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INDECOMPOSABLE (1,3)-GROUPS AND A MATRIX PROBLEM

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Abstract. Almost completely decomposable groups with a critical typeset of type $(1,3)$ and a p-primary regulator quotient are studied. It is shown that there are, depending on the exponent of the regulator quotient p^k, either no indecomposables if $k \leqslant 2$; only six near isomorphism types of indecomposables if $k = 3$; and indecomposables of arbitrary large rank if $k \geqslant 4$.

Keywords: almost completely decomposable group, indecomposable, representation

MSC 2010: 20K15, 20K25, 20K35, 15A21, 16G20

1. Introduction

A torsion-free abelian group G is *completely decomposable* if G is isomorphic to a finite direct sum of subgroups of $\mathbb{Q}$, the additive group of rational numbers, and *almost completely decomposable* if G contains a completely decomposable subgroup A with G/A a finite group. Almost completely decomposable groups are a notoriously complicated class of torsion-free abelian groups of finite rank ([15], [2], [17]), the source of many pathological decompositions ([13]) and have been generalized to infinite rank ([18]).

A subgroup R of an almost completely decomposable group G is a *regulating subgroup* of G if and only if R is completely decomposable and $|G/R|$ is the least integer in the set $\{|G/A|\colon$ *A is completely decomposable with G/A finite*$\}$ ([15]).

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The *regulator* $\mathrm{R}(G)$ is the intersection of all regulating subgroups of G. Burkhardt ([8]) showed that the regulator is again completely decomposable, has finite index in G, and is fully invariant.

It can happen that an almost completely decomposable group contains exactly one regulating subgroup that then coincides with the regulator. In this case we have a *regulating regulator*.

The set of all types of elements of a torsion-free abelian group G is called the *typeset* of G. For almost completely decomposable groups the (finite) set of types of the direct summands of rank 1 of the regulator is called the *critical typeset*. This is an invariant. The typeset of an almost completely decomposable group is the closure of the critical typeset relative to the intersection of types.

An essential breakthrough came with the concept of "near-isomorphism" that is a weakening of isomorphism, ([16], [17, Chapter 9]). While a classification of almost completely decomposable groups up to isomorphism is hopeless some almost completely decomposable groups could be classified up to near-isomorphism. At the same time near-isomorphism is not so general that important properties become indistinguishable. To witness, the well-known and important theorem of Arnold ([1, 12.9, p. 144], [17, Theorem 10.2.5]) states that the decomposition properties of two near-isomorphic torsion-free groups of finite rank have (up to near-isomorphism of summands) the same decomposition properties.

The pathological decompositions of almost completely decomposable groups, see for example Corner's Theorem ([9]) derive from the presence of several primes in the order of the regulator quotient $G/\mathrm{R}(G)$. If the regulator quotient is a primary group (the "p-local" case), then, according to a result by Faticoni-Schultz ([12]) the direct decompositions of the group with indecomposable summands are unique up to near-isomorphism.

In this paper we completely settle a special case. Let p be a prime, $(1,3) = (\tau_0, \tau_1 < \tau_2 < \tau_3)$ a set of types, partially ordered as indicated with $\tau_i(p) \neq \infty$. Let $R = R_0 \oplus R_1 \oplus R_2 \oplus R_3$ where R_i is homogeneous completely decomposable of finite rank $\geqslant 1$ and type τ_i. A p-reduced, almost completely decomposable group G is called a $(1,3)$-*group* if $\mathrm{R}(G) \cong R$ and $G/\mathrm{R}(G)$ is p-primary. Such a group has a regulating regulator ([19]) and, up to near-isomorphism, unique indecomposable decompositions. Hence, for $(1,3)$-groups, the main problem is to determine the near-isomorphism classes of indecomposable $(1,3)$-groups. We show that

- there are no indecomposable $(1,3)$-groups G with $\exp(G/\mathrm{R}(G)) \leqslant p^2$ (Theorem 30),
- there are six near-isomorphism classes of indecomposable $(1,3)$-groups G with $\exp(G/\mathrm{R}(G)) = p^3$. The regulator quotients are isomorphic to $\mathbb{Z}/p^3\mathbb{Z}$, $(\mathbb{Z}/p^3\mathbb{Z}) \oplus (\mathbb{Z}/p^3\mathbb{Z})$, $(\mathbb{Z}/p^3\mathbb{Z}) \oplus (\mathbb{Z}/p^2\mathbb{Z})$ or $(\mathbb{Z}/p^3\mathbb{Z}) \oplus (\mathbb{Z}/p\mathbb{Z})$ (Theorem 32),

- there exist indecomposable $(1,3)$-groups of rank $5n$ for any integer $n \geqslant 1$ with regulator quotient of exponent p^4 (Theorem 33) ([6, Theorem 1]). Our proof is alternate using the techniques of this paper.

Clearly, similar questions arise for other small typesets. For the case $(1,2)$ partial results can be found ([3], [11], [20], [26]). The cases $(1,2)$, $(2,2)$ and $(1,n)$ are in work.

A few words about the methods used in this paper. A p-reduced, almost completely decomposable group G with regulator quotient a finite p-group is associated with an integer coordinate matrix (Section 3). Two such groups are nearly isomorphic if and only if their coordinate matrices are equivalent via an equivalence relation defined by certain row and column operations (Theorem 12). A group G with no 1-rank summands is indecomposable if and only if its coordinate matrix is not equivalent to a matrix direct sum (Sections 4 and 5). As a result, indecomposable $(1,3)$-groups with regulator quotients bounded by p^3 can be characterized, up to near isomorphism, by using specified row and column operations to reduce the coordinate matrices to a coordinate matrix of an indecomposable group (Sections 6 and 7).

This classification procedure is similar to the solution to "matrix problems" for representations of finite posets over a field and algebras over a field, often an algebraically closed field ([21], [22], [10], [7], [23], [24]), and there is a survey of matrix problems over fields and division rings ([25]). Since matrix problems in this paper concern integer matrices, solutions and techniques for matrix problems over fields do not apply directly. Some matrix problems for matrices over discrete valuation rings and their factor rings are solved ([2, Chapter 4], [4], [5]).

The problem of characterizing those almost completely decomposable S-groups with regulator quotients bounded by p^m and bounded or unbounded representation type is also dealt with ([6]).

2. Preliminaries

Throughout, p denotes an arbitrary but fixed prime number. Let G be an almost completely decomposable group. Recall that G contains the completely decomposable, fully invariant regulator $\mathrm{R}(G)$ and that the regulator quotient $G/\mathrm{R}(G)$ is a finite p-primary abelian group. The isomorphism classes of the regulator and the regulator quotient are near-isomorphism invariants ([17, 8.1.13 and 8.2.8]). The *critical typeset*, $T_{\mathrm{cr}}(G)$, of the almost completely decomposable group G is

$$T_{\mathrm{cr}}(G) = \{\tau\colon\ G(\tau)/G^{\sharp}(\tau) \neq 0\}.$$

Recall that a partially ordered set T is said to be $\vee$-free (or an inverted forest) if for all $\tau \in T$, the subsets $T(>\tau)$ are chains. In particular, anti-chains are $\vee$-free.

