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THE ROTHE METHOD FOR THE MCKENDRICK-VON FOERSTER EQUATION

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Abstract. We present the Rothe method for the McKendrick-von Foerster equation with initial and boundary conditions. This method is well known as an abstract Euler scheme in extensive literature, e.g. K. Rektorys, The Method of Discretization in Time and Partial Differential Equations, Reidel, Dordrecht, 1982.

Various Banach spaces are exploited, the most popular being the space of bounded and continuous functions. We prove the boundedness of approximate solutions and stability of the Rothe method in L^∞ and L^1 norms. Proofs of these results are based on comparison inequalities. Our theory is illustrated by numerical experiments. Our research is motivated by certain models of mathematical biology.

Keywords: Rothe method; stability; comparison

MSC 2010: 65M12, 65M99, 92B99

1. Introduction

The model proposed by Gurtin and MacCamy [4] is well known in mathematical biology and is used for various problems related to age-structured populations, see [3], [12]. The birth process is described by a renewal condition with a fertility function. The death process is governed by a differential equation with a mortality function. A detailed introduction to age structured population models can be found in [5]. There is rich literature concerning the existence and uniqueness of solutions of such problems, see for example [6] and the bibliography therein. It is possible to consider populations with a structure given by some other measurable features such as size of individuals, their maturity or level of nutrition, etc. These types of models are discussed in [22]. Many other models of mathematical biology can be found in the monograph [14].

Solutions of both the age and size structured models can be found analytically only in special cases. Therefore, numerical approximation plays an important role in any analysis of these problems. Their solutions are approximated by finite difference schemes [1], discretized methods of characteristics [2] or finite element methods [13].

Solutions of a partial differential equation can be approximated by solutions of systems of ordinary differential equations, provided that one transforms the particular PDE to a system of ODE's using the method of lines (MOL) or the Rothe method. In the former case, spatial partial derivatives are replaced by appropriate finite difference quotients. In the latter case, the time derivative is replaced by a finite difference quotient. If the space variable is one dimensional, then we can interchange the time variable and the space variable, so that the Rothe method becomes MOL. Then one can take advantage of various programs designed for numerical approximation of systems of ODE's.

The Rothe method can be regarded as an abstract Euler method. Indeed, an evolutionary problem $\frac{\mathrm{d}}{\mathrm{d}t}u + A(t)u = F(t,u)$, $u(0) = u_0$ is approximated by

$$\frac{u^{(i)} - u^{(i-1)}}{h} + A(t^{(i)})u^{(i)} = F(t^{(i)}, u^{(i-1)}), \quad u^{(0)} = u_0.$$

The right-hand side can be modified for a particular equation. Moreover, if $F(t^{(i)}, u^{(i-1)})$ is replaced by $F(t^{(i)}, u^{(i)}, u^{(i-1)})$, then one obtains an abstract generalized Euler method, see [19]. The L^∞-dynamics leads to classical maximum principles and convergence statements. Examples of abstract Runge-Kutta methods for parabolic equations are described in [15].

In [8] the existence of solutions of a linear hyperbolic equation equipped with initial and boundary conditions is proved by means of the Rothe method and variational inequalities. An application of the Rothe method to parabolic problems can be found in [9], [18]. The monograph [16] provides existence results for hyperbolic and parabolic problems, proved by means of energy functionals. The book contains many illustrative examples.

Semidiscretization methods are applied to problems of mathematical biology. In [20] the author considered an age-dependent model with an additional structure. This structure can represent for example the size or weight of individuals. A boundary condition, given by a renewal equation, includes a dependence on time and the additional structure mentioned above. The main result of [20] is a convergence theorem concerning a semidiscrete scheme with discrete time, age and with a continuous parameter.

1.1. Formulation of the PDE. Let $E = [0,T] \times \mathbb{R}_+$, $\mathbb{R}_+ = [0,+\infty)$, $c, T > 0$ and $\lambda\colon E \times \mathbb{R}_+^2 \to \mathbb{R}$. Suppose that $v\colon \mathbb{R}_+ \to \mathbb{R}_+$, $\tilde{v}\colon [0,T] \to \mathbb{R}_+$ are given. Consider the differential equation

$$\frac{\partial u}{\partial t}(t,x)+c\frac{\partial u}{\partial x}(t,x)=u(t,x)\lambda\Big(t,x,u(t,x),\int_0^\infty u(t,y)\,\mathrm{d}y\Big) \tag{1.1}$$

with the initial and boundary conditions

$$u(0,x)=v(x)\quad\text{for } x\in\mathbb{R}_+,\qquad u(t,0)=\tilde v(t)\quad\text{for } t\in[0,T]. \tag{1.2}$$

We assume that the functions v and $\tilde v$ satisfy the consistency condition $v(0)=\tilde v(0)$.

