(H)$. Let $\mathscr{C}(e)$ be the set of all cycles containing e. Then we have*

$$\Phi(Q_H(G)) = \Phi(Q_H(G-e)) - \Phi(Q_{H,u}(G-e)) - \Phi(Q_{H,v}(G-e)) - 2\sum_{\substack{Z\in\mathscr{C}(e);\\ V(Z)\cap V(H)=\emptyset}} \Phi(Q_{H,Z}(G)).$$

3. The normalized Laplacian characteristic polynomial of a graph

Recall that the normalized Laplacian matrix of G is $\mathcal{L}(G) = D(G)^{-1/2}(D(G) - A(G))D(G)^{-1/2} = (n_{ij})$. Note that $n_{ii} = 1$ for $i = 1, 2, \ldots, n$; $n_{ij} = -1/\sqrt{d_i d_j}$, if v_i is adjacent to v_j; and $n_{ij} = 0$, if $i \neq j$ and v_i is not adjacent to v_j. Suppose that $Z = v_1 v_2 \ldots v_l v_1$ is a cycle of G. Let $d_G(Z) = \prod_{i=1}^{l} d(v_i)$ or simply $d_Z = \prod_{i=1}^{l} d(v_i)$ if G is clear from the context. By reasoning similar to that of Theorems 2.1 and 2.2, we also have the following result.

Theorem 3.1. *Let v be a non isolated vertex of a graph G, let $\varphi(v)$ be the set of all cycles containing v. Then the normalized Laplacian characteristic polynomial $\Theta(G)$ satisfies*

$$\Theta(G) = (x-1)\Phi(\mathcal{L}_v(G)) - \sum_{w} \frac{1}{d(v)d(w)}\Phi(\mathcal{L}_{vw}(G)) - 2\sum_{Z\in\varphi(v)} (-1)^{|Z|}\frac{1}{d_Z}\Phi(\mathcal{L}_Z(G)),$$

where the first summation extends over those vertices w adjacent to v, and the second summation extends over all $Z \in \varphi(v)$.

Corollary 3.1. *Let H be a proper subgraph of G, and let v be a non isolated vertex of $G - H$. Then we have*

$$\begin{aligned}\Phi(\mathcal{L}_H(G)) = (x-1)\Phi(\mathcal{L}_{H,v}(G)) &- \sum_{\substack{uv\in E(G);\\ u\notin V(H)}} \frac{1}{d(u)d(v)}\Phi(\mathcal{L}_{H,uv}(G))\\ &- 2\sum_{\substack{Z\in\varphi(v);\\ V(Z)\cap V(H)=\emptyset}} (-1)^{|Z|}\frac{1}{d_Z}\Phi(\mathcal{L}_{H,Z}(G)).\end{aligned}$$

In particular, if v is a pendent vertex of G, then we have

Corollary 3.2. *Let v be a pendant vertex of a graph G, and let u be the vertex adjacent to v. Then*

$$\Theta(G) = \frac{(d(u)-1)(x-1)}{d(u)}\Theta(G-v) + \frac{x^2-2x}{d(u)}\Phi(\mathcal{L}_{uv}(G)).$$

P r o o f. From Theorem 3.1 we have

$$\Theta(G) = (x-1)\Phi(\mathcal{L}_v(G)) - \frac{1}{d(u)}\Phi(\mathcal{L}_{uv}(G)). \tag{3.1}$$

Note that

$$\begin{aligned}\Phi(\mathcal{L}_v(G)) &= \frac{d(u)-1}{d(u)}\Phi(\mathcal{L}_v(G-e)) + \frac{x-1}{d(u)}\Phi(\mathcal{L}_e(G-e))\\ &= \frac{d(u)-1}{d(u)}\Theta(G-v) + \frac{x-1}{d(u)}\Phi(\mathcal{L}_{uv}(G)).\end{aligned}$$