An almost completely decomposable group G has a regulating regulator if $T_{\mathrm{cr}}(G)$ is $\vee$-free ([19, 1.4, p. 212], [17, Proposition 4.5.4]).

A torsion-free abelian group is called *p-reduced* if the maximal p-divisible subgroup is trivial. An almost completely decomposable group G is *p-local* if $G/\mathrm{R}(G)$ is a p-group. We consider exclusively p-reduced and p-local almost completely decomposable groups.

3. Coordinate matrices

The goal of this section is to describe almost completely decomposable groups by means of an integer matrix, the "coordinate matrix". We consider groups with fixed regulator and regulator quotient. The coordinate matrix is obtained by means of "bases" of $R = \mathrm{R}(G)$ and G/R.

Let $R = \bigoplus_{i=1}^{n} S_i x_i$ be a completely decomposable group. The ordered set $(x_1, \ldots, x_n)$ is a *decomposition basis* of R with *coefficient groups* S_i when $x_i \in R$ for each i and $S_i = \{s \in \mathbb{Q} \colon sx_i \in R\}$. The *type* of a subgroup $S \subset \mathbb{Q}$ is denoted by $\mathrm{tp}(S)$, and $\leqslant$ is the order relation in the lattice of types. Note that the purification of $\langle x_i \rangle$ in R is $\langle x_i \rangle_*^R = S_i x_i$, $1 \in S_i$, and $\mathrm{tp}(x_i) = \mathrm{tp}(S_i)$. The decomposition basis $(x_1, \ldots, x_n)$ is a *p-basis* of R if $p \notin S_i$.

We study transitions from one decomposition basis of R to another.

Lemma 1. *Let $Y = [Y_{i,j}]$ be a rational matrix, let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be decomposition bases of R such that $\mathrm{tp}(x_i) = \mathrm{tp}(y_i)$ for every i. Assume that $y_i = \sum_{j=1}^{n} Y_{i,j} x_j$. Then $Y_{i,j} \neq 0$ implies that $\mathrm{tp}(x_i) \leqslant \mathrm{tp}(x_j)$.*

Proof. Write $R = \bigoplus_{i=1}^{n} S_i x_i = \bigoplus_{i=1}^{n} T_i y_i$. By hypothesis $S_i \cong T_i$. We first note that for all $t \in T_i$ we get $ty_i = \sum_{j=1}^{n} tY_{i,j} x_j \in R$ and hence $tY_{i,j} \in S_j$. So $T_i Y_{i,j} \subset S_j$. Therefore, if $Y_{i,j} \neq 0$, then $\mathrm{tp}(S_i) = \mathrm{tp}(T_i) \leqslant \mathrm{tp}(S_j)$. □

The rank of an integer matrix modulo p is called its *p-rank*. A square integer matrix Y is *p-invertible* if $\gcd(p, \det Y) = 1$, equivalently, if there is an integer matrix Z such that $YZ = ZY \equiv I \bmod p^k$ for any integer $k > 0$.

Definition. Let $\mathfrak{T} = (\tau_1, \ldots, \tau_n)$ be a sequence of types. A p-invertible $n \times n$ matrix $Y = [Y_{i,j}]$ is *conforming* with $\mathfrak{T}$ if $Y_{i,j} \neq 0$ implies that $\tau_i \leqslant \tau_j$.

Remark 2. Suppose that the type sequence $\mathfrak{T} = (\tau_1, \ldots, \tau_n)$ is such that $\tau_i < \tau_j$ implies that $i < j$, i.e., if two different types are comparable; then the larger type has the larger index. Moreover, if $\tau_i = \tau_j$ for $i < j$, then $\tau_l = \tau_i = \tau_j$ for all $i \leqslant l \leqslant j$. We will always label types to satisfy this condition. In particular, a $\mathfrak{T}$-conforming matrix is upper block triangular. If in addition the types τ_i are pairwise different, then a $\mathfrak{T}$-conforming matrix is upper triangular. In the other extreme, if the types τ_i are all equal, then any p-invertible matrix is conforming.

The following characterization of conforming matrices will come in handy.

Lemma 3. *Let $R = S_1x_1 \oplus \ldots \oplus S_nx_n$, set $\tau_i = \operatorname{tp}(S_i)$, and let $Y = [Y_{i,j}]$ be a p-invertible $n \times n$ (integer) matrix. The matrix Y determines an invertible linear transformation $\tilde{Y}\colon \mathbb{Q}R \to \mathbb{Q}R$ by setting $\tilde{Y}(x_i) = \sum\limits_{j=1}^{n} Y_{i,j}x_j$. Then $Y_{i,j} \neq 0$ implies $\tau_i \leqslant \tau_j$ if and only if $\tilde{Y}(\mathbb{Q}R(\tau_i)) = \mathbb{Q}R(\tau_i)$ for $1 \leqslant i \leqslant n$. In particular, if Y is conforming, then so is* $\operatorname{adj}(Y)$.

P r o o f. Assume first that $\tilde{Y}(\mathbb{Q}R(\tau_i)) \subset \mathbb{Q}R(\tau_i)$ for every i. We have that $x_i \in R(\tau_i)$. From the definition of the transformation $\tilde{Y}$ we get that $\tilde{Y}(x_i) = Y_{i,1}x_1 + \ldots + Y_{i,n}x_n \in \mathbb{Q}R(\tau_i) \cap R = R(\tau_i)$. Hence if $Y_{i,j} \neq 0$, then $\tau_i \leqslant \tau_j$.

Conversely, assume that Y is conforming. Let $x \in \mathbb{Q}R(\tau_i)$. Then there is $0 \neq k \in \mathbb{N}$ such that $kx \in R(\tau_i)$. In terms of the basis of R we get $kx = \sum\{s_jx_j\colon \tau_j \geqslant \tau_i\}$. Applying $\tilde{Y}$ we get $k\tilde{Y}(x) = \sum\{s_j\tilde{Y}(x_j)\colon \tau_j \geqslant \tau_i\}$. As Y is conforming, $\tilde{Y}(x_j) = \sum\{Y_{j,t}x_t\colon \tau_t \geqslant \tau_j\}$ for $\tau_j \geqslant \tau_i$, hence $\tilde{Y}(x_j) \in \sum\{R(\tau_t)\colon \tau_t \geqslant \tau_j\} \subset R(\tau_j)$. Hence $k\tilde{Y}(x) \in \sum\{R(\tau_j)\colon \tau_j \geqslant \tau_i\} \subset R(\tau_i)$ and $\tilde{Y}(x) \in \mathbb{Q}R(\tau_i)$. This shows that $\tilde{Y}(\mathbb{Q}R(\tau_i)) \subset \mathbb{Q}R(\tau_i)$ and equality follows because $\mathbb{Q}R(\tau_i)$ is a finite dimensional $\mathbb{Q}$-vector space and Y is injective. Finally, letting $\tilde{Y}'$ denote the linear transformation determined by $\operatorname{adj}(Y)$, we have $\tilde{Y}'(\mathbb{Q}R(\tau_i)) = \det(Y)\tilde{Y}^{-1}(\mathbb{Q}R(\tau_i)) = \mathbb{Q}R(\tau_i)$. $\square$

Conforming matrices are connected with endomorphisms of completely decomposable groups.