1.2. Semidiscretization. Suppose that N is a given natural number. Then $h=T/N>0$ is the discretization parameter. Denote $t^{(i)}=i\cdot h$ for $i=0,\ldots,N$. We will approximate the solution $u(t^{(i)},x)$ of problem (1.1)–(1.2) by a function $u^{(i)}(x)$ which satisfies an ordinary differential equation. We write the Rothe method for (1.1)–(1.2) as follows:

$$\begin{aligned}\frac{u^{(i)}(x)-u^{(i-1)}(x)}{h}&+c\frac{\mathrm{d}}{\mathrm{d}x}u^{(i)}(x) \\ &=u^{(i)}(x)\lambda\Big(t^{(i)},x,u^{(i-1)}(x),\int_0^\infty u^{(i-1)}(x)\,\mathrm{d}x\Big),\quad i=1,\ldots,N,\end{aligned} \tag{1.3}$$

$$u^{(0)}(x)=v(x)\quad\text{for } x\in\mathbb{R}_+,\qquad u^{(i)}(0)=\tilde v(t^{(i)})\quad\text{for } i=1,\ldots,N. \tag{1.4}$$

By $L^1(\mathbb{R}_+)$ we denote the class of all real-valued Lebesgue integrable functions on $\mathbb{R}_+$ with the standard L^1-norm $\|\cdot\|_1$. The symbols $L^\infty(\mathbb{R}_+)$ and $L^\infty([0,T])$ stand for the classes of all real-valued measurable and essentially bounded functions on $\mathbb{R}_+$ and $[0,T]$, respectively, equipped with the essential supremum norms $\|\cdot\|$.

The data satisfy the following assumptions:

Assumption $[v,\tilde v]$**.** Suppose that $v\colon\mathbb{R}_+\to\mathbb{R}_+$ is bounded, continuous, integrable, vanishing at $+\infty$, $\tilde v\colon[0,T]\to\mathbb{R}_+$ is continuous, and there is a constant $M_v>0$ such that $\|v\|,\|\tilde v\|\leqslant M_v$.

Assumption $[\lambda]$**.** $\lambda\colon E\times\mathbb{R}_+^2\to\mathbb{R}$ is continuous, $|\lambda|$ is bounded by a positive constant M_λ, and

$$|\lambda(t,x,\bar p,\bar q)-\lambda(t,x,p,q)|\leqslant L_\lambda(|\bar p-p|+|\bar q-q|)$$

for $(t,x,p,q),(t,x,\bar p,\bar q)\in E\times\mathbb{R}_+^2$ with some $L_\lambda>0$.

We know from the theory of linear ODE's that, if the data λ and v are continuous, then the solutions $u^{(i)}$ of the Cauchy problems (1.3), (1.4) are continuous too. Moreover, $u^{(i)}(x)\geqslant 0$ for $i=1,\ldots,N$, $x\in\mathbb{R}_+$, provided that v, $\tilde v$ are nonnegative. We will prove that $u^{(i)}$ is bounded and integrable for $i=1,\ldots,N$.

Suppose that $|\lambda| \leqslant M_\lambda$ on $E\times\mathbb{R}_+^2$, where $M_\lambda > 0$. Consider the following auxiliary comparison problem with respect to (1.3)–(1.4):

$$\frac{F^{(i)} - F^{(i-1)}}{h} = F^{(i)}M_\lambda, \quad i = 1, \dots, N, \qquad F^{(0)} = M_v. \tag{1.5}$$

This is a recurrence equation, however, it can be regarded as a system of ODE's, whose solutions are constant.

Lemma 1.1. *Suppose that $c > 0$, λ is continuous, $|\lambda|$ is bounded by a constant $M_\lambda > 0$, and Assumption $[v, \tilde{v}]$ is satisfied. If $M > M_\lambda$ for some $M > 0$, then there is $h_0 > 0$ such that $h_0 M_\lambda < 1$ and*

$$u^{(i)}(x) \leqslant F^{(i)} \leqslant M_v \mathrm{e}^{Mt^{(i)}}$$

for $x \in \mathbb{R}_+$, $i = 0, \dots, N$, $h < h_0$, sufficiently small.

P r o o f. The first inequality is proved by induction on $i \in \{0, \dots, N\}$. The inequality $u^{(0)}(x) \leqslant F^{(0)}$ for $x \in \mathbb{R}_+$ is obvious. Suppose that for some $i = 1, \dots, N$ we have $u^{(i-1)}(x) \leqslant F^{(i-1)}$ on $\mathbb{R}_+$. Then we derive from (1.3) the differential inequality

$$c\frac{\mathrm{d}}{\mathrm{d}x}u^{(i)}(x) + \frac{u^{(i)}(x)}{h} \leqslant u^{(i)}(x)M_\lambda + \frac{F^{(i-1)}}{h}, \quad u^{(i)}(0) \leqslant F^{(i)}.$$