Substituting the above equation into equation (3.1), we have

$$\Theta(G) = \frac{(d(u)-1)(x-1)}{d(u)}\Theta(G-v) + \frac{x^2-2x}{d(u)}\Phi(\mathcal{L}_{uv}(G)).$$

□

Corollary 3.3. *If G and H are two rooted graphs with roots r and s, respectively, then the normalized Laplacian characteristic polynomial of the coalescence $G \cdot H$ satisfies*

$$\Theta(G\cdot H) = \frac{d_G(r)\Theta(G)\Phi(\mathcal{L}_s(H)) + d_H(s)\Theta(H)\Phi(\mathcal{L}_r(G))}{d_G(r)+d_H(s)}.$$

P r o o f. We apply Theorem 3.1 to $G \cdot H$ with the coalesced vertex w as the vertex v to get

$$\begin{aligned}\Theta(G\cdot H) &= (x-1)\Phi(\mathcal{L}_w(G\cdot H)) - \sum_{wu\in E(G\cdot H)} \frac{\Phi(\mathcal{L}_{wu}(G\cdot H))}{d_{G\cdot H}(u)d_{G\cdot H}(w)}\\ &\quad - 2\sum_{Z\in\varphi_{G\cdot H}(w)} (-1)^{|Z|}\frac{1}{d_{G\cdot H}(Z)}\Phi(\mathcal{L}_Z(G\cdot H))\\ &= (x-1)\Phi(\mathcal{L}_r(G))\Phi(\mathcal{L}_s(H)) - \sum_{ru\in E(G)} \frac{\Phi(\mathcal{L}_{ru}(G))\Phi(\mathcal{L}_s(H))}{d_G(u)(d_G(r)+d_H(s))}\\ &\quad - \sum_{sv\in E(H)} \frac{\Phi(\mathcal{L}_{sv}(H))\Phi(\mathcal{L}_r(G))}{d_H(v)(d_G(r)+d_H(s))}\\ &\quad - 2\sum_{Z\in\varphi_G(r)} (-1)^{|Z|}\frac{d_G(r)\Phi(\mathcal{L}_Z(G))\Phi(\mathcal{L}_s(H))}{(d_G(r)+d_H(s))d_G(Z)}\\ &\quad - 2\sum_{Z\in\varphi_H(s)} (-1)^{|Z|}\frac{d_H(s)\Phi(\mathcal{L}_r(G))\Phi(\mathcal{L}_Z(H)))}{(d_G(r)+d_H(s))d_H(Z)},\end{aligned} \tag{3.2}$$

where $\varphi_G(r)$ denotes the collection of all cycles containing the vertex r of G.

Applying Theorem 3.1 to G and H, respectively, we have

$$\begin{aligned}\Theta(G) &= (x-1)\Phi(\mathcal{L}_r(G)) - \sum_{ru\in E(G)} \frac{1}{d_G(r)d_G(u)}\Phi(\mathcal{L}_{ru}(G))\\ &\quad - 2\sum_{Z\in\varphi_G(r)} (-1)^{|Z|}\frac{1}{d_G(Z)}\Phi(\mathcal{L}_Z(G)).\end{aligned}$$

Then

$$\sum_{ru\in E(G)} \frac{\Phi(\mathcal{L}_{ru}(G))}{d_G(u)(d_G(r)+d_H(s))} + 2\sum_{Z\in\varphi_G(r)} (-1)^{|Z|} \frac{d_G(r)\Phi(\mathcal{L}_Z(G))}{d_G(Z)(d_G(r)+d_H(s))} = -\frac{d_G(r)}{d_G(r)+d_H(s)}(\Theta(G)-(x-1)\Phi(\mathcal{L}_r(G))). \tag{3.3}$$

Similarly,

$$\sum_{sv\in E(H)} \frac{\Phi(\mathcal{L}_{sv}(H))}{d_H(v)(d_G(r)+d_H(s))} + 2\sum_{Z\in\varphi_H(s)} (-1)^{|Z|} \frac{d_H(s)\Phi(\mathcal{L}_Z(H))}{d_H(Z)(d_G(r)+d_H(s))} = -\frac{d_H(s)}{d_G(r)+d_H(s)}(\Theta(H)-(x-1)\Phi(\mathcal{L}_s(H))). \tag{3.4}$$

