Mathematica Bohemica

Marcel Erné
$\mathcal{Z}$-distributive function lattices

$\mathcal{Z}$-DISTRIBUTIVE FUNCTION LATTICES

Marcel Erné, Hannover

Dedicated to Dana Scott on his 80th birthday

Abstract. It is known that for a nonempty topological space X and a nonsingleton complete lattice Y endowed with the Scott topology, the partially ordered set $[X, Y]$ of all continuous functions from X into Y is a continuous lattice if and only if both Y and the open set lattice $\mathcal{O}X$ are continuous lattices. This result extends to certain classes of $\mathcal{Z}$-distributive lattices, where $\mathcal{Z}$ is a subset system replacing the system $\mathcal{D}$ of all directed subsets (for which the $\mathcal{D}$-distributive complete lattices are just the continuous ones). In particular, it is shown that if $[X, Y]$ is a complete lattice then it is supercontinuous (i.e. completely distributive) iff both Y and $\mathcal{O}X$ are supercontinuous. Moreover, the Scott topology on Y is the only one making that equivalence true for all spaces X with completely distributive topology. On the way to these results, we find necessary and sufficient conditions for $[X, Y]$ to be complete, and some new, purely topological characterizations of continuous lattices by continuity conditions on their (infinitary) lattice operations.

Keywords: completely distributive lattice, continuous function, continuous lattice, Scott topology, subset system, $\mathcal{Z}$-continuous, $\mathcal{Z}$-distributive

MSC 2010: 06B35, 06D10, 54F05, 54H10

1. Introduction: continuous posets, supercontinuous lattices, and C-spaces

The underlying set of any topological space X is endowed with a natural quasi-order, the so-called *specialization order* defined by $x \leqslant y$ iff x belongs to each closed set containing y. The resulting quasi-ordered set is denoted by ΩX. Since continuous maps preserve the specialization order, Ω may be regarded as a functor from the category of topological spaces to that of quasi-ordered sets. If not otherwise stated, order-theoretical statements about spaces will refer to the specialization order. A topology *induces* a quasi-order if this is the specialization order of the space.

A space is T_0 iff its specialization order is antisymmetric, that is, a partial order. Conversely, any partially ordered set P with underlying set $|P|$ carries several 'intrinsic' order-inducing topologies; the weakest one is the *upper* or *weak* topology υP, generated by the complements of principal ideals $\downarrow y = \{x \in P \colon x \leqslant y\}$, while the strongest one, the *Alexandroff topology* αP, consists of all *upper sets*, i.e., subsets U that are equal to $\uparrow U = \{x \in P \colon \exists y \in U \ (x \geqslant y)\}$; the dually defined *lower sets* $A = \downarrow A$ are just the complements of the upper sets; hence, they are the closed sets w.r.t. αP. In the present context, the most important order-inducing topology is the *Scott topology* σP, whose closed sets are those lower sets which contain all existing joins of their directed subsets (see Gierz et al. [14], [15], Scott [24]). We write ΣP for the space $(|P|, \sigma P)$ and note the equation $\Omega \Sigma P = P$; similarly, we have $\Omega \Upsilon P = P$ for $\Upsilon P = (|P|, \upsilon P)$ and $\Omega A P = P$ for $AP = (|P|, \alpha P)$. A topological space X is said to be *order complete* if ΩX is a complete lattice.

For any topological spaces X and Y, the set $[X, Y]$ of all continuous functions from X into Y is (quasi-)ordered pointwise by the specialization order of Y. Our main purpose is an investigation of certain completeness properties and infinite distributive laws for such ordered function spaces (topologies on $[X, Y]$ will not concern us here). We do not assume a priori that Y carries the Scott topology—but that coincidence will often be a consequence of the continuity or distributivity properties under consideration.

Let us recall first a few basic order-theoretical definitions and facts (see e.g. [5], [7], [14], [15], [23], [24]). Given elements x, y of a poset P, write $x \ll y$ (respectively, $x \lll y$) if x belongs to every directed (every nonempty) lower set having a join (= supremum) that dominates y. A complete lattice L is said to be *continuous* (*supercontinuous*) if each element y of L is the join of elements $x \ll y$ ($x \lll y$). More generally, a *continuous poset* is a poset in which each of the sets $\ll y = \{x \colon x \ll y\}$ is directed with join y, and a *(continuous) domain* is an up-complete continuous poset (in which all directed subsets have joins). It is not hard to see (cf. Raney [23]) that a complete lattice is supercontinuous if and only if it is *completely distributive*, i.e., the identity

$$\bigwedge \{\bigvee A \colon A \in \mathcal{X}\} = \bigvee \bigcap \mathcal{X}$$

holds for all collections $\mathcal{X}$ of lower sets. The same identity for all collections of directed lower sets characterizes continuous lattices. It will be convenient to use the abbreviation cdl for '*completely distributive complete lattice*'. The relation $\ll$ on a continuous lattice and the relation $\lll$ on a cdl are always idempotent ('interpolation property'). Consequently, in a cdl, the complements of the sets $x \lll = \{y \colon x \lll y\}$ are principal ideals, being lower sets closed under arbitrary joins, whereas the complements of the sets $x \ll = \{y \colon x \ll y\}$ in continuous lattices or domains are only

closed under directed joins; in other words, the sets $x \ll$ are Scott open; moreover, they form a base for the Scott topology, while in a cdl, the sets $x \lll$ only form a *subbase* for the Scott topology (and for the upper topology).

A topological space is called *locally supercompact* or a *C-space* [5], [7], [10] if each point has a neighborhood base of supercompact sets, where a set C is *supercompact* if any open covering of C has a member that contains C. Supercompactness is equivalent to the existence of a least element w.r.t. the specialization order (warning: in other contexts, supercompactness may have a different, weaker meaning). The *core* $\uparrow x$ of a point x is the intersection of all neighborhoods of x. The cores are precisely the supercompact upper sets (w.r.t. the specialization order); thus, a space is locally supercompact iff every point has a neighborhood base consisting of cores. As in the well-known situation of locally compact spaces, one verifies:

Proposition 1.1. *A product of nonempty topological spaces is locally supercompact iff each factor is locally supercompact and almost all factors are supercompact. In particular, arbitrary powers of an order complete C-space are again order complete C-spaces.*

A *monotone convergence space* (see [14], [15]) or *d-space* (see Wyler [27]) is a T_0-space Y such that every directed subset of ΩY has a supremum and converges (as a net) to that supremum in the specialization order. By definition, the topology of a d-space Y is always weaker than (i.e. contained in) the Scott topology of ΩY. Thus, an order complete space Y is a d-space iff the topology $\mathcal{O}Y$ is weaker than the Scott topology $\sigma \Omega Y$. For more background concerning these notions and Theorem 1.1, see [5], [7], [10], [12], [14], [15], [16].

Theorem 1.1. *A topological space Y is locally supercompact iff its open set lattice $\mathcal{O}Y$ is a cdl. For locally supercompact T_0-spaces Y, the Scott topology of ΩY is weaker than the topology of Y. Moreover, Y is a locally supercompact d-space iff Y is the Scott space ΣP of a (unique) continuous domain P.*

A central result in the theory of continuous lattices is the following statement about the function posets $[X, \Sigma L]$ (see [14, II–4.8] or [15, II–4.7]):

Theorem 1.2. *Let X be a nonempty topological space and L a complete nonsingleton lattice. Then $[X, \Sigma L]$ is a continuous lattice if and only if $\mathcal{O}X$ and L are continuous lattices.*

In 1998, Klaus Keimel asked whether an analogous result would hold for 'completely distributive' ('supercontinuous') instead of 'continuous'. In a direct response,

I gave an affirmative answer in an unpublished first draft of this note, which contained Theorems 3.1 and 4.1–4.3 (see [18] for a citation). The present extended version includes additional results on completeness, continuity and infinite distributivity properties of ordered function spaces $[X, Y]$.

An important observation will be that all existing joins (but not all meets) in $[X, Y]$ are formed pointwise. Using that fact, we shall find several conditions on a space Y that are necessary and sufficient for $[X, Y]$ to be a complete lattice for each space X. One such condition is that all join operations of arbitrary arity exist in ΩY and are continuous. As a byproduct, we obtain new descriptions of continuous lattices by topological continuity properties of the (finitary and infinitary) lattice operations. More generally, we characterize order complete C-spaces as those spaces Y in which all meet operations of arbitrary arity exist and are continuous for the box topology on powers of Y. The Scott spaces of continuous lattices are exactly those C-spaces Y for which all spaces X with continuous open set lattices $\mathcal{O}X$ (and only these) give rise to continuous lattices $[X, Y]$. For continuous lattices L with the weak topology υL, join operations of arbitrary arity are continuous and the poset $[\Upsilon L, \Upsilon L]$ is a complete lattice, but in contrast to $[\Sigma L, \Sigma L]$, it need not be continuous.

Theorem 3.1 is a strengthened 'supercontinuous analogue' of Theorem 1.2: if X and Y are nontrivial spaces such that $[X, Y]$ is complete, then supercontinuity of $[X, Y]$ is equivalent to that of $\mathcal{O}X$ and ΩY. But we give an example of a space Y with supercontinuous lattices $\mathcal{O}Y$ and ΩY whose function poset $[Y, Y]$ is not even complete. On the other hand, if Y is a space such that for all spaces with supercontinuous topology $\mathcal{O}X$ the function posets $[X, Y]$ are supercontinuous lattices, too, then Y must necessarily carry the Scott topology. It remains open whether such a conclusion holds for 'continuous' instead of 'supercontinuous'.

In the last section, some of the results are extended, mutatis mutandis, to the so-called $\mathcal{Z}$-distributive complete lattices. Here $\mathcal{Z}$ is a subset system, that is, a function associating with all posets certain distinguished collections of subsets that are preserved under isotone maps. The notions of $\mathcal{Z}$-distributivity and $\mathcal{Z}$-continuity provide, respectively, obvious common generalizations of continuity and supercontinuity for complete lattices. But the proofs of the '$\mathcal{Z}$-generalized' statements often require additional arguments. For subset systems $\mathcal{Z}$ and spaces Y with suitable completeness and closedness properties, $\mathcal{Z}$-distributivity of $[X, Y]$ turns out to be equivalent to $\mathcal{Z}$-distributivity of $\mathcal{O}X$ and ΩY. Surprisingly, this does not lead to an essential generalization of the two special cases of continuous and supercontinuous lattices, as long as all directed subsets of bounded posets P are required to be members of $\mathcal{Z}P$: if $\mathcal{Z}$ is a union complete subset system with that property then the $\mathcal{Z}$-distributive complete lattices are either exactly the continuous ones or exactly the supercontinuous ones. Finally, an example will show that, in contrast to the situation with (super-)

continuous lattices, the lattice of all $\mathcal{Z}$-join preserving self-maps of a $\mathcal{Z}$-continuous complete lattice need not be $\mathcal{Z}$-continuous in case $\mathcal{Z}$ is the union complete system of finite subsets.

2. Completeness and continuity of $[X, Y]$

For any set I, the I-ary join and meet operations of a complete lattice L are given by

$$\bigvee\nolimits_I \colon L^I \to L, \quad (x_i \colon i \in I) \mapsto \bigvee\{x_i \colon i \in I\},$$

$$\bigwedge\nolimits_I \colon L^I \to L, \quad (x_i \colon i \in I) \mapsto \bigwedge\{x_i \colon i \in I\}.$$

We need a few basic facts concerning the ordered function sets $[X, Y]$. The 'only if' part of Theorem 1.2 and its analogue for the supercontinuous case is a consequence of

Proposition 2.1. *Let X, Y be nonempty topological spaces and suppose ΩY is a poset containing two elements $y < z$. Then the Sierpinski space $S = \Sigma(\{y, z\}, \leqslant)$ is a subspace of Y, and $[X, S]$ is a complete sublattice of $[X, Y]$ isomorphic to $\mathcal{O}X$. Hence, if $[X, Y]$ is a complete lattice then every identity built by finitary or infinitary lattice operations and valid in $[X, Y]$ must also hold in $\mathcal{O}X$ and in the complete lattice ΩY, which is isomorphic to the join- and meet-closed subposet $c[X, Y]$ of all constant functions from X to Y.*

P r o o f.   In order to prevent misunderstandings, we point out that by a *complete sublattice* of a partially ordered set (poset) P we mean here a nonempty subset C such that any nonempty subset of C has a join and a meet in P, and these joins and meets belong to C (but the least and the greatest element of C need not be the same as those of P).