By the comparison theorem on linear differential inequalities (see [21]) we obtain the estimate $u^{(i)}(x) \leqslant F^{(i)}$ for $x \in \mathbb{R}_+$. From (1.5) we obtain

$$F^{(i)} = M_v\Big(1 + \frac{hM_\lambda}{1 - hM_\lambda}\Big)^i \leqslant M_v(1 + hM)^i.$$

The last inequality is satisfied for sufficiently small h. Therefore, the second assertion is a consequence of the inequality $1 + \varepsilon \leqslant \mathrm{e}^\varepsilon$. □

Lemma 1.2. *Suppose that the assumptions of Lemma 1.1 are satisfied. Then*

$$\int_0^\infty u^{(i)}(x)\,\mathrm{d}x \leqslant (\|v\|_1 + ct^{(i)}\|\tilde{v}\|M/M_\lambda)\mathrm{e}^{Mt^{(i)}}$$

for $i = 0, \dots, N$ and sufficiently small h.

P r o o f. The assertion for $i = 0$ is obvious. If $u^{(i-1)} \in L^1(\mathbb{R}_+)$ for some $i = 1, \dots, N$, then the solution of (1.3) is given by

$$\begin{aligned} u^{(i)}(x) &= \tilde{v}(t^{(i)})\exp\Big(-\int_0^x \frac{1 - h\lambda^{(i)}(y)}{ch}\,\mathrm{d}y\Big) \\ &\quad + \frac{1}{ch}\int_0^x u^{(i-1)}(y)\exp\Big(-\int_y^x \frac{1 - h\lambda^{(i)}(\tau)}{ch}\,\mathrm{d}\tau\Big)\,\mathrm{d}y, \end{aligned}$$

where

$$\lambda^{(i)}(x) = \lambda\Big(t^{(i)}, x, u^{(i-1)}(x), \int_0^\infty u^{(i-1)}(x)\,\mathrm{d}x\Big). \tag{1.6}$$

Since $|\lambda(t,x,p,q)| \leqslant M_\lambda$ for $(t,x,p,q) \in E \times \mathbb{R}_+^2$, we obtain the inequality

$$u^{(i)}(x) \leqslant \tilde{v}(t^{(i)})\mathrm{e}^{-((1-hM_\lambda)/ch)x} + \frac{1}{ch}\int_0^x u^{(i-1)}(y)\mathrm{e}^{-((1-hM_\lambda)/ch)(x-y)}\,\mathrm{d}y.$$

If we integrate this relation over $\mathbb{R}_+$, we get

$$\int_0^\infty u^{(i)}(x)\,\mathrm{d}x \leqslant \frac{ch}{1-hM_\lambda}\tilde{v}(t^{(i)}) + \frac{1}{1-hM_\lambda}\int_0^\infty u^{(i-1)}(x)\,\mathrm{d}x.$$

By Assumption $[v,\tilde{v}]$ we have $0 \leqslant \tilde{v}(t^{(i)}) \leqslant \|\tilde{v}\| < +\infty$, hence we obtain the recurrence inequalities

$$\|u^{(i)}\|_1 \leqslant \frac{ch}{1-hM_\lambda}\|\tilde{v}\| + \frac{1}{1-hM_\lambda}\|u^{(i-1)}\|_1 \quad \text{for} \quad i = 1,\ldots,N$$

with the initial condition $\|u^{(0)}\|_1 = \|v\|_1$. By an elementary argument on recurrence inequalities we get the estimate

$$\|u^{(i)}\|_1 \leqslant \Big(1 + \frac{hM_\lambda}{1-hM_\lambda}\Big)^i \|v\|_1 + c\|\tilde{v}\| \left(\Big(1 + \frac{hM_\lambda}{1-hM_\lambda}\Big)^i - 1\right) \Big/ M_\lambda$$

for $i = 0,\ldots,N$. Thus by the inequalities $1+\varepsilon \leqslant \mathrm{e}^\varepsilon$, $M_\lambda/(1-hM_\lambda) \leqslant M$, we obtain the desired assertion, provided that h is sufficiently small. $\square$

2. Stability

Consider the perturbed Rothe method

$$\begin{aligned} &\frac{\bar{u}^{(i)}(x) - \bar{u}^{(i-1)}(x)}{h} + c\,\frac{\mathrm{d}}{\mathrm{d}x}\bar{u}^{(i)}(x) \\ &\quad = \bar{u}^{(i)}(t)\lambda\Big(t^{(i)}, x, \bar{u}^{(i-1)}(x), \int_0^\infty \bar{u}^{(i-1)}(x)\,\mathrm{d}x\Big) + \xi_h^{(i)}(x), \quad i = 1,\ldots,N \end{aligned} \tag{2.1}$$

with the initial condition

$$\bar{u}^{(0)}(x) = v(x) + \xi_h^{(0)}(x) \quad \text{for } x \in \mathbb{R}_+, \tag{2.2}$$

and the boundary condition

$$\bar{u}^{(i)}(0) = \tilde{v}(t^{(i)}) + \hat{\xi}_h^{(i)} \quad \text{for } i = 1, \ldots, N. \tag{2.3}$$