Substituting equations (3.3) and (3.4) into equation (3.2), we have

$$\Theta(G\cdot H) = \frac{d_G(r)\Theta(G)\Phi(\mathcal{L}_s(H)) + d_H(s)\Theta(H)\Phi(\mathcal{L}_r(G))}{d_G(r)+d_H(s)}.$$

□

Similarly, the rooted graphs G_1, r_1 and G_2, r_2 are called normalized Laplacian cospectrally rooted if not only $\Theta(G_1) = \Theta(G_2)$, but also $\Phi(\mathcal{L}_{r_1}(G_1)) = \Phi(\mathcal{L}_{r_2}(G_2))$ and $d_{G_1}(r_1) = d_{G_2}(r_2)$. From Corollary 3.3 we have

Corollary 3.4. *If G_1, r_1 and G_2, r_2 are normalized Laplacian cospectrally rooted and H is any rooted graph, then $\Theta(G_1\cdot H) = \Theta(G_2\cdot H)$.*

Remark 3.1. Corollary 3.4 asserts that a normalized Laplacian cospectrally rooted pair can be used to build larger normalized Laplacian cospectral graphs. Figure 3 displays two normalized Laplacian cospectrally rooted graphs.

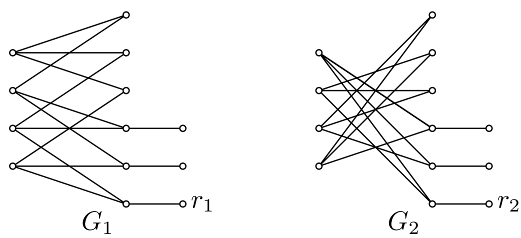

Figure 3. Normalized Laplacian cospectrally rooted graphs G_1 and G_2 with roots r_1 and r_2, respectively.

Remark 3.2. In [2], Butler posed the following problem:

Are there more general constructions which can be used to make cospectral graphs with respect to the normalized Laplacian which have arbitrarily high chromatic number?

From Corollary 3.4 and Remark 3.1, we completely answer the above problem: Let H be a connected graph with arbitrarily high chromatic number. Then $G_1 \cdot H$ and $G_2 \cdot H$ are two cospectral graphs with respect to the normalized Laplacian which have arbitrarily high chromatic number, where G_1 and G_2 are the graphs shown in Figure 3.

Theorem 3.2. *Let $e = uv$ be an edge of G, and let $\mathscr{C}(e)$ be the set of all cycles containing e. The normalized Laplacian characteristic polynomial of G satisfies*

$$\begin{aligned}\Theta(G) &= \frac{(d(u)-1)(d(v)-1)}{d(u)d(v)}\Theta(G-e) + \frac{(d(v)-1)(x-1)}{d(u)d(v)}\Phi(\mathcal{L}_u(G-e)) \\ &\quad + \frac{(d(u)-1)(x-1)}{d(u)d(v)}\Phi(\mathcal{L}_v(G-e)) + \frac{x(x-2)}{d(u)d(v)}\Phi(\mathcal{L}_{uv}(G)) \\ &\quad - 2\sum_{Z}(-1)^{|Z|}\frac{1}{d_Z}\Phi(\mathcal{L}_Z(G)),\end{aligned}$$

where the summation extends over all $Z \in \mathscr{C}(e)$.