Given $y < z$ in ΩY and $U \subseteq Y$, the function $c_U \colon X \to S = \{y, z\} \subseteq Y$ with $c_U(x) = z$ for $x \in U$ and $c_U(x) = y$ for $x \in X \setminus U$ is continuous iff $U \in \mathcal{O}X$. For any nonempty family $(c_U \colon U \in \mathcal{V})$ with $\mathcal{V} \subseteq \mathcal{O}X$, the continuous function $c_{\bigcup \mathcal{V}}$ is the pointwise supremum; hence, it is the join in $[X, Y]$. Their meet is not obtained pointwise in general but is the continuous function c_W where W is the interior of the intersection $\bigcap \mathcal{V}$. Indeed, if f is any lower bound of $\{c_U \colon U \in \mathcal{V}\}$ in $[X, Y]$ then $f(x) \leqslant y$ for all $x \in X \setminus \bigcap \mathcal{V}$, and the continuity of f forces $f(x) \leqslant y$ for all $x \in \overline{X \setminus \bigcap \mathcal{V}} = X \setminus W$, because $f^{-1}[\downarrow y] = f^{-1}[\overline{\{y\}}]$ is closed; thus, $f \leqslant c_W$. Sending U to c_U yields an isomorphism between $\mathcal{O}X$ and the complete sublattice $[X, S]$ of $[X, Y]$. Thus, all lattice identities valid in $[X, Y]$ are inherited by $[X, S]$ and then transferred to $\mathcal{O}X$. The corresponding statement about $\Omega Y \simeq c[X, Y]$ is obvious.

$\square$

E x a m p l e 2.1. We have already remarked that meets in $[X, Y]$ need not be formed pointwise. This is witnessed by a quick inspection of the (supercontinuous) complete chain

$$X = Y = \omega + 1 = \omega \cup \{\omega\} = \{0, 1, 2, \ldots, \omega\}$$

endowed with the Scott topology, and the continuous functions $f_n \colon X \to Y$ with $f_n(x) = 0$ for $x < n$ and $f_n(x) = \omega$ for $x \geqslant n$, whose pointwise meet f_ω is not continuous.

In contrast to the situation with meets, joins in $[X, Y]$ behave stably:

Theorem 2.1. *Let X and Y be spaces such that ΩY is a poset with greatest element.*
(1) *The isotone (= order preserving) functions from ΩX to ΩY are exactly the pointwise meets of continuous functions from X to Y.*
(2) *All joins in $[X, Y]$ are formed pointwise; that is, if f is the join of a set G in $[X, Y]$ then, for each $x \in X$, $f(x)$ is the join of $\{g(x) \colon g \in G\}$ in ΩY.*
(3) *$[X, Y]$ is a complete lattice iff ΩY is such and $[X, Y]$ is closed under poinwise joins.*

P r o o f.   (1) Let f be an isotone function from ΩX to ΩY. Then f is the pointwise meet of the continuous functions $f_z \colon X \to Y$ with $f_z(x) = f(z)$ for $x \leqslant z$ and $f_z(x) = \bigvee \Omega Y$ otherwise: continuity of f_z follows from the identities $f_z^{-1}[V] = X$ if $f(z) \in V = {\uparrow}V$ and $f_z^{-1}[V] = X \setminus {\downarrow}z$ for all other nonempty open (hence upper) sets V. We have $f \leqslant f_z$, because $x \leqslant z$ implies $f(x) \leqslant f(z) = f_z(x)$, and $x \not\leqslant z$ implies $f(x) \leqslant \bigvee \Omega Y = f_z(x)$. And if $g \colon X \to Y$ satisfies $g \not\leqslant f$ then there is a z with $g(z) \not\leqslant f(z) = f_z(z)$, hence $g \not\leqslant f_z$. Thus, f is the greatest lower bound of continuous functions.

Conversely, recall that all continuous functions from X to Y are isotone as functions from ΩX to ΩY, and that pointwise joins and meets of isotone functions are isotone.

(2) follows from (1) and a general observation about meet-dense subposets, applied to $P = [X, Y]$ and the pointwise ordered set Q of all isotone maps $f \colon \Omega X \to \Omega Y$:

If P is meet-dense in a poset Q and f is the join of S in P then f is also the join of S in Q.

(3) is an immediate consequence of (2) and Proposition 2.1.   □

The hypothesis that ΩY has a greatest element cannot be omitted in Theorem 2.1:

E x a m p l e 2.2. Let S be a 3-element meet-semilattice but not a chain, and let P be the poset obtained from S by replacing the least element with the chain ω. The map $f \colon P \to S$, sending the maximal elements to themselves and all other elements

to the least element of S, is maximal among the isotone maps from P to S, but it is not continuous as a map from $X = \Upsilon P$ to $Y = \Upsilon S$ (both posets endowed with the upper topology). Consequently, f cannot be represented as a pointwise meet of continuous functions from X to Y. On the other hand, f is the pointwise join of two continuous functions.

We are now in a position to characterize those spaces Y for which $[X, Y]$ is always complete.

Theorem 2.2. *For a topological space Y, the following conditions are equivalent:*
(a) *Y is order complete, and join operations of arbitrary (not only of finite) arity are continuous as functions from (topological) powers of Y to Y.*
(b) *Y is a d-space and a topological bounded $\vee$-semilattice in the specialization order.*
(c) *$[Y^I, Y]$ is a complete lattice for each set I.*
(d) *$[X, Y]$ is a complete lattice for each space X.*
(e) *Y is order complete, and $[X, Y]$ is closed under pointwise joins, for each space X.*

P r o o f.   (a) $\Rightarrow$ (b): If $\bigvee D \in V$ for some directed set $D \subseteq Y$ and some open set $V \subseteq Y$ then, by continuity of the join operation $\bigvee_D \colon Y^D \to Y$, we find a finite subset E of D and open neighborhoods U_y of $y \in D$ with $U_y = Y$ for $y \in D \setminus E$ such that the join map $\bigvee_D$ sends $\prod_{y \in D} U_y$ into V; in particular, since Y has a least element, $\bigvee E \in V$. Now, choosing a $z \in D$ with $y \leqslant z$ for all $y \in E$ and observing that V is an upper set with respect to the specialization order, we get $z \in D \cap V$. Thus, Y is a d-space.

(b) $\Rightarrow$ (e): If Y is a d-space and X is an arbitrary space then $[X, Y]$ is closed under directed pointwise joins (see [15, II–3.14]). But for a topological bounded $\vee$-semilattice Y, the poset $[X, Y]$ is also closed under finitary pointwise joins, because for any finite $F \subseteq [X, Y]$, the maps $g \colon X \to Y^F$, $x \mapsto (f(x) \colon f \in F)$ and $\bigvee_F \colon Y^F \to Y$ are continuous, so the same holds for the composite map, which is the pointwise join of F. Combining the two closedness properties, we see that $[X, Y]$ is a complete lattice closed under arbitrary pointwise joins.

(e) $\Rightarrow$ (d): See Theorem 2.1 (3).

(d) $\Rightarrow$ (c) is a trivial specialization.

(c) $\Rightarrow$ (a): ΩY is complete, being isomorphic to $[Y^0, Y]$. Since $[Y^I, Y]$ is closed under pointwise joins and the projections from Y^I to Y are continuous, so is their pointwise supremum, the join operation $\bigvee_I \colon Y^I \to Y$. $\square$

A useful sufficient condition for the properties listed in Theorem 2.2 is given in

Lemma 2.1. *Let Y be an order complete d-space admitting a quasi-approximating relation $\prec$ (satisfying $y = \bigvee\{x\colon x \prec y\}$ for all $y \in Y$) such that the sets $x\!\prec\, = \{y\colon x \prec y\}$ are open. Then arbitrary join operations of Y are continuous.*

P r o o f.   For any element $x = (x_i\colon i \in I) \in Y^I$, let D denote the set of all joins of finite sets $F \subseteq Y$ such that for each $z \in F$ there is $i \in I$ with $z \prec x_i$. Then D is directed, and if x has the join $y = \bigvee\{x_i\colon i \in I\}$ then D has the same join by the approximation property of $\prec$; thus, as a net, D converges to y. Hence, if $y \in V \in \mathcal{O}Y$, some finite join $\bigvee F \in D$ must belong to V. Choose a finite $N \subseteq I$ such that for each $z \in F$ there is an $i \in N$ with $z \prec x_i$. Then the cartesian product of the sets U_i with $U_i = \bigcap\{z\!\prec\colon z \in F, z \prec x_i\}$ if $i \in N$ and $U_i = Y$ if $i \in I \setminus N$ is an open neighborhood of $(x_i\colon i \in I)$ in the product space Y^I, and this neighborhood is mapped into the upper set V by the I-ary join operation $\bigvee_I$. $\square$

Such a relation $\prec$ as required in Lemma 2.1 exists if Y is an order complete d-space and

(1) ΩY is a cdl (take $\prec\, = \lll$), or
(2) ΩY is a continuous lattice and $\sigma\Omega Y \subseteq \mathcal{O}Y$ (take $\prec\, = \ll$), or
(3) $\mathcal{O}Y$ is a cdl (take $x \prec y \Leftrightarrow y \in (\uparrow x)^\circ$, the interior of the core of x).

Note that (1) implies (2) (since $\sigma L = \upsilon L$ for cdl L; see [12] or [15, VII–3.4 and 3.12]), and (2) implies (3) for d-spaces, since then $\mathcal{O}Y \subseteq \sigma\Omega Y$; see Theorem 1.1, which also tells us that the interior relation $\prec$ of a space Y with completely distributive $\mathcal{O}Y$ (that is, of a C-space) has the required approximation property.

Under certain distributivity assumptions, Theorem 2.2 may be improved:

Corollary 2.1. *Let Y be a nonsingleton order complete space so that ΩY or $\mathcal{O}Y$ is a cdl. Then the statements (a)–(e) in Theorem 2.2 are tantamount to each of the following ones:*
 (f) *Y is the Scott space ΣL of a continuous lattice L.*
 (g) *$Y = \Sigma\Omega Y$.*
 (h) *Y is sober.*
 (i) *Y is a d-space.*
 (j) *$\mathcal{O}Y \subseteq \sigma\Omega Y$.*

P r o o f.   (f) $\Rightarrow$ (g) is trivial.
(g) $\Rightarrow$ (h): If $Y = \Sigma\Omega Y$ and ΩY or $\mathcal{O}Y$ is completely distributive then ΩY is a continuous lattice (Theorem 1.1), hence $\Sigma\Omega Y$ is sober (see [15, II–1.12]).
(h) $\Rightarrow$ (i) $\Rightarrow$ (j): See, for example, [15] or [27].
(j) $\Rightarrow$ (f): If $\mathcal{O}Y$ is a cdl and contained in $\sigma\Omega Y$ then, by Theorem 1.1, Y must coincide with $\Sigma\Omega Y$, and ΩY is a continuous lattice. The same conclusion holds when

$\mathcal{O}Y \subseteq \sigma\Omega Y$ and $L = \Omega Y$ is a cdl, but the argument is slightly different; in that case, $\mathcal{O}Y \subseteq \sigma L = \upsilon L \subseteq \mathcal{O}Y$.

Finally, by Lemma 2.1 and the remarks thereafter, (i) implies (a) in Theorem 2.2 if ΩY or $\mathcal{O}Y$ is a cdl. Relating the items in Theorem 2.2 to those in Corollary 2.1, we arrive at the implication circuit

(a) $\Rightarrow$ (b) $\Rightarrow$ (i) $\Rightarrow$ (j) $\Rightarrow$ (f) $\Rightarrow$ (g) $\Rightarrow$ (h) $\Rightarrow$ (i) $\Rightarrow$ (a). $\qquad\square$

E x a m p l e 2.3. For the A-space $Y = \mathrm{A}(\omega + 1)$ (cf. Example 2.1), both ΩY and $\mathcal{O}Y$ are complete chains, hence completely distributive, but none of the equivalent properties in Theorem 2.2 and Corollary 2.1 is fulfilled ($\mathcal{O}Y$ is not contained in $\sigma\Omega Y$). In particular, the infinitary join operations $\bigvee_I \colon Y^I \to Y$ are not continuous!

On the other hand, all complete lattices L equipped with the upper topology υL have the equivalent properties in Theorem 2.2 (since preimages of principal ideals under I-ary join operations are principal ideals in the powers L^I). Order complete Scott spaces ΣL are d-spaces (but not always sober, see Isbell [17]), so they share these properties if and only if they are topological $\vee$-semilattices. This happens for all continuous lattices (see Corollary 2.1 or [14], [15]) but not for all join- and meet-continuous complete lattices. In [13] it is shown that for bounded semilattices S, the square space $\Sigma(S^2)$ is a topological semilattice iff ΣS is such and $\Sigma(S^2)$ coincides with $(\Sigma S)^2$, and examples of atomless complete (hence join- and meet-continuous) Boolean algebras are given for which the latter coincidence fails.