The perturbations $\xi_h^{(i)}$ for $i = 0, \ldots, N$ and $\hat{\xi}_h^{(i)}$ for $i = 1, \ldots, N$ may appear as a result of inaccurate computations or a substitution into the semidiscrete scheme (1.3)–(1.4) of other functions, for instance: the solution of the PDE (1.1)–(1.2), restricted to the lines $(t^{(i)}, x)$, $i = 0, 1, \ldots, N$, $x \in \mathbb{R}_+$. Denote $\varepsilon^{(i)}(x) = \bar{u}^{(i)}(x) - u^{(i)}(x)$ for $i = 0, \ldots, N$. This is the error function, i.e. the difference between the solutions of the perturbed problem (2.1)–(2.3) and the exact solution of the Rothe method (1.3)–(1.4). Recall that $\lambda^{(i)}(x)$ is defined by (1.6) and denote

$$\bar{\lambda}^{(i)}(x) = \lambda\biggl(t^{(i)}, x, \bar{u}^{(i-1)}(x), \int_0^\infty \bar{u}^{(i-1)}(x)\,\mathrm{d}x\biggr).$$

If we subtract (2.1)–(2.3) and (1.3)–(1.4), then we obtain the recurrence error relations

$$\frac{\varepsilon^{(i)}(x) - \varepsilon^{(i-1)}(x)}{h} + c\frac{\mathrm{d}}{\mathrm{d}x}\varepsilon^{(i)}(x) = \bar{u}^{(i)}(t)\bar{\lambda}^{(i)}(x) - u^{(i)}(t)\lambda^{(i)}(x) + \xi_h^{(i)}(x) \tag{2.4}$$

for $i = 1, \ldots, N$, equipped with the initial and boundary conditions

$$\varepsilon^{(0)}(x) = \xi_h^{(0)}(x) \quad \text{for } x \in \mathbb{R}_+, \quad \varepsilon^{(i)}(0) = \hat{\xi}_h^{(i)} \quad \text{for } i = 1, \ldots, N.$$

Assumption $[\xi, \hat{\xi}]$**.** Suppose that the perturbations $\xi_h^{(i)}$, $i = 0, \ldots, N$, $\hat{\xi}_h^{(i)}$, $i = 1, \ldots, N$, satisfy the conditions

$$\begin{gathered} \xi_h^{(i)} \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) \quad \text{for } i = 0, \ldots, N, \\ \|\xi_h^{(i)}\| \leqslant \alpha_h, \quad \|\xi_h^{(i)}\|_1 \leqslant \alpha_h^{(1)} \quad \text{for } i = 0, \ldots, N, \\ |\hat{\xi}_h^{(i)}| \leqslant \hat{\alpha}_h \quad \text{for } i = 1, \ldots, N, \end{gathered}$$

where $\alpha_h, \alpha_h^{(1)}, \hat{\alpha}_h \to 0$ as $h \to 0$.

Remark 2.1. If λ is Lipschitz continuous then the above Assumption $[\xi, \hat{\xi}]$ is satisfied for the exact solution $\bar{u}$ of the PDE (1.1)–(1.2), restricted to the grids, provided that $\bar{u}, \frac{\partial}{\partial t}\bar{u}$ are of the class $L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ (uniformly in t) and $\frac{\partial}{\partial t}\bar{u}$ satisfies the "tempered" Hölder condition

$$\Bigl|\frac{\partial}{\partial t}\bar{u}(t + \Delta t, x) - \frac{\partial}{\partial t}\bar{u}(t, x)\Bigr| \leqslant |\Delta t|^\alpha H_\alpha(x)$$

for some $\alpha \in (0, 1]$, $H_\alpha \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ and for all admissible $(t + \Delta t, x), (t, x)$. Of course these sufficient conditions are simple but not optimal. There are many cases of less regular solutions with nice $L^\infty \cap L^1$ perturbations.

Theorem 2.2. *Suppose that $c > 0$ and Assumptions $[v, \tilde{v}]$, $[\lambda]$, $[\xi, \hat{\xi}]$ are satisfied. Then*

$$\max_i \|\varepsilon^{(i)}\|,\ \max_i \|\varepsilon^{(i)}\|_1 \to 0 \quad \text{as} \quad h \to 0.$$

Proof. Without loss of generality, in the same way as in Lemmas 1.1, 1.2, one can prove that solutions of (2.1)–(2.3) satisfy the L^∞ and L^1 estimates:

$$\|\bar{u}^{(i)}\| \leqslant U, \quad \|\bar{u}^{(i)}\|_1 \leqslant U_1 \quad \text{for} \quad i = 0, \dots, N. \tag{2.5}$$