Lemma 4. *Let $R = \bigoplus\limits_{i=1}^{n} S_ix_i$ where $(x_1, \ldots, x_n) \subset R$ is a p-basis of R. Let $Y = [Y_{i,j}]$ be a conforming matrix. Set $y_i = \sum\limits_{j=1}^{n} Y_{i,j}x_j$. Then:*

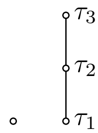
(1) *$\tilde{Y}\colon \mathbb{Q}R \to \mathbb{Q}R\colon \tilde{Y}(x_i) = y_i$ defines an invertible linear transformation whose matrix with respect to the basis $X := (x_1, \ldots, x_n)$ is Y^{tr}.*
(2) *$\tilde{Y}(R) \subset R$, i.e., $\tilde{Y} \in \operatorname{End}(R)$, if and only if $S_iY_{i,j} \subset S_j$ for all i, j.*
(3) *There exists an integer d relatively prime to p such that $d\tilde{Y} \in \operatorname{End}(R)$.*

P r o o f. (1) Recall that the columns of the matrix of $\tilde{Y}$ are formed by the X-coordinates of the images $\tilde{Y}(x_i) = y_i$. The map $\tilde{Y}$ is bijective because $\det Y \neq 0$.

(2) $\tilde{Y}(R) \subset R \iff \forall\, i\colon\ \tilde{Y}(S_i x_i) \subset R \iff \forall\, i, \forall\, s \in S_i\colon\ \tilde{Y}(s x_i) = s\tilde{Y}(x_i) = \sum_{j=1}^{n} s Y_{i,j} x_j \in R \iff \forall\, i, j\colon\ S_i Y_{i,j} \subset S_j$.

(3) The condition of (2) is satisfied if $Y_{i,j} = 0$. Assume that $Y_{i,j} \neq 0$. Then by the definition of the conforming matrix we have $\mathrm{tp}(S_i) \leqslant \mathrm{tp}(S_j)$ and this means that there exists $d \neq 0$ such that $dS_i \subset S_j$. This d may be chosen to be relatively prime to p because X is a p-basis, i.e., $p \notin S_i$, and d may be chosen large enough to work for all i, j. $\square$

Example 5. Let $R = R_0 \oplus R_1 \oplus R_2 \oplus R_3$ where R_i is homogeneous completely decomposable of rank $r_i \geqslant 1$ and type τ_i with the critical typeset $T_{\mathrm{cr}}(R) = \{\tau_0, \tau_1, \tau_2, \tau_3\}$ partially ordered as shown.

$T_{\mathrm{cr}}(R) =$ 

Let $n = r_0 + r_1 + r_2 + r_3$. An $n \times n$ integer matrix Y is conforming with $\mathfrak{T} = (\tau_0, \ldots, \tau_0, \tau_1, \ldots, \tau_1, \tau_2, \ldots, \tau_2, \tau_3, \ldots, \tau_3)$, τ_i repeated $r_i$times if it has the form

$$Y = \begin{bmatrix} Y_{0,0} & 0 & 0 & 0 \\ 0 & Y_{1,1} & Y_{1,2} & Y_{1,3} \\ 0 & 0 & Y_{2,2} & Y_{2,3} \\ 0 & 0 & 0 & Y_{3,3} \end{bmatrix}$$

where $Y_{i,j}$ is an $r_i \times r_j$ integer matrix and the diagonal blocks $Y_{i,i}$ are p-invertible.

Definition. Let G be a p-reduced, p-local, almost completely decomposable group with a completely decomposable subgroup R of finite index. A matrix $\alpha = [\alpha_{i,j}]$ is a *coordinate matrix* of G modulo R if α is integral, there is a basis $(\gamma_1, \ldots, \gamma_r)$ of G/R, there are representatives $g_i \in G$ of γ_i, and there is a p-basis $(x_1, \ldots, x_n)$ of R such that

$$g_i = p^{-k_i} \left(\sum_{j=1}^{n} \alpha_{i,j} x_j \right) \quad \text{where } \langle \gamma_i \rangle \cong \mathbb{Z}_{p^{k_i}}.$$

If $\gamma_i = g_i + R$, we will call $(g_1, \ldots, g_r)$ *a basis of* G *modulo* R. Since $(\gamma_1, \ldots, \gamma_r)$ is a basis of G/R, an $r \times n$ coordinate matrix has p-rank r, i.e., it has a p-invertible $r \times r$ submatrix.

The diagonal matrix $S = \mathrm{diag}(p^{k_1}, \ldots, p^{k_r})$ is called the *structure matrix of* G *modulo* R *corresponding* to the basis $(\gamma_1, \ldots, \gamma_r)$ of G/R if $p^{k_i} = \mathrm{ord}(\gamma_i)$. The sequence $\mathfrak{T} = (\mathrm{tp}(x_1), \ldots, \mathrm{tp}(x_n))$ is called the *type sequence corresponding* to the p-basis $(x_1, \ldots, x_n)$ of R.

Coordinate matrices exist in abundance.

Lemma 6. *Let G be a p-local, p-reduced, almost completely decomposable group with regulator $R = \bigoplus_{i=1}^{n} S_i x_i$ where $(x_1, \dots, x_n)$ is a p-basis of R. Let $G/R = \bigoplus_{i=1}^{r} \langle \gamma_i \rangle$ with $\langle \gamma_i \rangle \cong \mathbb{Z}_{p^{k_i}}$ be the regulator quotient of G, let $\gamma_i = g_i + R$, where $g_i = p^{-k_i}\Bigl(\sum_{j=1}^{n} \alpha'_{i,j} x_j\Bigr)$ for some $\alpha'_{i,j} \in S_j$.*

Then $G = R + \sum_{i=1}^{r} \mathbb{Z} g_i \subset \mathbb{Q}R$ and there exists an integer d relatively prime to p such that

- *$G/R = \bigoplus_{i=1}^{r} \langle d\gamma_i \rangle$,*
- *$\langle d\gamma_i \rangle \cong \mathbb{Z}_{p^{k_i}}$,*
- *$d\gamma_i = dg_i + R$,*
- *$dg_i = p^{-k_i}\Bigl(\sum_{j=1}^{n} d\alpha'_{i,j} x_j\Bigr)$ with $d\alpha'_{i,j} \in \mathbb{Z}$,*
- *$\gcd(p, d\alpha'_{i,1}, \dots, d\alpha'_{i,n}) = 1$.*

Proof. As $(x_1, \dots, x_n)$ is a p-basis, the denominator of $\alpha'_{i,j}$ as a reduced fraction is relatively prime to p. Hence there is $d \in \mathbb{N}$ with $\gcd(p, d) = 1$ such that $d\alpha'_{i,j} \in \mathbb{Z}$ for all i, j. With this d the first four claims are immediate. The last statement follows from the fact that $\operatorname{ord}(\gamma_i) = p^{k_i}$. □

Coordinate matrices are uniquely determined by the bases (x_i) and (g_i).