Now, we say a space Y is a CL-space (continuous lattice space) if

(C1) Y is order complete;

(C2) join operations of arbitrary arity are continuous;

(C3) meet operations of arbitrary arity are box continuous.

Here, box continuity of an operation $f \colon Y^I \to Y$ refers to the box topology and means that for any $x \in Y^I$ and $V \in \mathcal{O}Y$ with $f(x) \in V$ there is a family $(U_i \colon i \in I) \in (\mathcal{O}Y)^I$ such that $x \in U = \prod_{i \in I} U_i$ and $f[U] \subseteq V$. Demanding continuity for all (infinitary) meet operations with respect to the product topology would be far too strong: this property holds only for singleton spaces! On the other hand, condition (C2) is fulfilled for all order complete spaces with the upper topology. Notice also that every CL-space is, by definition, a topological lattice in the specialization order.

A rather surprising connection between local supercompactness and property (C3) is established in the next proposition, generalizing Corollary 6 in [12].

Proposition 2.2. *An order complete space is locally supercompact (that is, a C-space) iff it has box continuous meet operations of arbitrary arity.*

P r o o f.   Assume first that Y is an order complete C-space, and consider the meet operation $\bigwedge_I \colon Y^I \to Y$ sending $x = (x_i \colon i \in I)$ to $\bigwedge\{x_i \colon i \in I\}$.

If $\bigwedge_I x = y \in V \in \mathcal{O}Y$, pick an element w with $y \in (\uparrow w)^{\circ} \subseteq \uparrow w \subseteq V$. Then $U = ((\uparrow w)^{\circ})^I$ is a box open set with $x \in U$ and $\bigwedge_I[U] \subseteq \uparrow w \subseteq V$, proving box continuity of $\bigwedge_I$.

Conversely, assume that meet operations of arbitrary arity exist in ΩY and are box continuous w.r.t. $\mathcal{O}Y$. Let $^-$ denote the closure operator of Y and $^{-b}$ the closure operator of the box product space. Using the Axiom of Choice, one easily verifies the identity

$$\prod_{i \in I} A_i^- = \left(\prod_{i \in I} A_i \right)^{-b}$$

for all families of subsets $A_i \subseteq Y$; if, in addition, each A_i is a lower set then so is A_i^-, and

$$\bigcap_{i \in I} A_i^- = \left\{ \bigwedge_I x \colon x \in \prod_{i \in I} A_i^- \right\} = \left\{ \bigwedge_I x \colon x \in \left(\prod_{i \in I} A_i \right)^{-b} \right\}$$

$$\subseteq \left\{ \bigwedge_I x \colon x \in \prod_{i \in I} A_i \right\}^- \subseteq \left(\bigcap_{i \in I} A_i \right)^- \subseteq \bigcap_{i \in I} A_i^-.$$

It follows that the closure operator of Y preserves arbitrary meets of lower sets, and this condition is equivalent to complete distributivity of the lattice of closed open sets, whence Y is a C-space (see [5], [7], [10]). $\qquad \square$

Summarizing the previous thoughts, we now arrive at new topological characterizations of continuous lattices, justifying the chosen nomenclature *a posteriori*:

Theorem 2.3. *The CL-spaces are exactly the Scott spaces ΣL of continuous lattices L.*

Among all C-spaces Y, they are those with the following equivalent properties:
(a) *$[Y^I, Y]$ is a complete lattice for all sets I.*
(b) *$[Y^I, Y]$ is a continuous lattice for all sets I.*
(c) *$[X, Y]$ is a complete lattice for all spaces X.*
(d) *$[X, Y]$ is a continuous lattice for all spaces X with continuous topologies $\mathcal{O}X$.*

P r o o f. By Theorem 2.2 and Proposition 2.2, CL-spaces are order complete C-d-spaces. By Theorem 1.1, these are precisely the Scott spaces of continuous lattices. And these in turn are CL-spaces satisfying (c) and (d), by Theorem 1.2, Proposition 2.2 and Corollary 2.1.

Let Y be a C-space. Then (a) implies that Y is a CL-space: $\Omega Y \simeq [Y^0, Y]$ must be a complete lattice; in other words, Y is order complete. Furthermore, by Theorem 2.2, completeness of $[Y^I, Y]$ implies that join operations of any arity are continuous, and by Corollary 2.1, this entails $Y = \Sigma L$ for a continuous lattice L, using the hypothesis that Y is a C-space, i.e., $\mathcal{O}Y$ is completely distributive (Theorem 1.1).

(b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) are trivial.

(d) $\Rightarrow$ (b): $\Omega Y \simeq [Y^0, Y]$ is a complete lattice. By Proposition 1.1, Y^I is an order complete C-space if Y is one, and by Theorem 1.1, every C-space has a (super)continuous topology. $\qquad\square$

For a fixed space X, it may happen very well that $[X, Y]$ is a complete lattice for *all* order complete spaces Y (and only for these). This is certainly the case whenever X is an Alexandroff discrete space, alias *A-space* (see e.g. [7], [8], [10]), meaning that all upper sets (that is, all intersections of open sets) are open in X; consequently, the isotone maps from ΩX to ΩY are just the continuous functions from X to Y; and arbitrary pointwise joins of isotone maps are isotone. Surprisingly, the completeness of $[X, Y]$ for all order complete spaces Y is characteristic for A-spaces X:

Theorem 2.4. *A topological space X is an A-space if and only if $[X, Y]$ is a complete lattice for all order complete (A-)spaces Y.*

P r o o f.   It remains to verify that a space X with the property that $[X, Y]$ is complete whenever Y is an order complete A-space must be an A-space, too. To that aim, denote the open neighborhood filter of x in X by $\mathcal{O}_x$, fix a point $z \in X$ and put $L = \{\bigcap \mathcal{U} : \mathcal{U} \subseteq \mathcal{O}_z\}$. Being closed under arbitrary intersections and ordered by dual inclusion, L becomes a complete lattice with the least element X and the greatest element $\uparrow z = \bigcap \mathcal{O}_z$. Hence, $Y = \mathrm{A}L$ is an order complete A-space. Now, define a map $f \colon X \to Y$ by

$$f(x) = {\uparrow}x \cup {\uparrow}z = \bigcap \mathcal{O}_x \cup \bigcap \mathcal{O}_z = \bigcap(\mathcal{O}_x \cap \mathcal{O}_z) \in L.$$

Further, for each $U \in \mathcal{O}_z$, define $f_U \colon X \to Y$ by

$$f_U(x) = U \cup {\uparrow}z = \bigcap\{V \in \mathcal{O}_z \colon U \subseteq V\} \text{ for } x \in U \text{ and } f_U(x) = X \text{ for } x \in X \setminus U.$$

Each of these functions is continuous, since Y is an A-space (i.e., the cores form an open base) and for any $y \in Y$, the preimage of the open core ${\uparrow}y$ is

$$f_U^{-1}[{\uparrow}y] = \{x \in U \colon U \cup {\uparrow}z \subseteq y\} \cup \{x \in X \setminus U \colon X \subseteq y\},$$

which is equal to X if $y = X$, to U if $U \cup {\uparrow}z \subseteq y$ but $y \neq X$, and to $\emptyset$ if $U \cup {\uparrow}z \not\subseteq y$. Moreover, the function f is the pointwise join (= intersection!) of the family $(f_U \colon U \in \mathcal{O}_z)$:

$$f(x) = f(x) \cup {\uparrow}z = \bigcap\{U \cup {\uparrow}z \colon U \in \mathcal{O}_x \cap \mathcal{O}_z\} = \bigcap\{f_U(x) \colon U \in \mathcal{O}_z\}.$$

Now, if $[X, Y]$ is complete then, by Theorem 2.1, the pointwise join f of the functions $f_U \in [X, Y]$ must be a member of $[X, Y]$, hence continuous. In particular, the preimage of the open singleton $\{\uparrow z\}$ under f has to be open, and this preimage is just the core $\uparrow z$, because $x \in \uparrow z$ is tantamount to $f(x) = \uparrow x \cup \uparrow z = \uparrow z$.

As z was chosen arbitrarily in X, we see that all cores in X are open; in other words, X is an A-space. $\qquad\square$

3. Complete distributivity of $[X, Y]$

This section deals with results concerning supercontinuity (= complete distributivity) instead of continuity. The considerations in the previous section together with some additional arguments lead to a strengthened analogue of Theorem 1.2:

Theorem 3.1. *Let X be a nonempty topological space and Y a nonsingleton space such that $[X, Y]$ is a complete lattice. Then $[X, Y]$ is completely distributive iff so are the complete lattices $\mathcal{O}X$ and ΩY.*

P r o o f. Although a more general result will be established in Theorem 4.1, we give an ad hoc proof for the reader's convenience.

Suppose first that $[X, Y]$ is a cdl. Then $\mathcal{O}X$ and ΩY are cdl's, too, by Proposition 2.1. For the converse implication, suppose that $\mathcal{O}X$ and $L = \Omega Y$ are completely distributive lattices. Given $g \not\leqslant h$ in $[X, Y]$, we have to find an $f \in [X, Y]$ with $f \lll g$ and $f \not\leqslant h$. Choose $u \in X$ with $g(u) \not\leqslant h(u)$ and then a $y \in L$ with $y \lll g(u)$ but $y \not\leqslant h(u)$. As mentioned earlier, the set $y \lll$ is open (being complementary to a principal ideal, that is, to a point closure). By continuity of g, we have $u \in g^{-1}[y \lll] \in \mathcal{O}X$, and since $\mathcal{O}X$ is supercontinuous (i.e., X is a C-space), we find $w \in g^{-1}[y \lll]$ with $u \in U = (\uparrow w)^\circ$.

Now define $f \colon X \to Y$ by $f(x) = y$ if $x \in U$ and $f(x) = 0$ otherwise. Then the inverse image of any upper set in L under f is $\emptyset$, U or X. In particular, f is continuous. Clearly, $f \not\leqslant h$ since $f(u) = y \not\leqslant h(u)$. In order to prove $f \lll g$, consider any subset Z of $[X, Y]$ with $g \leqslant \bigvee Z$. By the pointwise formation of suprema in $[X, Y]$ (see Theorem 2.1), we have the inequality $y \lll g(w) \leqslant \bigvee\{k(w) \colon k \in Z\}$, hence $y \leqslant k(w)$ for some $k \in Z$. But then

$$f(x) = y \leqslant k(w) \leqslant k(x) \quad \text{for } x \in U \quad \text{and} \quad f(x) = 0 \leqslant k(x) \quad \text{for } x \in X \setminus U$$

(notice that $x \in U = (\uparrow w)^\circ$ entails $w \leqslant x$ and that the continuous map k preserves the specialization order). Thus, $f \leqslant k$, as desired. $\qquad\square$

A rather simple instance of Theorem 3.1 is this: for nonempty A-spaces X and any space Y, the ordered set ΩY is a cdl iff $[X, Y]$ is such, because the latter consists of *all* isotone maps from ΩX to ΩY, and these form a complete sublattice of $(\Omega Y)^{|X|}$. But the completeness hypothesis on $[X, Y]$ cannot be omitted in Theorem 3.1, as the next example shows.

E x a m p l e 3.1. Let X be the complete chain $\omega + 1 = \omega \cup \{\omega\}$ equipped with the Scott topology, and let Y be the same chain equipped with the Alexandroff topology of upper sets. Then X and Y are C-spaces, and both $\mathcal{O}X$ and ΩY are completely distributive lattices. But $[X, Y]$ is not even complete: the continuous functions $f_n \colon X \to Y$ with $f_n(x) = \min\{x, n\}$ ($n \in \omega$) have a pointwise supremum, viz the identity map from X to Y; however, this function is not continuous, because $\mathcal{O}X$ is properly contained in $\mathcal{O}Y$. Thus, $[X, Y]$ is not closed under pointwise suprema, and by Theorem 2.1, it cannot be a complete lattice.

Though the completeness hypothesis in Theorem 3.1 is essential, it may be replaced by several other conditions on Y alone, as listed in Theorem 2.2 and Corollary 2.1. We have seen that completeness of $[X, Y]$ for all spaces X forces Y to be an order complete d-space. The hypothesis of this conclusion may be slightly weakened as follows:

Lemma 3.1. *If Y is a topological space such that $[X, Y]$ is a complete lattice for each C-space X then Y is an order complete d-space.*

P r o o f.   For the A-space $\mathrm{A}\Omega Y$, whose topology consists of all upper sets, any power $X = (\mathrm{A}\Omega Y)^I$ is a C-space (but not an A-space unless I is finite). In fact, every A-space is a C-space, and a product of C-spaces having least elements in the specialization order is a C-space (see Proposition 1.1). Consequently, $[X, Y]$ is complete by hypothesis and closed under pointwise suprema by Theorem 2.1. The projections from X to Y are continuous; indeed, they are even continuous as maps from Y^I (which has a weaker topology than X) to Y. Hence, their pointwise supremum, the join map $\bigvee_I \colon X \to Y$, is continuous, too. Now, the same argument as for (a) $\Rightarrow$ (b) in Theorem 2.2 shows that Y must be a d-space. $\square$

We are now ready to characterize completely distributive complete lattices equipped with the Scott topology by distributivity properties of the associated function lattices.