Assumption $[\lambda]$ applied to (2.4) yields

$$\begin{aligned} \Big|\frac{\varepsilon^{(i)}(x)}{h} + c\frac{\mathrm{d}}{\mathrm{d}x}\varepsilon^{(i)}(x)\Big| &\leqslant \frac{|\varepsilon^{(i-1)}(x)|}{h} \\ &+ L_\lambda \bar{u}^{(i)}(x)\Big[|\varepsilon^{(i-1)}(x)| + \int_0^\infty |\varepsilon^{(i-1)}(x)|\,\mathrm{d}x\Big] + M_\lambda|\varepsilon^{(i)}(x)| + |\xi_h^{(i)}(x)| \end{aligned} \tag{2.6}$$

for $i = 1, \dots, N$. It follows from Assumption $[\xi, \hat{\xi}]$ that $|\hat{\xi}_h^{(i)}| \leqslant \hat{\alpha}_h$ for $i = 1, \dots, N$, thus we consider the following auxiliary problem associated with (2.6):

$$\frac{e^{(i)}(x)}{h} + c\frac{\mathrm{d}}{\mathrm{d}x}e^{(i)}(x) = \frac{e^{(i-1)}(x)}{h} + \Gamma^{(i)}(x, e^{(i)}), \quad e^{(i)}(0) = \hat{\alpha}_h \tag{2.7}$$

for $i = 1, \dots, N$, where

$$\Gamma^{(i)}(x, e^{(i)}(x)) = L_\lambda U e^{(i-1)}(x) + L_\lambda \bar{u}^{(i)}(x)\|e^{(i-1)}\|_1 + M_\lambda e^{(i)}(x) + |\xi_h^{(i)}(x)|$$

with $e^{(0)}(x) = |\xi_h^{(0)}(x)|$ for $x \in \mathbb{R}_+$. Note that the linear ODE (2.7) leads to the integral representation of $e^{(i)}$:

$$\begin{aligned} e^{(i)}(x) &= \mathrm{e}^{-x/ch}e^{(i)}(0) + \frac{\mathrm{e}^{-x/ch}}{ch}\int_0^x e^{(i-1)}(y)\mathrm{e}^{y/ch}\,\mathrm{d}y \\ &+ \frac{\mathrm{e}^{-x/ch}}{c}\int_0^x \Gamma^{(i)}(y, e^{(i)}(y))\mathrm{e}^{y/ch}\,\mathrm{d}y. \end{aligned} \tag{2.8}$$

The proof of the estimates $|\varepsilon^{(i)}| \leqslant e^{(i)}$ goes by induction on $i \in \{0, \dots, N\}$. The induction step

$$|\varepsilon^{(i-1)}| \leqslant e^{(i-1)} \Longrightarrow |\varepsilon^{(i)}| \leqslant e^{(i)}$$

is based on the comparison theorem [21] (p. 96) as follows. Since $|\varepsilon^{(i-1)}| \leqslant e^{(i-1)}$, $|\varepsilon^{(i)}(0)| \leqslant e^{(i)}(0)$ and (2.7) is a comparison ODE with respect to (2.6), we have $|\varepsilon^{(i)}| \leqslant e^{(i)}$ on $\mathbb{R}_+$.

In order to get effective estimates of $\|e^{(i)}\|$ and $\|e^{(i)}\|_1$ let us integrate (2.8) over $\mathbb{R}_+$. Then we obtain the recurrence equation

$$(2.9)\quad \|e^{(i)}\|_1 = che^{(i)}(0) + \|e^{(i-1)}\|_1 + h\{L_\lambda(U+U_1)\|e^{(i-1)}\|_1 + M_\lambda\|e^{(i)}\|_1 + \|\xi_h^{(i)}\|_1\}$$

with the initial condition $\|e^{(0)}\|_1 = \|\xi_h^{(0)}\|_1$. By analogous means as in the proof of Lemma 1.1, using (2.5), we derive from (2.7) the recurrence inequality

$$\|e^{(i)}\| \leqslant \|e^{(i-1)}\| + h\{L_\lambda U(\|e^{(i-1)}\| + \|e^{(i-1)}\|_1) + M_\lambda\|e^{(i)}\| + \alpha_h\}$$

with the initial condition

$$\|e^{(0)}\| \leqslant \max\{\alpha_h, \hat{\alpha}_h\}.$$

We find the following estimates of $\|e^{(i)}\|_1$, $\|e^{(i)}\|$:

$$\begin{aligned}\|e^{(i)}\|_1 &\leqslant \alpha_h^{(1)}\mathrm{e}^{t^{(i)}M} + (c\hat{\alpha}_h + \alpha_h^{(1)})t^{(i)}\mathrm{e}^{t^{(i)}M},\\ \|e^{(i)}\| &\leqslant \max\{\alpha_h, \hat{\alpha}_h\}\mathrm{e}^{t^{(i)}M} + t^{(i)}\mathrm{e}^{t^{(i)}M}(\alpha_h + L_\lambda U_1\|e^{(N)}\|_1),\end{aligned}$$

where $M > M_\lambda + L_\lambda(U+U_1)$. Hence we have

$$\|\varepsilon^{(i)}\|_1 \leqslant \|e^{(i)}\|_1, \quad \|\varepsilon^{(i)}\| \leqslant \|e^{(i)}\| \quad \text{for } i = 0, \ldots, N,$$

and $\|\varepsilon^{(i)}\|_1$, $\|\varepsilon^{(i)}\|$ tend to zero as $h \to 0$, uniformly with respect to i. □