Lemma 7. *Let G be p-local, p-reduced almost completely decomposable, let $(x_1, \dots, x_n)$ be a p-basis of R, and let $(g_1, \dots, g_r)$ be a basis of G modulo R. If α and β are coordinate matrices of G relative to the bases (x_i) and (g_i), then $\beta = \alpha$.*

Proof. By definition $g_i = p^{-k_i}\Bigl(\sum_{j=1}^{n} \alpha_{i,j} x_j\Bigr) = p^{-k_i}\Bigl(\sum_{j=1}^{n} \beta_{i,j} x_j\Bigr)$. Because $(x_1, \dots, x_n)$ is a basis of $\mathbb{Q}R = \mathbb{Q}G$ it follows that $\alpha_{i,j} = \beta_{i,j}$. □

We check next how the choice of the basis of G/R affects the coordinate matrix.

Definition. Let $S = \operatorname{diag}(p^{k_1}, \dots, p^{k_r})$.

(1) Two integer matrices M, M' (of equal size) are called *S-congruent* if $m_{i,j} \equiv m'_{i,j} \bmod p^{k_i}$ for all i, j. If so, we write $M \equiv_S M'$.
(2) A pair (U, U') of integer matrices that are p-invertible is called an *S-pair* if $US = SU'$.

Note that M is always S-congruent to a matrix M' where $0 \leqslant m'_{i,j} < p^{k_i}$. Also note: if (U, U') is an S-pair, then $((U')^{\mathrm{tr}}, U^{\mathrm{tr}})$ is an S-pair.

It is straightforward to verify that the integer matrix $U = [u_{i,j}]$ is the first component of an S-pair if and only if $u_{i,j} \in p^{k_i - k_j}\mathbb{Z}$.

The significance of S-pairs lies in their connection with automorphisms of finite abelian groups ([14, Theorem 3.15]).

Theorem 8. *Let $G = \langle g_1 \rangle \oplus \ldots \oplus \langle g_r \rangle$ be a finite p-group and set $S = \operatorname{diag}(\operatorname{ord}(g_1), \ldots, \operatorname{ord}(g_r))$.*

(1) *An $r \times r$ matrix $[u_{i,j}] \in \mathrm{M}_r(\mathbb{Z})$ induces an endomorphism of G given by $\tilde{U}(g_i) = u_{1,i}g_1 + u_{2,i}g_2 + \ldots + u_{r,i}g_r$ if and only if there exists $U' \in \mathrm{M}_r(\mathbb{Z})$ such that $US = SU'$.*

(2) *Suppose that $\tilde{U} \in \operatorname{End}(G)$. Then $\tilde{U} \in \operatorname{Aut}(G)$ if and only if U is p-invertible.*

We record some basic properties of the S-congruence that we will use without explicit reference.

Lemma 9. *Let $S = \operatorname{diag}(p^{k_1}, \ldots, p^{k_r})$.*

(1) *$M \equiv_S M'$ if and only if there is an integer matrix N such that $M = M' + SN$.*

(2) *$\equiv_S$ is an equivalence relation.*

(3) *If $M \equiv_S M'$ and K is an integer matrix, then $MK \equiv_S M'K$.*

(4) *If (U, U') is an S-pair, then $M \equiv_S M'$ if and only if $UM \equiv_S UM'$. In particular, if d is an integer relatively prime to p, then $M \equiv_S M'$ if and only if $dM \equiv_S dM'$.*

(5) *If d is an integer relatively prime to p and $dM \equiv_S M'$, then $M \equiv_S d'M'$ for some integer d' relatively prime to p.*

P r o o f. All verifications being straightforward, we only check (4).

$$\begin{aligned} M \equiv_S M' &\Longleftrightarrow M = M' + SN \Longleftrightarrow UM = UM' + USN = UM' + SU'N \\ &\Longleftrightarrow UM \equiv_S UM'. \end{aligned}$$

□

We clarify next how the coordinate matrix changes if the basis of G/R changes. Moreover, we show that for a fixed p-basis of R and a fixed basis of G/R the coordinate matrices form an equivalence class modulo S.

Lemma 10. *Let G be a p-reduced, p-local, almost completely decomposable group, let α be the coordinate matrix of G relative to the p-basis $(x_1, \ldots, x_n)$ of the regulator R and the basis $(g_1, \ldots, g_r)$ of G modulo R, and let β be the coordinate matrix of G relative to the same p-basis $(x_1, \ldots, x_n)$ of the regulator R and the basis $(h_1, \ldots, h_r)$ of G modulo R. Set $\gamma_i = g_i + R$ and $\delta_i = h_i + R$ and assume that*

$\operatorname{ord}(\gamma_i) = \operatorname{ord}(\delta_i) = p^{k_i}$. *Let* $S = \operatorname{diag}(p^{k_1}, \ldots, p^{k_r})$. *Then there is an* S*-pair* (U, U') *such that* $\beta \equiv_S U\alpha$.

In particular, if $g_i + R = h_i + R$ *for all* i*, then* $\beta \equiv_S \alpha$*. Conversely, if* $\beta \equiv_S \alpha$ *for two coordinate matrices relative to the same* p*-basis of* R*, then the corresponding groups are equal.*

P r o o f. We have

$$G/R = \bigoplus_{i=1}^{r} \langle \gamma_i \rangle = \bigoplus_{i=1}^{r} \langle \delta_i \rangle$$

and there is an automorphism of G/R with $\delta_i \mapsto \gamma_i$. By Theorem 8 this automorphism is given by an S-pair (V, V') where $V = (v_{i,j})$ and $\gamma_i = \sum_{j=1}^{r} v_{j,i}\delta_j$. By definition

$$g_i = p^{-k_i}\left(\sum_{j=1}^{n} \alpha_{i,j} x_j\right), \quad \text{and } h_i = p^{-k_i}\left(\sum_{j=1}^{n} \beta_{i,j} x_j\right).$$

Let $R = \bigoplus_{j=1}^{n} S_j x_j$. We have

$$g_i + R = \gamma_i = \sum_{j=1}^{r} v_{j,i}\delta_j = \sum_{j=1}^{r} v_{j,i} h_j + R$$

and hence

$$g_i = \sum_{j=1}^{r} v_{j,i} h_j + \sum_{j=1}^{n} \xi_{i,j} x_j, \quad \text{where } \xi_{i,j} \in S_j.$$