Theorem 3.2. *For a topological space Y, the following conditions are equivalent:*
(a) *Y is the Scott space ΣL (or the weak space ΥL) of a cdl L.*
(b) *ΩY is a cdl, and Y is a d-space.*

(c) $[Y,Y]$ *is a cdl, and Y is a nonempty d-space.*

(d) $[Y^F,Y]$ *is a cdl for all finite sets F, and Y is a d-space.*

(e) $[Y^I,Y]$ *is a cdl for all sets I.*

(f) $[X,Y]$ *is a cdl for all C-spaces X (and only for these).*

In (b), (c) and (d), 'd-space' may be replaced by 'sober space'.

P r o o f.   (a) $\Rightarrow$ (f): $[X,Y]$ is complete by Corollary 2.1; and by Theorem 3.1, $[X,Y]$ is completely distributive iff so are $\mathcal{O}X$ and $L = \Omega Y$.

(f) $\Rightarrow$ (b): ΩY is a cdl by Theorem 3.1, and Lemma 3.1 gives the d-space property.

(b) $\Rightarrow$ (a): See the proof of (j) $\Rightarrow$ (f) in Corollary 2.1.

(a) $\Rightarrow$ (e): Since Y is an order complete C-space by Theorem 1.1, so is Y^I by Proposition 1.1, and as we know from the proved implication (a) $\Rightarrow$ (f), $[Y^I,Y]$ is then a cdl.

(e) $\Rightarrow$ (d): See Theorem 2.2 for the d-space property.

(d) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (b): See Proposition 2.1.

In all, we have closed the implication circuits

(a) $\Rightarrow$ (f) $\Rightarrow$ (b) $\Rightarrow$ (a) and (a) $\Rightarrow$ (e) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a). The last statement in Theorem 3.2 follows from the known fact that ΣL is sober for continuous lattices L, and that sober spaces are always d-spaces. $\square$

Corollary 3.1. *Let Y be a nonempty T_0-space whose topology is the Alexandroff topology $\alpha\Omega Y$ or any topology weaker than the Scott topology $\sigma\Omega Y$ (like the upper topology $\upsilon\Omega Y$). Then $[Y,Y]$ is a cdl iff both $\mathcal{O}Y$ and ΩY are cdl's.*

In case ΩY is complete and $\mathcal{O}Y \subseteq \sigma\Omega Y$, we have a d-space Y, and Theorem 3.2 applies. The argument for $\alpha\Omega Y$ is different: here $[Y,Y]$ is a complete sublattice of the cdl $(\Omega Y)^{|Y|}$.

Another consequence is obtained for nonempty *finite* spaces Y such that $L = \Omega Y$ is a distributive lattice (hence a cdl): here, $Y = \Sigma L = AL$ and each power of Y is a C-space; moreover, $[X,Y]$ is a cdl for all C-spaces X (and only for these).

We have mentioned earlier that in any space Y with underlying cdl $L = \Omega Y$, each of the sets $x \lll$ is open, being the complement of a point closure. Moreover, for $y \in V \in \mathcal{O}Y$, we have $\bigvee\{x \colon x \lll y\} = y \in V \in \sigma\Omega Y$, hence $\bigvee F \in V$ for a finite set $F \subseteq \{x \colon x \lll y\}$, and it follows that $y \in \bigcap\{x \lll \colon x \in F\} \subseteq \{\uparrow x \colon x \lll y\} = \uparrow\bigvee F \subseteq V$, showing that Y is a C-space with subbasic open sets $x \lll$. But, in contrast to the sets $x \ll$ in continuous lattices, the sets $x \lll$ rarely form a *base*!

E x a m p l e 3.2. The real unit square $[0,1]^2$ is a completely distributive lattice L in which the sets $x \lll$ form a subbase but not a base for $\sigma L = \upsilon L$. Indeed, in the

present example, $(x_1, x_2) \lll (y_1, y_2)$ means

$$(x_1 = 0 \text{ and } x_2 < y_2) \quad \text{or} \quad (x_1 < y_1 \text{ and } x_2 = 0)$$
$$\text{or} \quad (x_1 = x_2 = y_1 = y_2 = 0).$$

Another prominent difference between continuous and supercontinuous lattices is that for a d-space Y with supercontinuous underlying lattice ΩY, the function poset $[Y, Y]$ is a supercontinuous complete lattice, too (see Theorem 3.2), whereas for d-spaces with continuous lattices $\mathcal{O}Y$ and ΩY, the function poset $[Y, Y]$ may be complete but not continuous.

E x a m p l e 3.3. Let L be any infinite lattice of height 2. Then L is an algebraic, hence continuous lattice; but L is not distributive, all the less supercontinuous. Equipped with the upper topology, this gives a sober space (hence a d-space) $Y = \Upsilon L$ in which arbitrary join operations are continuous (because preimages of principal ideals are principal ideals). The closed subsets are the finitely generated lower sets. Hence, the upper topology is here strictly weaker than the Scott topology, which coincides with the Alexandroff topology. By Theorem 2.2, $[X, Y]$ is a complete lattice for arbitrary spaces X. But, surprisingly, the complete lattice $[Y, Y]$ is *not* continuous! This can be seen as follows.

Consider a $g \in [Y, Y]$ with $g(0) = 0$; for the sake of simplicity, we assume that the top element 1 of L does not belong to the range of g (continuous maps g on Y with these properties exist in abundance). We claim that the only function $f \in [Y, Y]$ with $f \ll g$ is the zero function. Assume $f \in [Y, Y]$ is not the zero function and observe that $F = f^{-1}[\{0\}]$ must be finite by continuity. Choose a function $h \in [Y, Y]$ with $h(0) = 0$, $h(1) = 1$ and $f(a) \neq h(a) \neq 1$ for all $a \in A \setminus F$, where $A = L \setminus \{0, 1\}$ is the set of all atoms (such a function may be constructed from f by a suitable fixpoint-free permutation of $A \setminus F$). For each finite $E \subseteq A$, define $h_E \colon Y \to Y$ by $h_E(x) = 1$ for $x \in E$ and $h_E(y) = h(y)$ for $y \in L \setminus E$. Then, in particular, $h_E(0) = h(0) = g(0) = f(0) = 0$. For continuity of h_E, note that if C is closed but does not contain 1 then C must be finite, and $1 \notin h_E^{-1}[C] = h^{-1}[C]$, so that this preimage is a finite lower set and therefore closed. Thus, $h_E \in [Y, Y]$.

The pointwise (directed!) supremum of the functions h_E has the constant value 1 for $x \neq 0$. Hence, $g \leqslant \bigvee \{h_E \colon E \subseteq A, E \text{ finite}\}$, but no h_E can satisfy $f \leqslant h_E$, since for $a \in A \setminus (E \cup F)$ one obtains $0 < f(a) \neq h(a) = h_E(a) < 1$, which excludes $f(a) \leqslant h_E(a)$. Thus, f cannot be way below g, as claimed.

On the other hand, a variant of Example 3.1 will show that there are spaces Y for which $\mathcal{O}Y$ and ΩY are algebraic completely distributive lattices, and still the function poset $[Y, Y]$ is not even a complete lattice.

E x a m p l e 3.4. As in [7] and [10], we mean by a *B-space* a topological space with a minimal base (which then consists of all open cores). Every A-space is a B-space, and every B-space is a C-space, but not conversely. It is known and easy to see that the B-spaces are exactly those spaces whose lattice of open or closed sets is *superalgebraic*, i.e. supercontinuous and algebraic. Simple examples of B-spaces are the Scott spaces $\Sigma\kappa$ of ordinals κ, whereas $\Sigma\mathbb{R}$ and $\Sigma[0,1]$ are C-spaces but not B-spaces.

Specifically, let us consider the complete ordinal $L = \omega^2 + 1 = \omega^2 \cup \{\omega^2\}$ and the modified Scott space Y with $\Omega Y = L$ and $\mathcal{O}Y = \sigma L \cup \{\{\omega^2\}\}$. Then Y is neither a Scott space nor an Alexandroff space, but it is obviously a B-space. In fact, both $\mathcal{O}Y$ and ΩY are superalgebraic lattices, being algebraic complete chains.

Now, define the function $g\colon Y \to Y$ by $g(x) = x$ for $x < \omega$ and $g(y) = \omega^2$ for $x \geqslant \omega$. As $\{\omega^2\}$ is open while its preimage $g^{-1}[\{\omega^2\}] = L \setminus \omega$ is not, this g cannot be continuous. But g is the directed supremum of the functions $f_n\colon Y \to Y$ with $f_n(x) = x$ for $x \in \omega \cup \{\omega^2\}$ and $f_n(y) = \omega \cdot n + y$ otherwise ($n \in \omega$). Here $\omega \cdot n + y$ denotes the ordinal sum $\omega + \ldots + \omega + y$, putting the ordinal y above the ordinal sum of the n ω-chains. Each f_n is continuous, since preimages of open cores $\uparrow x$ (where x is a successor ordinal or $x = \omega^2$) are again open cores. This shows that $[Y, Y]$ is not even closed under suprema of ω-chains, hence certainly not complete, by Theorem 2.1.

Another subtle detail has to be pointed out. From Theorem 2.2 we know that if $[Y^I, Y]$ is a complete lattice for all sets I then Y must be an order complete d-space. But it does not suffice to require completeness or even supercontinuity of $[Y^F, Y]$ for all *finite F*.

E x a m p l e 3.5. Let Y be the Alexandroff space $\mathrm{A}L$ of some continuous lattice L (e.g. the real unit interval $[0, 1]$). For finite F, the power Y^F is an A-space, too, and consequently $[Y^F, Y]$ consists of all isotone maps from L^F to L. These form a complete sublattice of the continuous lattice L^I, where I is the underlying set of L^F, and therefore $[Y^F, Y]$ is again a continuous lattice. But Y fails to be a d-space unless L is noetherian (all ideals are principal), which is necessary and sufficient for the coincidence of αL with σL.

4. $\mathcal{Z}$-DISTRIBUTIVITY AND $\mathcal{Z}$-CONTINUITY OF $[X, Y]$

In this final section, we are going to establish a common generalization of Theorems 1.2 and 3.1, and also one of Theorems 2.3 and 3.2. For this purpose, we need the notion of a subset system in the sense of Wright, Wagner and Thatcher [26].

A *subset system* is a function $\mathcal{Z}$ on the class of all posets assigning to each poset P a collection $\mathcal{Z}P$ of subsets such that for any isotone function $f\colon P \to Q$ and for each $Z \in \mathcal{Z}P$ the image $f[Z]$ belongs to $\mathcal{Z}Q$ and at least one $\mathcal{Z}P_0$ contains a nonempty member, which entails that *each* $\mathcal{Z}P$ contains all singletons. The collection of all $\mathcal{Z}$-*ideals*, i.e., sets of the form $\downarrow Z$ with $Z \in \mathcal{Z}P$, is denoted by $\mathcal{Z}^\wedge P$. A subset system $\mathcal{Z}$ is called *union complete* if $\mathcal{Y} \in \mathcal{Z}^\wedge \mathcal{Z}^\wedge P$ implies $\bigcup \mathcal{Y} \in \mathcal{Z}^\wedge P$. Simple examples of union complete subset systems are $\mathcal{A}$ (arbitrary subsets), $\mathcal{D}$ (directed subsets), $\mathcal{E}$ (one-element subsets), and $\mathcal{F}$ (finite subsets), whereas the subset systems $\mathcal{B}$ (binary subsets, having at most two elements) and $\mathcal{C}$ (chains, i.e. linearly ordered subsets) are *not* union complete. By sending any isotone map $f\colon P \to Q$ to the map $\mathcal{Z}^\wedge f\colon \mathcal{Z}^\wedge P \to \mathcal{Z}^\wedge Q$ with $\mathcal{Z}^\wedge f(Z) = \downarrow f[Z]$, one may regard $\mathcal{Z}^\wedge$ as an endofunctor of the category of posets and isotone maps. Moreover, if $\mathcal{Z}$ is union complete, $\mathcal{Z}^\wedge$ gives rise to a standard construction or monad (see Erné [6] and Meseguer [20]).