Proof. Applying Theorem 3.1 to G and $G - e$, respectively, we have

$$\begin{aligned}\Theta(G) &= (x-1)\Phi(\mathcal{L}_v(G)) - \sum_{vw \in E(G)} \frac{1}{d(v)d(w)}\Phi(\mathcal{L}_{vw}(G)) \\ &\quad - 2\sum_{Z \in \varphi_G(v)} (-1)^{|Z|}\frac{1}{d_Z}\Phi(\mathcal{L}_Z(G))\end{aligned}$$

and

$$\begin{aligned}\Theta(G-e) &= (x-1)\Phi(\mathcal{L}_v(G-e)) - \sum_{vw \in E(G-e)} \frac{1}{(d(v)-1)d(w)}\Phi(\mathcal{L}_{vw}(G-e)) \\ &\quad - 2\sum_{\substack{Z \in \varphi_{G-e}(v) \\ u \notin V(Z)}} (-1)^{|Z|}\frac{d(v)}{(d(v)-1)d_Z}\Phi(\mathcal{L}_Z(G-e)) \\ &\quad - 2\sum_{\substack{Z \in \varphi_{G-e}(v) \\ u \in V(Z)}} (-1)^{|Z|}\frac{d(u)d(v)}{(d(u)-1)(d(v)-1)d_Z}\Phi(\mathcal{L}_Z(G-e)).\end{aligned}$$

Then

$$\begin{aligned}(3.5)\qquad &\frac{d(u)d(v)}{d(u)-1}\,\Theta(G)-(d(v)-1)\Theta(G-e)\\
&=(x-1)\Big[\frac{d(u)d(v)}{d(u)-1}\Phi(\mathcal{L}_v(G))-(d(v)-1)\Phi(\mathcal{L}_v(G-e))\Big]\\
&\quad-\Big[\sum_{vw\in E(G)}\frac{d(u)}{(d(u)-1)d(w)}\Phi(\mathcal{L}_{vw}(G))-\sum_{\substack{vw\in E(G)\\ w\neq u}}\frac{1}{d(w)}\Phi(\mathcal{L}_{vw}(G-e))\Big]\\
&\quad-2d(v)\Big[\sum_{Z\in\varphi_G(v)}\frac{d(u)(-1)^{|Z|}}{(d(u)-1)d_Z}\Phi(\mathcal{L}_Z(G))\\
&\quad-\sum_{\substack{Z\in\varphi_{G-e}(v)\\ u\notin V(Z)}}\frac{(-1)^{|Z|}}{d_Z}\Phi(\mathcal{L}_Z(G-e))\\
&\quad-\sum_{\substack{Z\in\varphi_{G-e}(v)\\ u\in V(Z)}}(-1)^{|Z|}\frac{d(u)}{(d(u)-1)d_Z}\Phi(\mathcal{L}_Z(G-e))\Big]\\
&=(x-1)\Big[\frac{d(u)d(v)}{d(u)-1}\Phi(\mathcal{L}_v(G))-(d(v)-1)\Phi(\mathcal{L}_v(G-e))\Big]\\
&\quad-2\sum_{Z\in\mathscr{C}(e)}\frac{d(u)d(v)(-1)^{|Z|}}{(d(u)-1)d_Z}\Phi(\mathcal{L}_Z(G))+\frac{1}{d(u)-1}\Phi(\mathcal{L}_{uv}(G))\\
&\quad-\sum_{vw\in E(G-e)}\frac{1}{d(w)}\Big[\frac{d(u)}{(d(u)-1)}\Phi(\mathcal{L}_{vw}(G))-\Phi(\mathcal{L}_{vw}(G-e))\Big]\\
&\quad-2\sum_{\substack{Z\in\varphi_G(v)\\ u\notin V(Z)}}(-1)^{|Z|}\Big[\frac{d(u)d(v)}{(d(u)-1)d_Z}\Phi(\mathcal{L}_Z(G))-\frac{d(v)}{d_Z}\Phi(\mathcal{L}_Z(G-e))\Big].\end{aligned}$$