There exists $d \in \mathbb{N}$ relatively prime to p such that $d\xi_{i,j} \in \mathbb{Z}$ for all i, j. Multiplying by dp^{k_i} and substituting for g_i and h_i we obtain

$$\begin{aligned}
d\sum_{t=1}^{n} \alpha_{i,t} x_t = dp^{k_i} g_i &= p^{k_i} \sum_{j=1}^{r} dv_{j,i} h_j + p^{k_i} \sum_{j=1}^{n} (d\xi_{i,j}) x_j \\
&= p^{k_i} \sum_{j=1}^{r} dv_{j,i}\left(p^{-k_j} \sum_{t=1}^{n} \beta_{j,t} x_t\right) + p^{k_i} \sum_{j=1}^{n} (d\xi_{i,j}) x_j \\
&= p^{k_i} \sum_{t=1}^{n}\left(\sum_{j=1}^{r} dv_{j,i} p^{-k_j} \beta_{j,t}\right) x_t + p^{k_i} \sum_{t=1}^{n} (d\xi_{i,t}) x_t \\
&= \sum_{t=1}^{n}\left(\sum_{j=1}^{r} p^{k_i} dv_{j,i} p^{-k_j} \beta_{j,t} + p^{k_i}(d\xi_{i,t})\right) x_t,
\end{aligned}$$

hence

$$d\alpha_{i,t} = \sum_{j=1}^{r} p^{k_i} dv_{j,i} p^{-k_j} \beta_{j,t} + p^{k_i}(d\xi_{i,t}).$$

In terms of matrices this means that

$$d\alpha \equiv_S dSV^{\mathrm{tr}}S^{-1}\beta = d(V')^{\mathrm{tr}}\beta \quad \text{where } V' = S^{-1}VS.$$

Noting that $((V')^{\mathrm{tr}}, V^{\mathrm{tr}})$ is an S-pair, we can set $U = (V')^{\mathrm{tr}}$ and $U' = V^{\mathrm{tr}}$ and obtain $\alpha \equiv_S U\beta$ as claimed.

In particular, if $g_i + R = h_i + R$ for all i, then $V \equiv_S V' \equiv_S I_r$, the identity matrix, thus $\beta \equiv_S \alpha$. Conversely, if $\beta \equiv_S \alpha$ for two coordinate matrices relative to the same p-basis of R, then by Lemma 6 the corresponding groups are equal. $\square$

Finally, we consider the effect of a change of the p-basis on the coordinate matrix.

Lemma 11. *Let G be a p-reduced, p-local, almost completely decomposable group given by a coordinate matrix α relative to a p-basis $(x_1, \ldots, x_n)$ of the regulator R and a basis $(g_1, \ldots, g_r)$ of G modulo R. Let $S = \operatorname{diag}(p^{k_1}, \ldots, p^{k_r})$ where $p^{k_i} = \operatorname{ord}(g_i + R)$.*

Let β be the coordinate matrix of G relative to the p-basis $(y_1, \ldots, y_n)$ of the regulator R and the basis $(g_1, \ldots, g_r)$ of G modulo R. Assume that the type sequences of the two p-bases are equal. Then there exists a conforming matrix Y such that $\beta \equiv_S \alpha Y$.

P r o o f. By definition

$$g_i = p^{-k_i}\sum_{j=1}^{n}\alpha_{i,j}x_j = p^{-k_i}\sum_{t=1}^{n}\beta_{i,t}y_t.$$

Write $R = \bigoplus\limits_{i=1}^{n} S_i y_i$. Then $x_j = \sum\limits_{t=1}^{n}\xi_{j,t}y_t$ for some $\xi_{j,t} \in S_t$. There exists $d \in \mathbb{N}$ relatively prime to p such that $d\xi_{j,t} \in \mathbb{Z}$. Note that $[d\xi_{j,t}]$ is conforming by Lemma 1. Now

$$d\sum_{t=1}^{n}\beta_{i,t}y_t = \sum_{j=1}^{n}\alpha_{i,j}\left(\sum_{t=1}^{n}d\xi_{j,t}y_t\right) = \sum_{t=1}^{n}\left(\sum_{j=1}^{n}\alpha_{i,j}d\xi_{j,t}\right)y_t.$$

It follows that $d\beta_{i,t} = \sum\limits_{j=1}^{n}\alpha_{i,j}d\xi_{j,t}$ and in terms of matrices that $d\beta = \alpha Y'$ where $Y' = [d\xi_{j,t}]$. As d is relatively prime to p there exist $u, v \in \mathbb{Z}$ such that $1 = ud + vp^k$ with $k \geqslant k_i$, and so $\beta = (ud + vp^k)\beta = u\alpha Y' + p^k v\beta$ and this says that $\beta \equiv_S \alpha Y$ where $Y = uY'$. $\square$

Combining Lemma 10 and Lemma 11, we obtain the first part of the following fundamental theorem.

Theorem 12. *Let G be a p-reduced, p-local, almost completely decomposable group with the coordinate matrix α relative to the p-basis $(x_1, \ldots, x_n)$ of the regulator R, and the basis $(g_1, \ldots, g_r)$ of G modulo R. Let $\mathfrak{T} = (\operatorname{tp}(x_1), \ldots, \operatorname{tp}(x_n))$ and let $S = \operatorname{diag}(\operatorname{ord}(g_1 + R), \ldots, \operatorname{ord}(g_r + R))$ be the structure matrix. Assume that G has a regulating regulator.*

(1) *Let β be the coordinate matrix of G relative to the p-basis $(y_1, \ldots, y_n)$ of the regulator R and the basis $(h_1, \ldots, h_r)$ of G modulo R such that $\operatorname{ord}(h_i + R) = \operatorname{ord}(g_i + R)$ and the type sequences corresponding to the two p-decomposition bases $(x_j \mid j)$ and $(y_j \mid j)$ are the same. Then there is an S-pair (U, U') and a conforming matrix Y such that $\beta \equiv_S U\alpha Y$.*

(2) *Conversely, suppose that an S-pair (U, U') and a conforming matrix Y are given. Then there is a group H nearly isomorphic to G containing R, a basis of H modulo R, and a p-basis of R such that H has the structure matrix S, the type sequence $\mathfrak{T}$, and $U\alpha Y$ is the corresponding coordinate matrix of H.*

Remark. We do not know whether the hypothesis that G has a regulating regulator is necessary in Theorem 12.

Proof. Write $R = \bigoplus_{i=1}^{n} S_i x_i$. By definition $g_i = p^{-k_i} \sum_{j=1}^{n} \alpha_{i,j} x_j$.