In [1], the notion of continuous posets has been generalized to the setting of arbitrary subset systems $\mathcal{Z}$ as follows: a poset P is $\mathcal{Z}$-*complete* if each $Z \in \mathcal{Z}P$ has a join in P. The $\mathcal{Z}$-*below relation* is then defined by

$$x \ll_{\mathcal{Z}} y \Leftrightarrow x \in \ll_{\mathcal{Z}} y = \bigcap \{Z \in \mathcal{Z}^\wedge P\colon y \leqslant \bigvee Z\}.$$

Now, a $\mathcal{Z}$-complete poset P is called $\mathcal{Z}$-*distributive* if $y = \bigvee \ll_{\mathcal{Z}} y$ for all $y \in P$ (see [9]), and P is called $\mathcal{Z}$-*continuous* if, in addition, $\ll_{\mathcal{Z}} y \in \mathcal{Z}^\wedge P$ for all $y \in P$. If, moreover, the $\mathcal{Z}$-below relation is idempotent ('interpolative'), one speaks of a *strongly $\mathcal{Z}$-distributive* or *strongly $\mathcal{Z}$-continuous* poset, respectively (cf. Baranga [3], Venugopalan [25]; see also Erné [9] for results on more general subset selections $\mathcal{Z}$, and Novak [21] for a different approach).

A straightforward verification shows that a complete lattice L is $\mathcal{Z}$-distributive iff it satisfies the following distribution law for all $\mathcal{X} \subseteq \mathcal{Z}^\wedge L$:

$$\bigwedge \{\bigvee Z\colon Z \in \mathcal{X}\} = \bigvee \bigcap \mathcal{X}.$$

From this it easily follows that if a subset S of a $\mathcal{Z}$-distributive complete lattice is closed under $\mathcal{Z}$-joins and under arbitrary meets then S is $\mathcal{Z}$-distributive, too. More generally, the following facts have been established in [1] (see also [11] and [25] for related results on idempotent isotone operators):

Lemma 4.1. *Let Q be the range of a $\mathcal{Z}$-join preserving closure operation on a poset P. If P is a $\mathcal{Z}$-distributive, $\mathcal{Z}$-continuous or strongly $\mathcal{Z}$-continuous poset then Q has the same property. This applies to $\mathcal{Z}$-join- and meet-closed subsets of complete lattices, because they are the ranges of $\mathcal{Z}$-join preserving closure operations.*

But, surprisingly, an analogous statement for strong $\mathcal{Z}$-distributivity fails.

E x a m p l e 4.1. Powerset lattices $\mathcal{P}X$ are always strongly $\mathcal{B}$-distributive and strongly $\mathcal{F}$-distributive but not $\mathcal{B}$-continuous for $|X| > 2$; nevertheless, their $\mathcal{B}$-below relation coincides with the $\mathcal{F}$-below relation and is idempotent: $Y \ll_{\mathcal{B}} Z \Leftrightarrow Y \ll_{\mathcal{F}} Z \Leftrightarrow Y \subseteq Z$ and $|Y| \leqslant 1$. For an 8-element Boolean lattice $L \simeq \mathcal{P}3$ (where $3 = \{0, 1, 2\}$), the system $\mathcal{B}^{\wedge}L$ is not a complete lattice, being isomorphic to the poset obtained by omitting the median element

$$v = (x \vee y) \wedge (y \vee z) \wedge (z \vee y) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge y)$$

in the free distributive lattice FD_3 with three generators x, y, z, and adding a new bottom element. For the top element $\top = x \vee y \vee z$ of FD_3, it is easy to see that $v \ll_{\mathcal{B}} \top$ in FD_3, but no $w \in FD_3$ satisfies $v \ll_{\mathcal{B}} w \ll_{\mathcal{B}} \top$. Thus, FD_3 is $\mathcal{B}$-distributive but neither $\mathcal{B}$-continuous nor strongly $\mathcal{B}$-distributive, though being isomorphic to $\mathcal{A}^{\wedge}L \setminus \{\emptyset, L\}$, a complete sublattice of the strongly $\mathcal{B}$-distributive lattice $\{A \subseteq L \setminus \{\bigvee L\} \colon \bigwedge L \in A\} \simeq \mathcal{P}6$. Similar examples show that a strongly $\mathcal{F}$- or $\mathcal{A}_{m+1}$-distributive lattice need not be strongly $\mathcal{A}_m$-distributive.

$L \simeq \mathcal{P}3$ FD_3 $\mathcal{B}^{\wedge}L$
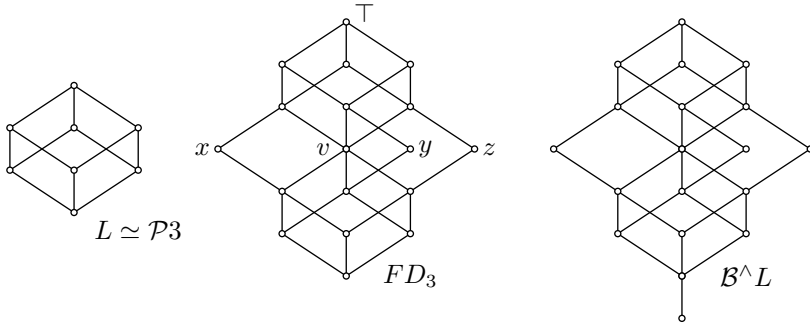
It is known and easy to see that for union complete $\mathcal{Z}$, every $\mathcal{Z}$-continuous poset is already strongly $\mathcal{Z}$-continuous; on the other hand, if $\mathcal{Z}^{\wedge}P$ is a complete lattice (equivalently, a closure system) then $\mathcal{Z}$-distributivity and $\mathcal{Z}$-continuity are equivalent properties for P (see [9]). In particular, $\mathcal{D}$-distributive complete lattices are already strongly $\mathcal{D}$-continuous; in fact, they are just the continuous lattices, while the (strongly) $\mathcal{A}$-continuous ($= \mathcal{A}$-distributive) posets are precisely the supercontinuous ($=$ completely distributive) lattices. However, for the union complete system $\mathcal{F}$ of finite subsets, an $\mathcal{F}$-distributive complete lattice need not be strongly $\mathcal{F}$-distributive: all strongly $\mathcal{F}$-distributive complete lattices are *cospatial*, i.e. *isomorphic* copies of closed set lattices of topological spaces, whereas the $\mathcal{F}$-distributive complete lattices are exactly the complete *homomorphic* images of such lattices (see [2], [9]); a dualized example of an $\mathcal{F}$-distributive but not cospatial complete lattice has been given

by Kříž and Pultr [19]. Clearly, all cospatial lattices are $\mathcal{F}$-distributive; but there are cospatial lattices that fail to be *strongly $\mathcal{F}$-distributive*:

E x a m p l e 4.2. For any infinite set X and any two-element subset Z of X, the set $\mathcal{L} = \{Y \subseteq X \colon Z \subseteq Y\} \cup \mathcal{F}X$ is the lattice of closed sets of a topological T_1-space, hence a cospatial lattice and therefore $\mathcal{F}$-distributive. However, the $\mathcal{F}$-below relation $\ll_{\mathcal{F}}$ on $\mathcal{L}$ is not idempotent, since $Z \ll_{\mathcal{F}} X$ holds but there is no $Y \in \mathcal{L}$ with $Z \ll_{\mathcal{F}} Y \ll_{\mathcal{F}} X$.

Below, we shall demonstrate that union completeness of $\mathcal{Z}$ is not necessary for the coincidence of $\mathcal{Z}$-continuity and strong $\mathcal{Z}$-continuity. In fact, sometimes one can show that $\mathcal{Z}$-continuous posets are even *$\mathcal{Z}$-algebraic*, that is, each element is the join of a set $Z \in \mathcal{Z}P$ consisting of elements $x \ll_{\mathcal{Z}} x$, referred to as *$\mathcal{Z}$-prime* or *$\mathcal{Z}$-compact* elements (cf. [8]). Clearly, all $\mathcal{Z}$-algebraic posets are strongly $\mathcal{Z}$-continuous. Notice that every $\mathcal{B}$- or $\mathcal{D}$-distributive complete Boolean algebra is already $\mathcal{A}$-algebraic (hence $\mathcal{A}$-continuous), but only *finite* Boolean algebras are $\mathcal{F}$-algebraic and $\mathcal{F}$-continuous. Hence, strongly $\mathcal{F}$-distributive complete lattices need not be $\mathcal{F}$-continuous; indeed, $\mathcal{F}$-continuity is equivalent to $\mathcal{F}$-algebraicity, meaning that each element is the join of *finitely many $\vee$-primes*.

Lemma 4.2. *Let P be a $\mathcal{Z}$-complete poset with $\mathcal{B}^{\wedge}P \subseteq \mathcal{Z}^{\wedge}P$. If $x \ll_{\mathcal{Z}} y$ and $u \vee x = y$ for some $u < y$ in P then $x \ll_{\mathcal{Z}} x$.*

P r o o f. Consider a $Z \in \mathcal{Z}P$ with $x \leqslant \bigvee Z$. We then have $y = u \vee x \leqslant \bigvee\{u \vee z \colon z \in Z\}$ and $\{u \vee z \colon z \in Z\} \in \mathcal{Z}P$ (because $z \mapsto u \vee z$ is isotone). Hence, $x \ll_{\mathcal{Z}} y$ entails $x \leqslant u \vee z$ for some $z \in Z$. But then $y = u \vee x \leqslant u \vee z$ and $x \not\leqslant u$ (since $u < y = u \vee x$). And now, $x \ll_{\mathcal{Z}} y$ together with $\downarrow \{u, z\} \in \mathcal{B}^{\wedge}P \subseteq \mathcal{Z}^{\wedge}P$ implies $x \leqslant z$, proving $x \ll_{\mathcal{Z}} x$. $\qquad\square$

For any cardinal m, let $\mathcal{A}_m P$ denote the collection of all subsets having cardinality less than m. As remarked in [6] and [8], the subset system $\mathcal{A}_m$ is union complete if and only if m is a regular cardinal. Specifically, $\mathcal{A}_\omega = \mathcal{F}$ is union complete, while $\mathcal{A}_3 = \mathcal{B}$ is not.
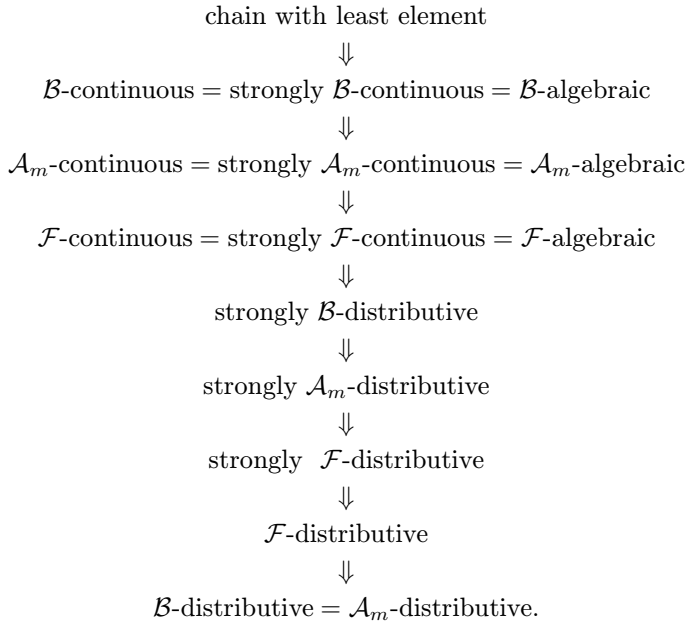
Corollary 4.1. *For any cardinal m with $2 < m \leqslant \omega$ and any poset P, the following conditions are equivalent:*

(a) *P is $\mathcal{A}_m$-continuous.*
(b) *P is strongly $\mathcal{A}_m$-continuous.*
(c) *P is $\mathcal{A}_m$-algebraic, i.e., it is a $\vee$-semilattice with 0 in which each element is a join of less than m $\vee$-primes.*

P r o o f. (a) $\Rightarrow$ (c) follows from Lemma 4.2: taking $Z \in \mathcal{A}_m P$ minimal with $\ll_{\mathcal{A}_m} y = \downarrow Z$, one obtains $u < y = u \vee x$ for $x \in Z$ and $u = \bigvee(Z \setminus \{x\})$. Hence, each $x \in Z$ is $\mathcal{A}_m$-prime.