We formulate a simple conclusion for a classical McKendrick-von Foerster model with a mortality coefficiet dependent on the total size of individuals.

Corollary 2.3. *Let $d\colon \mathbb{R}_+ \to \mathbb{R}_+$ be boundend and Lipschitz continuous. Suppose that $v\colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $|v|, |v'|, |v''| \leqslant V$ for some $V \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, vanishing at $+\infty$. Then the Rothe method corresponding to*

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = -u(t,x)d\Big(\int_0^\infty u(t,y)\,\mathrm{d}y\Big)$$

with the initial-boundary conditions $u(0,x) = v(x)$ and $u(t,0) = v(0)$ is convergent.

Proof. The function $\lambda(t,x,p,q) = -d(q)$ satisfies the assumption of the above theorem. The assumptions on v imply the desired estimates of the perturbations for the exact solution $\bar{u}$ of the PDE restricted to the grids. □

Remark 2.4. The results of the paper, under some slight modification of assumptions, remain true if we consider the following implicit Rothe method for (1.1)–(1.2):

$$
\begin{aligned}
(2.10)\qquad & \frac{u^{(i)}(x)-u^{(i-1)}(x)}{h}+c\frac{\mathrm{d}}{\mathrm{d}x}u^{(i)}(x)\\
&\qquad = u^{(i)}(t)\lambda\Big(t^{(i)},x,u^{(i)}(x),\int_0^\infty u^{(i)}(x)\,\mathrm{d}x\Big)\quad \text{for } i=1,\dots,N,
\end{aligned}
$$

$$
(2.11)\qquad u^{(0)}(x)=v(x)\quad \text{for } x\in\mathbb{R}_+,\quad u^{(i)}(0)=\tilde v(t^{(i)})\quad \text{for } i=1,\dots,N.
$$

To be more precise, if $u^{(i)}$, $i=0,\dots,N$, are solutions of (2.10)–(2.11), then $u^{(i)}\in L^1(\mathbb{R}_+)\cap L^\infty(\mathbb{R}_+)$ and the estimates of $\|u^{(i)}\|$, $\|u^{(i)}\|_1$ are the same as those in Lemmas 1.1, 1.2. In the stability analysis of (2.10)–(2.11) we consider a system similar to (2.6) and the corresponding comparison problems for errors in $\|\cdot\|$, $\|\cdot\|_1$. Under the assumptions of Theorem 2.2 with the additional condition $h_0(M_\lambda+L_\lambda(U+U_1))<1$, we obtain the stability criterion for the implicit Rothe method (2.10)–(2.11).

Remark 2.5. Due to the Lax-Richtmyer equivalence theorem, convergence of a scheme is equivalent to its stability and consistency, provided that the problem is linear, cf. [11]. The same statement holds also for nonlinear problems, see [7]. We provide here a brief description of the consistency for our problem.

Suppose that the data v, $\tilde v$ are such that the derivatives v', $\tilde v'$ exist, $v,v'\in L^1(\mathbb{R}_+)\cap L^\infty(\mathbb{R}_+)$, $\tilde v,\tilde v'$ are bounded on $[0,T]$ and there are $\alpha\in(0,1)$, $L^*\colon\ \mathbb{R}_+\to\mathbb{R}_+$ such that $L^*\in L^1(\mathbb{R}_+)\cap L^\infty(\mathbb{R}_+)$ and

$$
|v'(x+\Delta x)-v'(x)|\leqslant L^*(x)|\Delta x|^\alpha\quad \text{on } \mathbb{R}_+
$$

for $|\Delta x|\leqslant 1$. Moreover, $\lambda\colon\ E\times\mathbb{R}_+\to\mathbb{R}$ satisfies the Lipschitz condition with respect to (t,x,u,z) on $E\times\mathbb{R}^2$. Then the Rothe method (1.3)–(1.4) is consistent at a solution of (1.1)–(1.2), see [17].