Note that

$$\begin{aligned}(3.6)\qquad \Phi(\mathcal{L}_v(G))&=\frac{d(u)-1}{d(u)}\Phi(\mathcal{L}_v(G-e))+\frac{x-1}{d(u)}\Phi(\mathcal{L}_e(G-e))\\
&=\frac{d(u)-1}{d(u)}\Phi(\mathcal{L}_v(G-e))+\frac{x-1}{d(u)}\Phi(\mathcal{L}_e(G)).\end{aligned}$$

If $w\neq u$ and v, then

$$\Phi(\mathcal{L}_{vw}(G))=\frac{d(u)-1}{d(u)}\Phi(\mathcal{L}_{vw}(G-e))+\frac{x-1}{d(u)}\Phi(\mathcal{L}_{vw,u}(G)).$$

Thus, we have

$$\sum_{vw\in E(G-e)} \frac{1}{d(w)}\Big[\frac{d(u)}{d(u)-1}\Phi(\mathcal{L}_{vw}(G)) - \Phi(\mathcal{L}_{vw}(G-e))\Big] = \sum_{vw\in E(G-e)} \frac{x-1}{(d(u)-1)d(w)}\Phi(\mathcal{L}_{vw,u}(G)). \tag{3.7}$$

If $Z \in \varphi_G(v)$ and $u \notin V(Z)$, then

$$\Phi(\mathcal{L}_Z(G)) = \frac{d(u)-1}{d(u)}\Phi(\mathcal{L}_Z(G-e)) + \frac{x-1}{d(u)}\Phi(\mathcal{L}_{Z,u}(G-e)).$$

Thus, we have

$$\sum_{\substack{Z\in\varphi_G(v)\\ u\notin V(Z)}} (-1)^{|Z|}\Big[\frac{d(u)d(v)}{(d(u)-1)d_Z}\Phi(\mathcal{L}_Z(G)) - \frac{d(v)}{d_Z}\Phi(\mathcal{L}_Z(G-e))\Big] = \sum_{\substack{Z\in\varphi_{G-e}(v)\\ u\notin V(Z)}} \frac{d(v)(x-1)(-1)^{|Z|}}{(d(u)-1)}\Phi(\mathcal{L}_{Z,u}(G-e)). \tag{3.8}$$

Substituting equations (3.6)–(3.8) into equation (3.5), we have

$$\begin{aligned}
&\frac{d(u)d(v)}{d(u)-1}\Theta(G) - (d(v)-1)\Theta(G-e) \\
&\quad = (x-1)\Phi(\mathcal{L}_v(G-e)) + \frac{d(v)(x-1)^2}{d(u)-1}\Phi(\mathcal{L}_{uv}(G)) \\
&\qquad - \frac{1}{d(u)-1}\Phi(\mathcal{L}_{uv}(G)) - \sum_{vw\in E(G-e)} \frac{x-1}{(d(u)-1)d(w)}\Phi(\mathcal{L}_{vw,u}(G)) \\
&\qquad - 2\sum_{Z\in\mathscr{C}(e)} \frac{d(u)d(v)(-1)^{|Z|}}{(d(u)-1)d_Z}\Phi(\mathcal{L}_Z(G)) \\
&\qquad - 2\sum_{\substack{Z\in\varphi_G(v)\\ u\notin V(Z)}} \frac{d(v)(x-1)(-1)^{|Z|}}{(d(u)-1)d_Z}\Phi(\mathcal{L}_{Z,u}(G-e)).
\end{aligned} \tag{3.9}$$

From Corollary 3.1 we have

$$\begin{aligned}\Phi(\mathcal{L}_u(G-e)) &= (x-1)\Phi(\mathcal{L}_e(G-e)) - \sum_{vw\in E(G-e)} \frac{1}{(d(v)-1)d(w)}\Phi(\mathcal{L}_{u,vw}(G-e))\\
&\quad - 2\sum_{\substack{Z\in\varphi_{G-e}(v)\\ u\notin V(Z)}} \frac{d(v)(-1)^{|Z|}}{(d(v)-1)d_Z}\Phi(\mathcal{L}_{Z,u}(G-e))\\
&= (x-1)\Phi(\mathcal{L}_e(G)) - \sum_{vw\in E(G-e)} \frac{1}{(d(v)-1)d(w)}\Phi(\mathcal{L}_{vw,u}(G))\\
&\quad - 2\sum_{\substack{Z\in\varphi_{G}(v)\\ u\notin V(Z)}} \frac{d(v)(-1)^{|Z|}}{(d(v)-1)d_Z}\Phi(\mathcal{L}_{Z,u}(G-e)).\end{aligned}$$