(c) $\Rightarrow$ (b) $\Rightarrow$ (a) is obvious. $\qquad\square$

Note the following implications for $3 < m < \omega$:

$$\text{chain with least element}$$
$$\Downarrow$$
$$\mathcal{B}\text{-continuous} = \text{strongly } \mathcal{B}\text{-continuous} = \mathcal{B}\text{-algebraic}$$
$$\Downarrow$$
$$\mathcal{A}_m\text{-continuous} = \text{strongly } \mathcal{A}_m\text{-continuous} = \mathcal{A}_m\text{-algebraic}$$
$$\Downarrow$$
$$\mathcal{F}\text{-continuous} = \text{strongly } \mathcal{F}\text{-continuous} = \mathcal{F}\text{-algebraic}$$
$$\Downarrow$$
$$\text{strongly } \mathcal{B}\text{-distributive}$$
$$\Downarrow$$
$$\text{strongly } \mathcal{A}_m\text{-distributive}$$
$$\Downarrow$$
$$\text{strongly } \mathcal{F}\text{-distributive}$$
$$\Downarrow$$
$$\mathcal{F}\text{-distributive}$$
$$\Downarrow$$
$$\mathcal{B}\text{-distributive} = \mathcal{A}_m\text{-distributive}.$$

The non-invertibility of the first four implications may be checked by looking at powerset lattices of sets with 2, $m - 1$, m or ω elements, respectively. Induction shows that $\mathcal{B}$-distributive posets are $\mathcal{A}_m$-distributive for all $m < \omega$, but are they always $\mathcal{F}$-distributive?

By a $\mathcal{Z}^\vee$-*ideal* of a poset P we mean a $\mathcal{Z}$-join closed lower set $A \subseteq P$ (that is, if $Z \in \mathcal{Z}P$ is contained in A and has a join $\bigvee Z$ then $\downarrow \bigvee Z \subseteq A$; cf. [6], [8], [9]). For example, the usual ideals of a lattice are the $\mathcal{F}^\vee$-ideals, whereas the principal ideals of a complete lattice are its $\mathcal{A}^\vee$-ideals, and the Scott-closed subsets of a poset are its $\mathcal{D}^\vee$-ideals. The $\mathcal{Z}^\vee$-ideals of any poset P form a closure system $\mathcal{Z}^\vee P$, hence a complete lattice. Accordingly, the complements of the $\mathcal{Z}^\vee$-ideals of a poset P form a kernel system $\sigma_{\mathcal{Z}} P$ (closed under arbitrary unions), sometimes referred to as a *generalized Scott topology* (cf. Baranga [3], [4]). Note the inclusions

$$\sigma_{\mathcal{A}} P \subseteq \sigma_{\mathcal{D}} P = \sigma P \subseteq \sigma_{\mathcal{E}} P = \alpha P.$$

Observe that $\sigma_{\mathcal{Z}} P$ need not be a topology unless $\mathcal{Z}^{\wedge} P$ is contained in $\mathcal{D}^{\wedge} P$. Therefore, in the sequel, it will be reasonable to extend the considerations from topological spaces to arbitrary closure spaces. Notions like *open, closed, neighborhood, continuous function* and *specialization order* remain meaningful in that general context (see [11] for an extensive survey of these matters). A quick inspection shows that, for example, Proposition 2.1 and Theorem 2.1 keep their validity in the extended realm of closure spaces.

Generalizing the familiar notion of finitary closure systems, we say a (set-theoretical) closure system $\mathcal{X}$ on a poset P is $\mathcal{Z}$-*ary* if for each element x in the closure $\bigcap\{C \in \mathcal{X}\colon A \subseteq C\}$ of a lower set A there is a $Z \in \mathcal{Z} P$ with $Z \subseteq A$ whose closure contains x. Specifically, '$\mathcal{F}$-ary' means 'finitary'. For the next two propositions, see [2] and [9].

Proposition 4.1. *A $\mathcal{Z}$-complete poset P is strongly $\mathcal{Z}$-continuous iff the closure system $\mathcal{Z}^{\vee} P$ of $\mathcal{Z}^{\vee}$-ideals is $\mathcal{Z}$-ary and completely distributive.*

Proposition 4.2. *If $\mathcal{Z}$ is union complete and L as well as $\mathcal{Z}^{\wedge} L$ are complete lattices, then the following statements are all equivalent:*
(a) *L is $\mathcal{Z}$-distributive.*
(b) *L is strongly $\mathcal{Z}$-distributive.*
(c) *L is $\mathcal{Z}$-continuous.*
(d) *L is strongly $\mathcal{Z}$-continuous.*
(e) *The lattice $\mathcal{Z}^{\vee} L$ of all $\mathcal{Z}^{\vee}$-ideals is a cdl.*
(f) *L is a meet- and $\mathcal{Z}$-join closed subposet of a cdl.*

Let us denote by $\Sigma_{\mathcal{Z}} P$ the kernel space $(|P|, \sigma_{\mathcal{Z}} P)$ or, alternatively, the closure space $(|P|, \mathcal{Z}^{\vee} P)$, by $\tau_{\mathcal{Z}} P$ the topology generated by $\sigma_{\mathcal{Z}} P$, and by $\mathrm{T}_{\mathcal{Z}} P$ the topological space $(|P|, \tau_{\mathcal{Z}} P)$. A topological space or closure space Y is said to be $\mathcal{Z}$-*fine* if each $\mathcal{Z}^{\vee}$-ideal of ΩY is closed, that is, if $\sigma_{\mathcal{Z}} \Omega Y$ is contained in $\mathcal{O} Y$, the kernel system of all open sets. By definition, the spaces $\Sigma_{\mathcal{Z}} P$ and $\mathrm{T}_{\mathcal{Z}} P$ are always $\mathcal{Z}$-fine. In particular, all Scott spaces $\Sigma P = \Sigma_{\mathcal{D}} P = \mathrm{T}_{\mathcal{D}} P$ are $\mathcal{D}$-fine. Examples of $\mathcal{Z}$-fine C-spaces are all A-spaces $\mathrm{A} P = \Sigma_{\mathcal{A}} P = \mathrm{T}_{\mathcal{A}} P$, but also the spaces $\mathrm{T}_{\mathcal{Z}} P$ of strongly $\mathcal{Z}$-continuous $\vee$-semilattices P:

Lemma 4.3. *If P is a strongly $\mathcal{Z}$-continuous $\vee$-semilattice then $\mathrm{T}_{\mathcal{Z}} P$ is a C-space.*

P r o o f.   By Proposition 4.1, the complete lattice $\mathcal{Z}^{\vee} P$ and its dual $\sigma_{\mathcal{Z}} P$ are completely distributive, and the obvious generalization of Theorem 1.1 from topological spaces to closure spaces (see e.g. [9]) ensures that for $y \in V \in \sigma_{\mathcal{Z}} P$ there

are $x \in V$ and a $U \in \sigma_{\mathcal{Z}} P$ with $y \in U \subseteq \uparrow x$. Since $\tau_{\mathcal{Z}} P$ is the topology gener-
ated by $\sigma_{\mathcal{Z}} P$, we conclude that for $y \in W \in \tau_{\mathcal{Z}} P$ there are $V_1, \ldots, V_n \in \sigma_{\mathcal{Z}} P$ with
$y \in V_1 \cap \ldots \cap V_n \subseteq W$, and then we find $x_i \in V_i$ and $U_i \in \sigma_{\mathcal{Z}} P$ with $y \in U_1 \cap \ldots \cap U_n \subseteq$
$\uparrow x_1 \cap \ldots \cap \uparrow x_n = \uparrow z$ for $z = x_1 \vee \ldots \vee x_n$. Thus, $U = U_1 \cap \ldots \cap U_n \in \tau_{\mathcal{Z}} P$ satisfies
$y \in U \subseteq \uparrow z \subseteq W$, and $\mathrm{T}_{\mathcal{Z}} P$ is a C-space. $\qquad\square$

Now, to the common generalization of Theorems 1.2 and 3.1:

Theorem 4.1. *Let X, Y be nonempty closure spaces such that $[X, Y]$, the set
of all continuous functions from X to Y with the pointwise specialization order, is
a complete lattice.*

(1) *If Y is not a singleton and $[X, Y]$ is $\mathcal{Z}$-distributive or (strongly) $\mathcal{Z}$-continuous,
then $\mathcal{O}X$ and ΩY have the corresponding property.*

(2) *If $\mathcal{O}X$ is $\mathcal{Z}$-distributive, Y is $\mathcal{Z}$-fine and ΩY is strongly $\mathcal{Z}$-distributive then
$[X, Y]$ is $\mathcal{Z}$-distributive.*

P r o o f. ΩY is the underlying set of Y, ordered by $x \leqslant y$ iff x belongs to the
closure of $\{y\}$.

As in Proposition 2.1, one observes that $L = \Omega Y$ is a complete lattice if $[X, Y]$ is
such.

(1) Referring to Lemma 4.1 and the closure version of Proposition 2.1, one checks
that (no matter how meets are formed in $[X, Y]$) the complete sublattices $[X, S]$
and $c[X, Y]$ as well as their isomorphic copies, $\mathcal{O}X$ and L, must be $\mathcal{Z}$-distributive.
Similarly, these complete sublattices are (strongly) $\mathcal{Z}$-continuous whenever $[X, Y]$
has the respective property.

(2) Under the given hypotheses, we have to construct, for $g \not\leqslant h$ in $[X, Y]$, a func-
tion $f \in [X, Y]$ with $f \ll_{\mathcal{Z}} g$ but $f \not\leqslant h$. Choose $u \in X$ with $g(u) \not\leqslant h(u)$ and an
element $v \in L$ with $v \ll_{\mathcal{Z}} g(u)$ but $v \not\leqslant h(u)$. Then $V = \{z \in L \colon v \ll_{\mathcal{Z}} z\}$ is the
complement of a $\mathcal{Z}^{\vee}$-ideal: clearly, V is an upper set; for $Z \in \mathcal{Z}L$ with $v \ll_{\mathcal{Z}} \bigvee Z$, the
interpolation property of $\ll_{\mathcal{Z}}$ in $L = \Omega Y$ yields a $w \in L$ such that $v \ll_{\mathcal{Z}} w \ll_{\mathcal{Z}} \bigvee Z$,
whence $v \ll_{\mathcal{Z}} w \leqslant z$ for some $z \in Z$, i.e., V meets Z. By the hypothesis that Y
is $\mathcal{Z}$-fine, V is open, and by continuity of g, we conclude that $g^{-1}[V]$ is an open
neighborhood of u. By $\mathcal{Z}$-distributivity of $\mathcal{O}X$, there is an open neighborhood U of
u with $U \ll_{\mathcal{Z}} g^{-1}[V]$.

Define $f \colon X \to Y$ by $f(x) = v$ if $x \in U$ and $f(x) = 0$ if $x \in X \setminus U$. The only
inverse images of open (hence upper) sets under f are $\emptyset$, U and X; observe that
$X \in \mathcal{O}X$ if $Y \in \mathcal{O}Y$ (since $[X, Y]$ is nonempty, being a complete lattice) and that
$f^{-1}[V] \subseteq U$ for all $V \in \mathcal{O}Y$ otherwise. Thus, f is continuous, and we have $f \not\leqslant h$
since $f(u) = v \not\leqslant h(u)$.

In order to prove $f \ll_{\mathcal{Z}} g$, consider a $Z \in \mathcal{Z}[X,Y]$ with $g \leqslant \bigvee Z$. By the extension of Theorem 2.1 to closure spaces, joins are formed pointwise in $[X,Y]$; thus, for all $x \in X$ we have $g(x) \leqslant \bigvee\{k(x): k \in Z\}$. For each $x \in g^{-1}[V]$ we have $v \ll_{\mathcal{Z}} g(x)$, and since $\{k(x): k \in Z\}$ is a member of $\mathcal{Z}L$ (being the image of $Z \in \mathcal{Z}[X,Y]$ under the isotone projection onto the x-th coordinate), there exists $k \in Z$ with $v \ll_{\mathcal{Z}} k(x)$ (interpolation strikes again), that is, $x \in k^{-1}[V] \in \mathcal{O}X$. Thus, we have $g^{-1}[V] \subseteq \bigcup\{k^{-1}[V]: k \in Z\}$. The map $k \mapsto k^{-1}[V]$ from $[X,Y]$ to $\mathcal{O}X$ is isotone (because $k \leqslant l$ and $x \in k^{-1}[V]$ imply $v \ll_{\mathcal{Z}} k(x) \leqslant l(x)$ and so $x \in l^{-1}[V]$). Hence, $\{k^{-1}[V]: k \in Z\}$ belongs to $\mathcal{Z}\mathcal{O}X$, and from $U \ll_{\mathcal{Z}} g^{-1}[V]$ we conclude that U must be contained in $k^{-1}[V]$ for some $k \in Z$. But then $f(x) = v \ll_{\mathcal{Z}} k(x)$ for $x \in U$ and $f(x) = 0 \leqslant k(x)$ for $x \in X \setminus U$, whence $f \leqslant k$. $\qquad\square$

In view of Theorem 4.1, the following questions (and their strong analogues) arise:

Suppose X, Y are closure spaces such that Y is $\mathcal{Z}$-fine and $[X,Y]$ is a complete lattice.

(1) If $\mathcal{O}X$ and ΩY are $\mathcal{Z}$-distributive lattices, is $[X,Y]$ then $\mathcal{Z}$-distributive, too?