Remark 2.6. Suppose that real numbers $c,T,X>0$ and a function $d\colon\ \mathbb{R}\to\mathbb{R}_+$ are given. Consider the differential equation

$$
(2.12)\qquad \frac{\partial u}{\partial t}(t,x)+c\frac{\partial u}{\partial x}(t,x)=-d\Big(\int_0^X u(t,y)\,\mathrm{d}y\Big)u(t,x)
$$

on $[0,T]\times[0,X]$ with the initial and boundary conditions

$$
(2.13)\qquad u(0,x)=v(x)\quad \text{for } x\in[0,X],\quad u(t,0)=\tilde v(t)\quad \text{for } t\in[0,T],
$$

such that $v(0) = \tilde{v}(0)$. Problem (2.12)–(2.13) with $c = 1$ was considered in [10]. Denote $z(t) = \int_0^X u(t,y)\,\mathrm{d}y$. Using the method of characteristics one can write a solution of the above problem:

$$u(t,x) = \begin{cases} \tilde{v}\Big(t - \dfrac{x}{c}\Big)\exp\Big(-\displaystyle\int_{t-x/c}^{t} d(z(s))\,\mathrm{d}s\Big), & x < ct, \\ v(x-ct)\exp\Big(-\displaystyle\int_0^t d(z(s))\,\mathrm{d}s\Big), & X \geqslant x \geqslant ct. \end{cases} \tag{2.14}$$

If $t \in [0,T]$ and $0 \leqslant ct \leqslant X$, then z satisfies the integral equation

$$z(t) = c\int_0^t \tilde{v}(y)\exp\Big(-\int_y^t d(z(s))\,\mathrm{d}s\Big)\,\mathrm{d}y + \int_0^{X-ct} v(y)\exp\Big(-\int_0^t d(z(s))\,\mathrm{d}s\Big)\,\mathrm{d}y,$$

which is equivalent to the differential-integral equation

$$z'(t) + z(t)d(z(t)) = c\tilde{v}(t) - cv(X-ct)\exp\Big(-\int_0^t d(z(s))\,\mathrm{d}s\Big) \tag{2.15}$$

with the initial condition $z(0) = \int_0^X v(x)\,\mathrm{d}x$. If $t \in [X/c, T]$, then z satisfies the integral equation

$$z(t) = \int_0^X \tilde{v}\Big(t - \frac{x}{c}\Big)\exp\Big(-\int_{t-\frac{x}{c}}^t d(z(s))\,\mathrm{d}s\Big)\,\mathrm{d}x,$$

or equivalently

$$z'(t) + z(t)d(z(t)) = c\tilde{v}(t) - c\tilde{v}\Big(t - \frac{X}{c}\Big)\exp\Big(-\int_{t-X/c}^t d(z(s))\,\mathrm{d}s\Big). \tag{2.16}$$

Suppose that an initial function v is defined on $\mathbb{R}_+$ and $v \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$. Then we can consider equation (2.12) on $[0,T] \times \mathbb{R}_+$. A solution of problem (2.12) on $[0,T] \times \mathbb{R}_+$ can be computed by the same method as the case (2.12)–(2.13) on $[0,T] \times [0,X]$. Moreover, solutions of this problem on $[0,T] \times [0,X]$ tend to the solution on $[0,T] \times \mathbb{R}_+$.

Example. Consider problem (2.12)–(2.13) with $c = 1$, $T = 10$, $X = 2$ and $d(x) = x$, equipped with the initial and boundary conditions

$$v(x) = 1 \quad \text{for } x \in [0,X]; \quad \tilde{v}(t) = \mathrm{e}^{-t} \quad \text{for } t \in [0,T]. \tag{2.17}$$

This problem was discussed in [10], page 434. Solutions of (2.15), (2.16) are approximated by a finite difference method. Then we find an approximate solution of problem (2.12)–(2.13), based on formula (2.14). In our numerical experiment we set $h = 10^{-4}$, $N = 10^5$. The integrals in the above formulas are approximated by the trapezoidal rules. Figure 1 displays the approximation of z obtained by the above procedure, whereas Figure 2 presents the approximation of u.