Then

$$\begin{aligned}&\sum_{vw\in E(G-e)} \frac{x-1}{(d(u)-1)d(w)}\Phi(\mathcal{L}_{vw,u}(G))\\
&\qquad + 2\sum_{\substack{Z\in\varphi_{G}(v)\\ u\notin V(Z)}} \frac{d(v)(x-1)(-1)^{|Z|}}{(d(u)-1)d_Z}\Phi(\mathcal{L}_{Z,u}(G-e))\\
&\qquad = \frac{(d(v)-1)(x-1)^2}{d(u)-1}\Phi(\mathcal{L}_e(G)) - \frac{(d(v)-1)(x-1)}{d(u)-1}\Phi(\mathcal{L}_u(G-e)).\end{aligned}\tag{3.10}$$

Substituting equation (3.10) into equation (3.9), we have

$$\begin{aligned}&\frac{d(u)d(v)}{d(u)-1}\Theta(G) - (d(v)-1)\Theta(G-e)\\
&\quad = (x-1)\Phi(\mathcal{L}_v(G-e)) + \frac{d(v)(x-1)^2}{d(u)-1}\Phi(\mathcal{L}_{uv}(G)) - \frac{1}{d(u)-1}\Phi(\mathcal{L}_{uv}(G))\\
&\qquad - \frac{(d(v)-1)(x-1)^2}{d(u)-1}\Phi(\mathcal{L}_{uv}(G)) - 2\sum_{Z\in\mathscr{C}(e)} \frac{d(u)d(v)(-1)^{|Z|}}{(d(u)-1)d_Z}\Phi(\mathcal{L}_Z(G))\\
&\qquad + \frac{(d(v)-1)(x-1)}{d(u)-1}\Phi(\mathcal{L}_u(G-e))\\
&\quad = (x-1)\Phi(\mathcal{L}_v(G-e)) + \frac{(d(v)-1)(x-1)}{d(u)-1}\Phi(\mathcal{L}_u(G-e))\\
&\qquad + \frac{x(x-2)}{d(u)-1}\Phi(\mathcal{L}_{uv}(G)) - 2\sum_{Z\in\mathscr{C}(e)} \frac{d(u)d(v)(-1)^{|Z|}}{(d(u)-1)d_Z}\Phi(\mathcal{L}_Z(G)).\end{aligned}$$

The result follows. □

From Corollary 3.1, by reasoning similar to that of Theorem 3.2, we have

Corollary 3.5. *Let H be a proper subgraph of G, and let $e = uv$ be an edge of G such that $u, v \notin V(H)$. Let $\mathscr{C}(e)$ be the set of all cycles containing e. Then we have*

$$\begin{aligned}\Phi(\mathcal{L}_H(G)) = {} & \frac{(d(u)-1)(d(v)-1)}{d(u)d(v)}\Phi(\mathcal{L}_H(G-e)) + \frac{(d(v)-1)(x-1)}{d(u)d(v)}\Phi(\mathcal{L}_{H,u}(G-e)) \\ & + \frac{(d(u)-1)(x-1)}{d(u)d(v)}\Phi(\mathcal{L}_{H,v}(G-e)) + \frac{x(x-2)}{d(u)d(v)}\Phi(\mathcal{L}_{H,uv}(G)) \\ & - 2\sum_{Z\in\mathscr{C}(e)}(-1)^{|Z|}\frac{1}{d_Z}\Phi(\mathcal{L}_{H,Z}(G)).\end{aligned}$$

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