(1) $h_i = p^{-k_i} \sum_{j=1}^{n} \beta'_{i,j} x_j$ for some $\beta'_{i,j} \in S_j$. There is $d \in \mathbb{N}$ relatively prime to p such that $d\beta'_{i,j} \in \mathbb{Z}$ for all i, j. We now have

$$dg_i = p^{-k_i} \sum_{j=1}^{n} d\alpha_{i,j} x_j \quad \text{where } d\alpha_{i,j} \in \mathbb{Z},$$

and

$$dh_i = p^{-k_i} \sum_{j=1}^{n} d\beta'_{i,j} x_j \quad \text{where } d\beta'_{i,j} \in \mathbb{Z}.$$

Note that (dg_i) and (dh_i) are bases of G modulo R. Set $S = \operatorname{diag}(p^{k_1}, \ldots, p^{k_r})$. By Lemma 10 there is an S-pair (U, U') such that $d\beta' \equiv_S Ud\alpha$. We now have the coordinate matrix $d\beta'$ relative to the bases (dh_i) and (x_i) and the coordinate matrix $d\beta$ relative to the bases (dh_i) and (y_i). By Lemma 11 there is a conforming matrix Y such that $d\beta \equiv_S (d\beta')Y \equiv_S Ud\alpha Y$ and it follows that $\beta \equiv_S U\alpha Y$.

(2) clearly can be done in two steps: First we deal with U, then with Y.

(2.1) Let (U, U') be an S-pair, i.e., $US = SU'$. Then also $((U')^{\mathrm{tr}}, U^{\mathrm{tr}})$ is an S-pair. Set $V = (U')^{\mathrm{tr}} = (v_{i,j})$, thus $V' = U^{\mathrm{tr}}$. Let $\tilde{V}'$ be the endomorphism induced by the matrix V'. Then

$$\tilde{V}'(g_i) = \sum_{j=1}^{r} v'_{j,i} g_j = \sum_{j=1}^{r} v'_{j,i} \left(p^{-k_j} \sum_{s=1}^{n} \alpha_{j,s} x_s \right) = \sum_{s=1}^{n} \left(\sum_{j=1}^{r} p^{-k_j} v'_{j,i} \alpha_{j,s} \right) x_s.$$

The matrix $V^{\mathrm{tr}} S^{-1} \alpha = S^{-1}(V')^{\mathrm{tr}} \alpha = S^{-1} U \alpha$ has in a position (i,s) the entry $\sum_{j=1}^{r} p^{-k_j} v'_{j,i} \alpha_{j,s}$, and this shows that $U\alpha$ is the coordinate matrix of G with respect to the bases $(x_1, \ldots, x_n)$ of R and $(\tilde{V}'(g_1), \ldots, \tilde{V}'(g_r))$ of G modulo R.

(2.2) Using the conforming matrix $Y = [Y_{i,j}]$ we define an invertible linear transformation

$$\tilde{Y}\colon\ \mathbb{Q}R \to \mathbb{Q}R\colon\ \tilde{Y}(x_i) = \sum_{j=1}^{n} Y_{i,j} x_j.$$

Setting $p^{k_i} = \operatorname{ord}(g_i + R)$ we have by the definition of coordinate matrix that

$$g_i = p^{-k_i} \sum_{j=1}^{n} \alpha_{i,j} x_j.$$

Therefore

$$\tilde{Y}(g_i) = p^{-k_i} \sum_{j=1}^{n} \alpha_{i,j} \tilde{Y}(x_j).$$

We define

$$H := R + \langle \tilde{Y}(g_1) \rangle + \ldots + \langle \tilde{Y}(g_r) \rangle.$$

Then $(x_1, \ldots, x_n)$ is a p-basis of R by hypothesis. We claim that

(a) $H \cong_{\mathrm{nr}} G$,
(b) $\operatorname{ord}(\tilde{Y}(g_i) + R) = p^{k_i}$,
(c) R is the regulator of H,
(d) $H/R = \langle \tilde{Y}(g_1) + R \rangle \oplus \ldots \oplus \langle \tilde{Y}(g_r) + R \rangle$, and
(e) the coordinate matrix of H with respect to $(x_1, \ldots, x_n)$ and $(\tilde{Y}(g_1), \ldots, \tilde{Y}(g_r))$ is αY.

(a) By Lemma 4 there exists $d \in \mathbb{N}$ relatively prime to p such that $d\tilde{Y}(R) \subset R$. Hence also $d\tilde{Y}(G) \subset H$ and we have a monomorphism

$$d\tilde{Y}\colon\ G \to H.$$

We have $H \cong_{\mathrm{nr}} G$ if there is an integer f relatively prime to p such that $fH \subset d\tilde{Y}(G)$ ([17, Theorem 9.2.4.2]). By Lemma 3 the adjoint $\operatorname{adj}(Y)$ of Y is again conforming.

Let $\tilde{Y}_a$ be the linear transformation with matrix $\operatorname{adj}(Y)$. By Lemma 4 there is $d' \in \mathbb{N}$, relatively prime to p, such that $d'\tilde{Y}_a(R) \subset R$. We now have

$$d' \det(Y)R = d'\tilde{Y}\tilde{Y}_a(R) \subset \tilde{Y}(R) \tag{1}$$

and therefore

$$\begin{aligned} dd' \det(Y)H &= dd' \det(Y)R + d' \det(Y)(\langle d\tilde{Y}(g_1)\rangle + \ldots + \langle d\tilde{Y}(g_r)\rangle) \\ &\subset d\tilde{Y}(R) + \langle d\tilde{Y}(g_1)\rangle + \ldots + \langle d\tilde{Y}(g_r)\rangle \subset d\tilde{Y}(G). \end{aligned}$$

We have established that G and H are nearly isomorphic.

(b) and (e). We have

$$p^{k_i}\tilde{Y}(g_i) = \sum_{j=1}^{n} \alpha_{i,j}\tilde{Y}(x_j) = \sum_{j=1}^{n} \alpha_{i,j}\left(\sum_{s=1}^{n} Y_{j,s}x_s\right) = \sum_{s=1}^{n}\left(\sum_{j=1}^{n} \alpha_{i,j}Y_{j,s}\right)x_s.$$

If $p^{k_i-1}\tilde{Y}(g_i) \in R$, then $\sum_{j=1}^{n} p^{-1}\alpha_{i,j}Y_{j,s} \in S_s$. Since (x_i) is a p-basis, it follows that p is a factor of $\sum_{j=1}^{n} \alpha_{i,j}Y_{j,s}$ for every s. But these are the entries of the ith row of the matrix αY. The coordinate matrix α contains an $r \times r$ submatrix whose determinant is relatively prime to p, thus αY also contains an $r \times r$ submatrix whose determinant is relatively prime to p, and this precludes that a row of αY is divisible by p. This shows that $\operatorname{ord}(\tilde{Y}(g_i) + R) = p^{k_i}$. Our formulas also show that (e) holds provided that $(\tilde{Y}(g_1), \ldots, \tilde{Y}(g_r))$ is a basis of H modulo R.