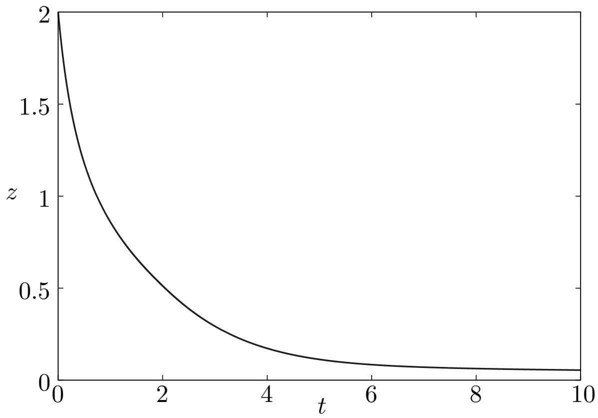

Figure 1. Numerical approximation of z with $h = 10^{-4}$

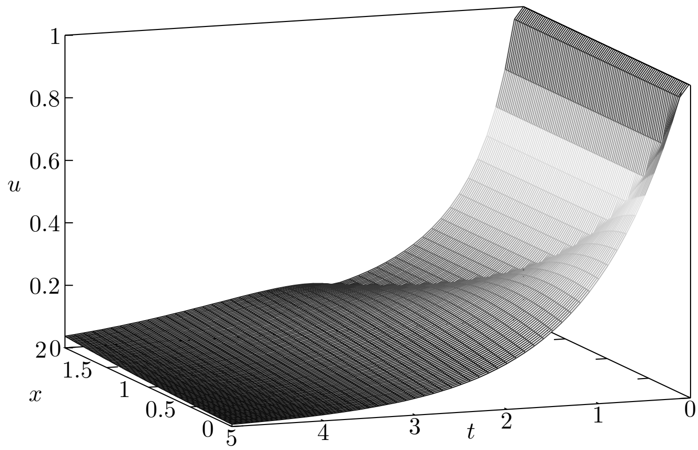

Figure 2. Approximate solution of (2.12), (2.13) with $v(x) = 1$, $\tilde{v}(t) = \mathrm{e}^{-t}$

Consider the Rothe method for (2.12)–(2.13) with the initial and boundary data (2.17). Then we get the recurrence relations

$$\frac{\mathrm{d}}{\mathrm{dx}} u^{(i)}(x) + u^{(i)}(x)\Big(z^{(i-1)} + \frac{1}{h}\Big) = \frac{u^{(i-1)}(x)}{h}, \quad u^{(i)}(0) = \tilde{v}^{(i)}$$

for $i = 1, \ldots, N$, where $z^{(i)} = \int_0^2 u^{(i)}(x)\,\mathrm{d}x$. If we integrate the above differential equations over $[0, 2]$, we obtain

$$z^{(i)} = \frac{\tilde{v}^{(i)} - u^{(i)}(2) + z^{(i-1)}/h}{z^{(i-1)} + 1/h}, \quad z^{(0)} = 2.$$

The solution is given by

$$u^{(i)}(x) = \tilde{v}^{(i)} \mathrm{e}^{-(z^{(i-1)}+1/h)x} + \frac{1}{h} \int_0^x u^{(i-1)}(y) \mathrm{e}^{-(x-y)(z^{(i-1)}+1/h)} \, \mathrm{d}y$$

for $i = 1, \ldots, N$. In our experiment we approximate the functions $u^{(i)}$ by piecewise linear functions $\hat{u}^{(i)}$, $i = 1, \ldots, N$, given by

$$\hat{u}^{(i)}(y) = u_j^{(i)} \frac{H(j+1) - y}{H} + u_{j+1}^{(i)} \frac{y - Hj}{H}, \quad y \in [Hj, H(j+1)]$$

for $j = 0, \ldots, 2/H - 1$, where $H = \sqrt{h}$ and

$$u_j^{(i)} = \tilde{v}^{(i)} \mathrm{e}^{-(z^{(i-1)}+1/h)Hj} + \frac{1}{h} \sum_{k=0}^{j-1} \int_{Hk}^{H(k+1)} \hat{u}^{(i-1)}(y) \mathrm{e}^{-(Hj-y)(z^{(i-1)}+1/h)} \, \mathrm{d}y$$

for $j = 1, \ldots, 2/H$ with the initial data $u_j^{(0)} = 1$, $j = 0, \ldots, 2/H$. Figure 3 shows the approximate solution of (2.12)–(2.13) obtained by the Rothe method. Computations were performed by Octave Ver. 3.4.3.

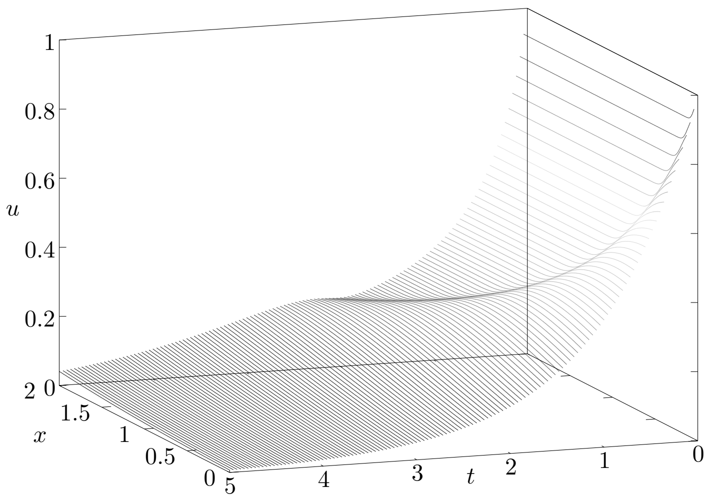

Figure 3. The Rothe method for (2.12), (2.13) with $v(x) = 1$, $\tilde{v}(t) = \mathrm{e}^{-t}$

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