(2) If $\mathcal{O}X$ and ΩY are $\mathcal{Z}$-continuous lattices, is $[X,Y]$ then $\mathcal{Z}$-continuous, too?

Whereas (1) remains open, we shall give a negative answer to (2) at the end of this note. Nevertheless, by Proposition 4.2 and Theorem 4.1, the answers to (1) and (2) are in the affirmative if the subset system $\mathcal{Z}$ is union complete and $\mathcal{Z}^{\wedge}L$ is complete for each complete lattice L. This happens if $\mathcal{Z}$ is the system $\mathcal{D}_m$ of all m-directed subsets, where m is any cardinal number. $\mathcal{D}_m$-continuous complete lattices have been called m-continuous in [2]; in particular, 'ω-continuous' means 'continuous', and '2-continuous' means 'supercontinuous'.

Corollary 4.2. *Suppose $\mathcal{Z}$ is a union complete subset system and X, Y are nonempty closure spaces such that Y is $\mathcal{Z}$-fine and $\mathcal{Z}^{\wedge}\Omega Y$ as well as $[X,Y]$ are complete lattices. Then $[X,Y]$ is a $\mathcal{Z}$-distributive (i.e. $\mathcal{Z}$-continuous) complete lattice iff so are both $\mathcal{O}X$ and ΩY.*

In particular, for nonempty spaces X, Y such that Y is $\mathcal{D}_m$-fine, $[X,Y]$ is an m-continuous lattice iff it is complete and both $\mathcal{O}X$ and ΩY are m-continuous lattices. The completeness hypothesis on $[X,Y]$ may be replaced by requiring that Y is a d-space.

For general subset systems $\mathcal{Z}$, we have not been able to weaken the assumption of *strong* $\mathcal{Z}$-distributivity for ΩY in Theorem 4.1 (2). However, in case Y carries the Scott topology and $\mathcal{Z}\Omega Y$ contains at least all directed sets, the interpolation property of the $\mathcal{Z}$-below relation is not needed for the desired conclusion.

Theorem 4.2. *Let X be a nonempty closure space and L a nonsingleton poset with $\mathcal{D}^\wedge L \subseteq \mathcal{Z}^\wedge L$. Then $[X, \Sigma L]$ is a $\mathcal{Z}$-distributive complete lattice iff so are both $\mathcal{O}X$ and L.*

P r o o f.   The arguments are similar to those for Theorem 4.1. First, observe that for a $\mathcal{Z}$-distributive complete lattice L, the inclusion $\mathcal{D}^\wedge L \subseteq \mathcal{Z}^\wedge L$ implies that L is a continuous lattice (because $x \ll_Z y$ entails $x \ll y$), so that $[X, \Sigma L]$ is complete by Theorem 2.3.

Now, given $g \nleqslant h$ in $[X, \Sigma L]$, pick $u \in X$ with $g(u) \nleqslant h(u)$ and use $\mathcal{Z}$-distributivity of L three times (interpolation is not needed!) to obtain elements $v, y, w \in L$ such that $v \ll_{\mathcal{Z}} y \ll_{\mathcal{Z}} w \ll_{\mathcal{Z}} g(u)$ and $v \nleqslant h(u)$. The inclusion $\mathcal{D}^\wedge L \subseteq \mathcal{Z}^\wedge L$ gives $v \ll y \ll w \ll g(u)$. As L is continuous, the sets $V = v \ll$ and $W = w \ll$ are Scott open; the continuity of g ensures that $g^{-1}[W]$ is an open neighborhood of u in X, and the $\mathcal{Z}$-distributivity of $\mathcal{O}X$ yields a $U \in \mathcal{O}X$ with $U \ll_{\mathcal{Z}} g^{-1}[W]$ and $u \in U$. For any $Z \in \mathcal{Z}[X, Y]$ with $g \leqslant \bigvee Z$ we obtain $g^{-1}[W] \subseteq \bigcup\{k^{-1}[V] \colon k \in Z\}$; indeed, if $x \in g^{-1}[W]$ then $y \ll w \leqslant g(x) \leqslant \bigvee\{k(x) \colon k \in Z\}$ and therefore $v \ll y \leqslant k(x)$ for some $k \in Z$, i.e. $x \in k^{-1}[V]$. Now, $\{k^{-1}[V] \colon k \in Z\}$ belongs to $\mathcal{Z}\mathcal{O}X$ ($k \mapsto k^{-1}[V]$ is isotone!) and we have $U \ll_{\mathcal{Z}} g^{-1}[W]$, so there is a $k \in Z$ such that $U \subseteq k^{-1}[V]$. For the continuous function f defined as in the proof of Theorem 4.1 we conclude that $f \leqslant k$ (because $x \in U$ implies $f(x) = v \ll k(x)$, and $x \in X \setminus U$ implies $f(x) = 0 \leqslant k(x)$). Thus, $f \ll_{\mathcal{Z}} g$ and $f \nleqslant h$ (indeed, $f(u) = v \nleqslant h(u)$), showing that $[X, \Sigma L]$ is $\mathcal{Z}$-distributive.

The reverse implication was obtained in Theorem 4.1 (1).   □

Notice that the inclusion $\mathcal{D}^\wedge L \subseteq \mathcal{Z}^\wedge L$ is trivially fulfilled for all noetherian lattices L (satisfying $\mathcal{D}^\wedge L = \{\downarrow x \colon x \in L\}$). However, only very few *union complete* subset systems $\mathcal{Z}$ are known with $\mathcal{D}^\wedge P \subseteq \mathcal{Z}^\wedge P$ for *all posets* P. Besides $\mathcal{D}$ and $\mathcal{A}$, one is the system $\mathcal{D}_\perp$ of all directed or empty sets, another one is the system $\mathcal{A}_0$ of all nonempty subsets. A third, less trivial one is the system $\mathcal{C}_1$ of all nonempty *connected* subsets (where connectivity refers to the order relation or, equivalently, to the Alexandroff topology of all upper sets). A fourth one is the system $\mathcal{C}_2$ of all consistent subsets, where $C \subseteq P$ is *consistent* if every finite subset of C has an upper bound in P (not necessarily in C). Clearly, in any bounded poset (having a least element and a greatest element), every subset is consistent and every nonempty lower set is connected. Given two subset systems $\mathcal{Y}$ and $\mathcal{Z}$, let us write $\mathcal{Y} \sqsubseteq \mathcal{Z}$ if $\mathcal{Y}^\wedge P \subseteq \mathcal{Z}^\wedge P$ holds at least for all bounded posets P, and $\mathcal{Y} \sqsubseteq\!\!\!\sqsupseteq \mathcal{Z}$ if $\mathcal{Y} \sqsubseteq \mathcal{Z}$ and $\mathcal{Z} \sqsubseteq \mathcal{Y}$. Thus, $\mathcal{C}_1 \sqsubseteq\!\!\!\sqsupseteq \mathcal{A}_0$ and $\mathcal{C}_2 \sqsubseteq\!\!\!\sqsupseteq \mathcal{A}$. Clearly, $\sqsubseteq$ is a quasi-order and $\sqsubseteq\!\!\!\sqsupseteq$ is an equivalence relation.

The next result shows that under the hypothesis of union completeness, Theorems 1.2 and 3.1 are essentially the only two 'global' instances of Theorem 4.2.

Proposition 4.3. *Let $\mathcal{Z}$ be a union complete subset system with $\mathcal{D} \sqsubseteq \mathcal{Z}$. Then*

$$\mathcal{Z} \,\square\, \mathcal{D} \quad \text{or} \quad \mathcal{Z} \,\square\, \mathcal{D}_\perp \quad \text{or} \quad \mathcal{Z} \,\square\, \mathcal{A}_0 \quad \text{or} \quad \mathcal{Z} \,\square\, \mathcal{A}.$$

Hence, for complete lattices L, $\mathcal{Z}^\wedge L$ is one of the systems $\mathcal{D}^\wedge L$, $\mathcal{D}_\perp^\wedge L$, $\mathcal{A}_0^\wedge L$, $\mathcal{A}^\wedge L$, and '$\mathcal{Z}$-distributive' means 'continuous' or 'completely distributive' ('supercontinuous').

P r o o f. We consider the case $\emptyset \in \mathcal{Z}Q$ for some bounded poset Q, which entails $\emptyset \in \mathcal{Z}P$ and $\mathcal{D}_\perp^\wedge P \subseteq \mathcal{Z}^\wedge P$ for *all* bounded posets P. If this inclusion is proper for at least one bounded poset P_0, choose a nonempty $Z \in \mathcal{Z}P_0 \setminus \mathcal{D}P_0$. Then:

(1) In any four-element Boolean lattice $L = \{0, a, a', 1\}$, the subset $\{0, a, a'\}$ is a $\mathcal{Z}$-ideal.

Indeed, since Z is not directed, we find $x, y \in Z$ such that no $z \in Z$ satisfies $x \leqslant z$ and $y \leqslant z$. Define the map $f \colon P_0 \to L$ as the pointwise supremum of $c_{x,a}$ and $c_{y,a'}$, where $c_{x,a}(u) = a$ if $x \leqslant u$ and $c_{x,a}(u) = 0$ otherwise. Then f is isotone and satisfies $f[Z] = \{a, a'\}$ or $f[Z] = \{0, a, a'\}$, in any case $\{0, a, a'\} = {\downarrow}f[Z] \in \mathcal{Z}^\wedge L$.

(2) $A, B \in \mathcal{Z}^\wedge P$ implies $A \cup B \in \mathcal{Z}^\wedge P$ for all bounded posets P.

Without loss of generality, we may assume $A \not\subseteq B \not\subseteq A$. Then $L = \{\{0\}, A, B, P\}$ is a four-element Boolean lattice and a subposet of $\mathcal{Z}^\wedge P$, whence $\{\{0\}, A, B\} \in \mathcal{Z}^\wedge L$, by (1). It follows that ${\downarrow}\{A, B\} = \{C \in \mathcal{Z}^\wedge P \colon C \subseteq A \text{ or } C \subseteq B\} \in \mathcal{Z}^\wedge \mathcal{Z}^\wedge P$, and then, by union completeness, $A \cup B \in \mathcal{Z}^\wedge P$. Now, induction gives (using ${\downarrow}u \in \mathcal{Z}^\wedge P$ for all $u \in P$):

(3) ${\downarrow}F \in \mathcal{Z}^\wedge P$ for all bounded posets P and all $F \in \mathcal{F}P$.

(4) $\mathcal{Z} \,\square\, \mathcal{A}$, that is, ${\downarrow}A \in \mathcal{Z}^\wedge P$ for all bounded posets P and all $A \subseteq P$.

Indeed, $\{{\downarrow}F \colon F \in \mathcal{F}A\} \in \mathcal{D}\mathcal{Z}^\wedge P$ implies ${\downarrow}A = \bigcup\{{\downarrow}F \colon F \in \mathcal{F}A\} \in \mathcal{Z}^\wedge P$, by the hypothesis $\mathcal{D}^\wedge \mathcal{Z}^\wedge P \subseteq \mathcal{Z}^\wedge \mathcal{Z}^\wedge P$ and the union completeness of $\mathcal{Z}$ (notice that $\mathcal{Z}^\wedge P$ is a bounded poset). Thus, $\emptyset \in \mathcal{Z}Q$ for some poset Q entails $\mathcal{Z} \,\square\, \mathcal{D}_\perp$ or $\mathcal{Z} \,\square\, \mathcal{A}$.

The case $\emptyset \notin \mathcal{Z}Q$ is treated analogously and leads to $\mathcal{Z} \,\square\, \mathcal{D}$ or $\mathcal{Z} \,\square\, \mathcal{A}_0$. $\square$

We come now to the announced common generalization of Theorems 2.3 and 3.2.

Theorem 4.3. *Let $\mathcal{Z}$ be a union complete subset system and Y a $\mathcal{Z}$-fine topological space such that $\mathcal{Z}^\wedge \Omega Y$ is a complete lattice. Then the following statements are equivalent:*

(a) *Y is the Scott space ΣL of a $\mathcal{Z}$-distributive complete lattice L.*

(b) *$[Y, Y]$ is a $\mathcal{Z}$-distributive complete lattice, and Y is a nonempty d-space.*

(c) *$[Y^I, Y]$ is a $\mathcal{Z}$-distributive complete lattice for all sets I.*

(d) *Y is a C-space, and $[X, Y]$ is a $\mathcal{Z}$-distributive complete lattice for all topological spaces X with $\mathcal{Z}$-distributive open set lattices $\mathcal{O}X$ (and only for these).*

(e) *$Y = \Sigma L = \mathrm{T}_{\mathcal{Z}} L$ for a continuous and strongly $\mathcal{Z}$-continuous complete lattice L.*

P r o o f. (a) $\Rightarrow$ (e): By the given hypotheses and Proposition 4.2, L is strongly $\mathcal{Z}$-continuous. By Lemma 4.3, $\mathrm{T}_{\mathcal{Z}}L$ is a C-space. Since Y is $\mathcal{Z}$-fine, it follows that $\sigma L \subseteq \tau_{\mathcal{Z}}L \subseteq \mathcal{O}Y = \sigma L$, hence $Y = \Sigma L = \mathrm{T}_{\mathcal{Z}}L$, and that L is a continuous lattice (Theorem 1.1).