(d) We have that $H/R = \langle\tilde{Y}(g_1) + R\rangle + \ldots + \langle\tilde{Y}(g_r) + R\rangle$ and need to show that the sum is direct. So suppose that $\sum_{i=1}^{r} m_i(\tilde{Y}(g_i) + R) = 0$ for some integers m_i. We must show that $m_i(\tilde{Y}(g_i) + R) = 0$ for every i.

Our assumption says that $\sum_{i=1}^{r} m_i\tilde{Y}(g_i) \in R$. Hence $\sum_{i=1}^{r} d' \det(Y)m_i\tilde{Y}(g_i) \in d' \det(Y)R \subset \tilde{Y}(R)$ by Formula (1). Therefore $\sum_{i=1}^{r} d' \det(Y)m_ig_i \in R$ which implies that $d' \det(Y)m_ig_i \in R$ for every i. Hence p^{k_i} divides $d' \det(Y)m_i$ and, $d' \det(Y)$ being relatively prime to p, the order p^{k_i} divides m_i. This means that $m_i(\tilde{Y}(g_i) + R) = 0$ as desired.

(c) By (b) and (d) we have that $H/R \cong G/R$. By assumption R is regulating. Then $|G/R|$ is the regulating index of G which is a near-isomorphism invariant. Hence R is a completely decomposable subgroup of H whose index in H is the regulating index. This means that R is a regulating subgroup of H and the regulator because H has a regulating regulator, this property being a near-isomorphism invariant. □

By Arnold's Theorem two near-isomorphic torsion-free groups of finite rank have, up to near-isomorphism of summands, the same decomposition properties. Hence, given a coordinate matrix we may manipulate the matrix in the ways described in Theorem 12, which means that we obtain coordinate matrices of the same group or of a nearly isomorphic group. If we arrive at a matrix that shows that the group to which it belongs decomposes or not, then the original group is decomposable or not, respectively.

We show next how one can recognize the regulator of an almost completely decomposable group.

Lemma 13 (Regulator Criterion)**.** *Let G be an almost completely decomposable group that is the finite extension of the completely decomposable R and assume that G/R is a p-group. Then the following statements are equivalent:*

(1) *R is the regulating regulator of G;*
(2) $\forall\, \tau \in T_{\mathrm{cr}}(G)\colon\ R(\tau) = G(\tau)$*;*
(3) *if α is a coordinate matrix of G with r rows and $\alpha \restriction_{\not\geq\tau}$ is the submatrix formed by the columns of α that belong to types $\not\geq \tau$, then the p-rank of $\alpha \restriction_{\not\geq\tau}$ is equal to r.*

Proof. (1) $\Longleftrightarrow$ (2). ([17, Proposition 4.5.1]).

(2) $\Longleftrightarrow$ (3). Let $(g_1, \ldots, g_r)$ be a basis of G modulo R and $(x_1, \ldots, x_n)$ a p-basis of R that come with the coordinate matrix α. Recall that $g_i = p^{-k_i}\Big(\sum_{j=1}^{n} \alpha_{i,j} x_j\Big)$ where $\operatorname{ord}(g_i + R) = p^{k_i}$.

Write $R = \bigoplus_{i=1}^{n} S_i x_i$ and set $R_{\not\geq\tau} = \bigoplus\{S_i x_i\colon\ \mathrm{tp}(S_i) \not\geq \tau\}$. Then

$$R = R(\tau) \oplus R_{\not\geq\tau} \quad \text{and} \quad G(\tau) + R = G(\tau) \oplus R_{\not\geq\tau}. \tag{2}$$

Let $x \in (G/R)[p]$. Then, for some integers λ_i,

$$x = \sum_{i=1}^{r} \lambda_i p^{k_i - 1} g_i + R = \sum_{i=1}^{r} \lambda_i p^{-1} \sum_{j=1}^{n} \alpha_{i,j} x_j + R = \sum_{j=1}^{n} \Big(p^{-1} \sum_{i=1}^{r} \lambda_i \alpha_{i,j}\Big) x_j + R. \tag{3}$$

Setting $\vec{\lambda} = [\lambda_1, \ldots, \lambda_r]$ and $\alpha_{*,j} = [\alpha_{1,j}, \ldots, \alpha_{r,j}]^{\mathrm{tr}}$, an arbitrary element $x \in (G/R)[p]$ is of the form

$$x = v + R \quad \text{where } v = \sum_{j=1}^{n} p^{-1}(\vec{\lambda}\alpha_{*,j}) x_j.$$

Let $v = \sum_{j=1}^{n} p^{-1}(\vec{\lambda}\alpha_{*,j}) x_j$. For a critical type τ, let $v = v|_{\not\geq\tau} + v|_{\geq\tau}$ where

$$v|_{\not\geq\tau} = \sum\{p^{-1}(\vec{\lambda}\alpha_{*,j}) x_j\colon\ \mathrm{tp}(S_j) \not\geq \tau\}$$

and

$$v|_{\geqslant\tau} = \sum\{p^{-1}(\vec{\lambda}\alpha_{*,j})x_j\colon\ \mathrm{tp}(S_j) \geqslant \tau\}.$$

Now we show that

$$v + R \in \frac{G(\tau)+R}{R} \Longleftrightarrow v|_{\ngeqslant\tau} \in R.$$

Suppose first that $v|_{\ngeqslant\tau} \in R$. Then $v + R = v|_{\ngeqslant\tau} + v|_{\geqslant\tau} + R = v|_{\geqslant\tau} + R \in (G(\tau)+R)/R$.

Conversely, assume that $v + R \in (G(\tau)+R)/R$. Then $v = v|_{\ngeqslant\tau} + v|_{\geqslant\tau} = y + z$ for some $y \in R_{\ngeqslant\tau}$ and some $z \in G(\tau)$ by Formula (2). Hence

$$v|_{\ngeqslant\tau} - y = z - v|_{\geqslant\tau} \in \mathbb{Q}R_{\ngeqslant\tau} \cap \mathbb{Q}R(\tau) = 0.$$

Thus $v|_{\ngeqslant\tau} = y \in R$.

Every element $x \in p^{-1}R/R$ has the form $x = \sum_{j=1}^{n} \mu_j p^{-1}x_j + R$ for integers μ_j. This enables us to obtain a well-defined homomorphism

$$\kappa\colon\ \frac{p^{-1}R}{R} \longrightarrow \Big(\frac{\mathbb{Z}}{p\mathbb{Z}}\Big)^n;\quad \kappa\Big(\sum_{j=1}^{n}\mu_j p^{-1}x_j + R\Big) = [\ldots,\overline{\mu_j},\ldots] \quad \text{where } \overline{\mu_j} = \mu_j + p\mathbb{Z}.$$

It is obvious that κ is injective.