(e) $\Rightarrow$ (d): $Y = \Sigma L$ is a C-d-space by Theorem 1.1. Completeness of $[X, Y]$ follows from Corollary 2.1. Moreover, by Corollary 4.2, $[X, Y]$ is $\mathcal{Z}$-distributive iff so are $\mathcal{O}X$ and ΩY.

(d) $\Rightarrow$ (c): Y is order complete by completeness of $[1, Y]$, and as Y is a C-space, so is any power $X = Y^I$, by Proposition 1.1. Hence, $\mathcal{O}X$ is completely distributive and, in particular, $\mathcal{Z}$-distributive, so that $[X, Y]$ is $\mathcal{Z}$-distributive by (d).

(c) $\Rightarrow$ (b): See Theorem 2.2.

(b) $\Rightarrow$ (a): $L = \Omega Y$ is a $\mathcal{Z}$-distributive complete lattice by Theorem 4.1; as before, Lemma 4.3 combined with Theorem 1.1 tells us that $\mathrm{T}_{\mathcal{Z}}L$ is a C-space with $\sigma L \subseteq \tau_{\mathcal{Z}}L \subseteq \mathcal{O}Y$; on the other hand, the inclusion $\mathcal{O}Y \subseteq \sigma L$ follows from the fact that Y is a d-space. Combining both inclusions, we obtain the equation $Y = \Sigma L = \mathrm{T}_{\mathcal{Z}}L$. $\quad\square$

In Example 4.1, the equivalent statements (a)–(d) are fulfilled for the subset system $\mathcal{Z} = \mathcal{B}$ (which is not union complete!) and the Scott spaces of the finite lattices $L \simeq \mathcal{P}3$ and $FD_3 \simeq \mathcal{A}L \setminus \{\emptyset, L\}$, whereas (e) is violated, although $Y = \Sigma L = \mathrm{T}_Z L$.

As a consequence of Theorems 2.3 and 3.2, we note that for continuous (respectively, supercontinuous) lattices L, the complete lattice of all maps from L to L that preserve directed (arbitrary) joins is again continuous (supercontinuous). We deduce a common generalization of these remarks from Theorem 4.1 (cf. Feng Qin [22] for a similar result). As observed already in [6], for any pair of $\mathcal{Z}$-complete posets P, Q,

$$J_{\mathcal{Z}}[P, Q] = [\Sigma_{\mathcal{Z}}P, \Sigma_{\mathcal{Z}}Q]$$

is the poset of all $\mathcal{Z}$-join preserving isotone maps from P to Q: these are exactly the continuous functions from $\Sigma_{\mathcal{Z}}P$ to $\Sigma_{\mathcal{Z}}Q$ (such that preimages of $\mathcal{Z}^{\vee}$-ideals are $\mathcal{Z}^{\vee}$-ideals).

Theorem 4.4. *If P and Q are strongly $\mathcal{Z}$-continuous complete lattices then the function poset $J_{\mathcal{Z}}[P, Q]$ of all $\mathcal{Z}$-join preserving maps from P to Q is a $\mathcal{Z}$-distributive complete lattice.*

P r o o f. By definition, the closure spaces $X = \Sigma_{\mathcal{Z}}P$ and $Y = \Sigma_{\mathcal{Z}}Q$ are $\mathcal{Z}$-fine. By Proposition 4.1, the closure system $\mathcal{Z}^{\vee}P$ and consequently the kernel system $\sigma_{\mathcal{Z}}P = \mathcal{O}X$ are completely distributive, a fortiori $\mathcal{Z}$-distributive. Furthermore, $J_{\mathcal{Z}}[P, Q] = [X, Y]$ is a complete lattice, because arbitrary pointwise joins of $\mathcal{Z}$-join preserving maps again preserve $\mathcal{Z}$-joins. By Theorem 4.1, it follows that $J_{\mathcal{Z}}[P, Q] = [X, Y]$ is $\mathcal{Z}$-distributive. $\quad\square$

We know that if $\mathcal{Z}$ is union complete and $\mathcal{Z}^\wedge P$ is a complete lattice then (strong) $\mathcal{Z}$-continuity of P is equivalent to $\mathcal{Z}$-distributivity. In view of that observation, it comes as a surprise that the modified conclusion in Theorem 4.4 with '$\mathcal{Z}$-continuous' instead of '$\mathcal{Z}$-distributive' may fail, even if $\mathcal{Z}$ is union complete and the specific poset $\mathcal{Z}^\wedge P$ is complete!

E x a m p l e 4.3. Another look at the chain $L = \omega + 1$ reveals a further puzzling property of that seemingly harmless object. This time, we consider the union complete system $\mathcal{F}$ of finite subsets. Note first that for the special lattice $L = \omega + 1$, the descending chain condition ensures that $\mathcal{F}^\wedge L$ is actually a closure system, hence a complete lattice. Like every chain, L is (strongly) $\mathcal{F}$-continuous, since all non-zero elements in a chain are $\vee$-prime.

The complete lattice $J_\mathcal{F}[L, L]$ consists of all functions $f \colon L \to L$ that preserve finite joins, which here simply means that f is isotone with $f(0) = 0$. Although the $\vee$-prime members of $J_\mathcal{F}[L, L]$ are rather specific, they form a join-dense subset: an easy verification shows that the $\vee$-prime elements of $J_\mathcal{F}[L, L]$ are exactly those isotone functions from L to L which have precisely two values, one of which is 0. (Indeed, if f has at least three values, say $f(0) = 0 < f(y) = u < f(z) = v$, take y and z minimal with images u and v, respectively; put $g(y) = f(y) - 1$, $h(z) = f(z) - 1$, and $g(x) = f(x)$; $h(x) = f(x)$ otherwise. Then $g, h \in J_\mathcal{F}[L, L]$, $g < f$, $h < f$ and $f = g \vee h$, whence f cannot be $\vee$-prime.)

Since each $f \in J_\mathcal{F}[L, L]$ is the join of the $\vee$-prime functions $f_y \in J_\mathcal{F}[L, L]$ with $f_y(x) = f(y)$ for $x \geqslant y$ and $f_y(x) = 0$ otherwise, we see that the complete lattice $J_\mathcal{F}[L, L]$ is cospatial, hence $\mathcal{F}$-distributive. But it cannot be $\mathcal{F}$-continuous, since finite pointwise joins of $\vee$-prime elements of $J_\mathcal{F}[L, L]$ have a finite range, whereas, for example, the identity map has infinite range.

Finally, consider the algebraic closure space Y whose closed sets are the ideals of the chain $L = \omega + 1$. For this choice, the lattice $[Y, Y]$ of continuous self-maps on Y is equal to $J_\mathcal{F}[L, L]$. The previous thoughts show that $[Y, Y]$ is not $\mathcal{F}$-continuous, although $L = \Omega Y$ and $\mathcal{O}Y$ are (strongly) $\mathcal{F}$-continuous, being complete chains. Taking $\mathcal{F}_0 L = \mathcal{F}L \setminus \{\emptyset\}$ instead of $\mathcal{F}L$, one obtains even an A-space $Y = \mathrm{A}L$ for which $[Y, Y]$ is not $\mathcal{F}_0$-continuous.

References

[1] *H.-J. Bandelt, M. Erné*: The category of Z-continuous posets. J. Pure Appl. Algebra *30* (1983), 219–226.

[2] *H.-J. Bandelt, M. Erné*: Representations and embeddings of M-distributive lattices. Houston J. Math. *10* (1984), 315–324.

[3] *A. Baranga*: Z-continuous posets. Discrete Math. *152* (1996), 33–45.

[4] *A. Baranga*: Z-continuous posets, topological aspects. Stud. Cercet. Mat. *49* (1997), 3–16.

[5] *M. Erné*: Scott convergence and Scott topology on partially ordered sets II. Continuous Lattices. Proc. Conf., Bremen 1979 (B. Banaschewski, R.-E. Hoffmann, eds.). Lect. Notes Math. 871, Springer, Berlin, 1981, pp. 61–96.

[6] *M. Erné*: Adjunctions and standard constructions for partially ordered sets. Contributions to General Algebra. Proc. Klagenfurt Conf. 1982 (G. Eigenthaler et al. Contributions to General Algebra, eds.). Contrib. Gen. Algebra 2, Hölder, Wien, 1983, pp. 77–106.

[7] *M. Erné*: The ABC of order and topology. Category Theory at Work. Proc. Workshop, Bremen 1991 (H. Herrlich, H.-E. Porst, eds.). Res. Expo. Math. 18, Heldermann, Berlin, 1991, pp. 57–83.

[8] *M. Erné*: Algebraic ordered sets and their generalizations. Algebras and Orders. Kluwer Academic Publishers. NATO ASI Ser. C, Math. Phys. Sci. 389 (I. Rosenberg, ed.). Kluwer Acad. Publ., Dordrecht, 1993, pp. 113–192.

[9] *M. Erné*: Z-continuous posets and their topological manifestation. Appl. Categ. Struct. *7* (1999), 31–70.

[10] *M. Erné*: Minimal bases, ideal extensions, and basic dualities. Topol. Proc. *29* (2005), 445–489.

[11] *M. Erné*: Closure. Beyond Topology. AMS (F. Mynard, E. Pearl, eds.). Contemporary Mathematics 486, Providence, R.I., 2009, pp. 163–238.

[12] *M. Erné*: Infinite distributive laws versus local connectedness and compactness properties. Topology Appl. *156* (2009), 2054–2069.

[13] *M. Erné, H. Gatzke*: Convergence and continuity in partially ordered sets and semilattices. Continuous Lattices and Their Applications. Proc. 3rd Conf., Bremen 1982. Lect. Notes Pure Appl. Math. 101, 1985, pp. 9–40.

[14] *G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott*: A Compendium of Continuous Lattices. Springer, Berlin, 1980.

[15] *G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott*: Continuous Lattices and Domains. Encyclopedia of Mathematics and Its Applications 93, Cambridge University Press, Cambridge, 2003.

[16] *R.-E. Hoffmann*: Continuous posets, prime spectra of completely distributive complete lattices, and Hausdorff compactifications. Continuous Lattices. Proc. Conf., Bremen 1979 (B. Banaschewski, R.-E. Hoffmann, eds.). Lect. Notes Math. 871, Springer, Berlin, 1981, pp. 159–208.

[17] *J. Isbell*: Completion of a construction of Johnstone. Proc. Am. Math. Soc. *85* (1982), 333–334.

[18] *K. Keimel*: Bicontinuous domains and some old problems in domain theory. Electronical Notes in Th. Computer Sci. *257* (2009), 35–54.

[19] *I. Kříž, A. Pultr*: A spatiality criterion and an example of a quasitopology which is not a topology. Houston J. Math. *15* (1989), 215–234.

[20] *J. Meseguer*: Order completion monads. Algebra Univers. *16* (1983), 63–82.

[21] *D. Novak*: Generalization of continuous posets. Trans. Am. Math. Soc. *272* (1982), 645–667.

[22] *F. Qin*: Function space of Z-continuous lattices. Fuzzy Syst. Math. *14* (2000), 31–35. (In Chinese.)

[23] *G. N. Raney*: A subdirect-union representation for completely distributive complete lattices. Proc. Am. Math. Soc. *4* (1953), 518–522.

[24] *D. S. Scott*: Continuous lattices. Toposes, Algebraic Geometry and Logic. Dalhousie Univ. Halifax 1971, Lect. Notes Math. 274. Springer, Berlin, 1972, pp. 97–136.

[25] *G. Venugopalan*: Z-continuous posets. Houston J. Math. *12* (1986), 275–294.

[26] *J. B. Wright, E. G. Wagner, J. W. Thatcher*: A uniform approach to inductive posets and inductive closure. Theor. Comput. Sci. *7* (1978), 57–77.

[27] *O. Wyler*: Dedekind complete posets and Scott topologies. Continuous Lattices. Proc. Conf., Bremen 1979, Lect. Notes Math. 871 (B. Banaschewski, R.-E. Hoffmann, eds.). Springer, Berlin, 1981, pp. 384–389.

Author's address: *Marcel Erné*, Leibniz University, Welfengarten 1, Hannover, Germany, e-mail: `erne@math.uni-hannover.de`.