Note that $(G/R)[p] \subset p^{-1}R/R$. Therefore κ acts on the elements of $(G/R)[p]$. Specifically we find that

$$\kappa(p^{k_i-1}g_i + R) = [\overline{\alpha_{i,1}},\ldots,\overline{\alpha_{i,n}}],\quad \kappa(v+R) = [\ldots,\vec{\lambda}\overline{\alpha_{*,j}},\ldots].$$

We observe that the rows of α are linearly independent modulo p because κ is injective and the $p^{k_i-1}g_i + R$ are linearly independent in $(G/R)[p]$.

Let $\alpha_{\ngeqslant\tau}$ be the submatrix of α with columns $\alpha_{*,j}$ such that $\mathrm{tp}(S_j) \ngeqslant \tau$ and let $\alpha_{\geqslant\tau}$ be the submatrix of α with columns $\alpha_{*,j}$ such that $\mathrm{tp}(S_j) \geqslant \tau$. Then

$$\kappa(v\,|_{\ngeqslant\tau} + R) = \vec{\lambda}\overline{\alpha_{\ngeqslant\tau}} \quad\text{and}\quad \kappa(v\,|_{\geqslant\tau} + R) = \vec{\lambda}\overline{\alpha_{\geqslant\tau}}.$$

Finally, we have

$$\begin{aligned} G(\tau) = R(\tau) &\Longleftrightarrow \frac{G(\tau)+R}{R}[p] = 0 \\ &\Longleftrightarrow \forall\, v = \sum_{j=1}^{n} p^{-1}(\vec{\lambda}\alpha_{*,j})x_j\colon\ (v|_{\ngeqslant\tau} \in R \Rightarrow v \in R) \\ &\Longleftrightarrow \forall\,\vec{\lambda}\colon\ (\vec{\lambda}\overline{\alpha_{\ngeqslant\tau}} = 0 \Rightarrow \vec{\lambda}\overline{\alpha} = 0). \end{aligned}$$

This is the case if and only if the rows of $\alpha_{\ngeqslant\tau}$ are linearly independent modulo p. $\square$

We want a reduced form for coordinate matrices and introduce some necessary notation. The term *line* means a row or a column. An integer u is a *p-unit* if $\gcd(p, u) = 1$. If so, for any integer $k > 0$ there is $u' \in \mathbb{Z}$ such that $uu' \equiv 1 \bmod p^k$. Often, we simply say “unit” in place of p-unit because there are no other units in use.

It is convenient to allow a matrix B to be of size $0 \times n$ (*to have no rows*) or of size $r \times 0$ (*to have no columns*) or of size 0×0 (*to have no lines*). For instance, let $M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ be a block matrix. If B has no rows, then $M = [A, 0]$. If B has no columns, then $M = \begin{bmatrix} A \\ 0 \end{bmatrix}$. If B has no lines, then $M = A$. We say that a matrix B is *absent* or *missing* if B has either no rows or no columns or both, and we say that B *appears* if it has rows and columns.

A diagonal matrix $S = \operatorname{diag}(p^{k_1}, \dots, p^{k_r})$ with natural numbers k_i is called a *structure matrix.* If $k_1 \geqslant \dots \geqslant k_r \geqslant 1$, then S is called an *ordered structure matrix.*

Let $A = [a_{i,j}]$ be an integer $r \times n$ matrix and let $S = \operatorname{diag}(p^{k_1}, \dots, p^{k_r})$ be a structure matrix. We extend an S-congruence to the entries of A by defining: $a_{i,j}$ is *S-congruent to a*, denoted as $a_{i,j} \equiv_S a$, if $a_{i,j} \equiv a \bmod p^{k_i}$.

A matrix is *decomposed* if it is of the form $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Here either one of the matrices A, B is allowed to have no rows or no columns, i.e., the decomposed matrices include the special cases

$$[0\ B], \quad \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad [A\ 0], \quad \begin{bmatrix} A \\ 0 \end{bmatrix}.$$

A matrix is *properly decomposed* if the blocks A, B both have rows and columns.

A matrix A is called *decomposable* if there are row and column permutations that transform it to a decomposed form, i.e., there are permutation matrices P, Q such that PAQ is decomposed. Similarly to the above we use the term *properly decomposable.*

A matrix is *S-decomposed* or *S-decomposable* if it is S-congruent to a decomposed or decomposable matrix, respectively. Note that A is S-decomposable if there are permutation matrices P, Q such that PAQ is PSP^{-1}-decomposed.

Let $A = [a_{i,j}]$ be an integer matrix and let S be an ordered structure matrix. Then A is called *S-reduced* if

(1) modulo p the matrix A has at most one entry $\neq 0$ in a line,
(2) if the nonzero entries of $A \bmod p$ are at the positions (i_s, j_s), then $a_{i_s,j} \equiv_S 0$ for all $j > j_s$ and $a_{i,j_s} \equiv_S 0$ for all $i > i_s$, and $a_{i_s,j}, a_{i,j_s} \in p\mathbb{Z}$ for all $j < j_s$ and all $i < i_s$.

If $S = p^s I$ we say *p^s-reduced* instead of S-reduced. Note that in an S-reduced matrix, the entries to the left of a unit are in $p\mathbb{Z}$ and the entries above a unit are in $p\mathbb{Z}$. In the group situation a coordinate matrix and an ordered structure matrix are given provided that the basis elements of the regulator quotient have non-increasing order.

A row or column transformation of a matrix is equivalent to the left or right multiplication by a corresponding matrix, respectively. We use both approaches simultaneously, the context clarifying what is meant. Often we use elementary row transformations that add a multiple of a row to a row below which is equivalent to a left multiplication by some lower triangular elementary matrix, and elementary column transformations that add a multiple of a column to a column to the right which is equivalent to a right multiplication by some upper triangular elementary matrix.

Lemma 14. *Let A be an $r \times n$ integer matrix and S an ordered structure matrix. Then there are two p-invertible matrices U, Y with the following properties.*

(1) *U is a product of lower triangular elementary matrices, where each elementary factor annihilates an entry $\not\equiv_S 0$,*
(2) *Y is a product of upper triangular elementary matrices, where each elementary factor annihilates an entry $\not\equiv_S 0$,*

such that UAY is S-reduced.

In particular, if the ith line of A is $\equiv_S 0$, then the ith line of UAY is $\equiv_S 0$.

P r o o f. Let $A = [a_{i,j}]$. We proceed by induction on the number of columns of A. Suppose that A has a single column. If $A \equiv 0 \bmod p$, then we take U and Y to be the identity matrices and the claims are trivially true. So suppose that A contains entries that are units. Let i_0 be the least index such that $a_{i_0,1}$ is a unit. By elementary row transformations this unit may be used to annihilate the entries $\not\equiv_S 0$ below, since the exponents k_i are decreasing, and this amounts to left multiplication of A by a product U of lower triangular elementary matrices. Note that the entries $\equiv_S 0$ are left unchanged. Thus $M = UA$ is S-reduced and 0-entries do not change.

Now suppose that A has more than one column. If the first column $a_{*,1}$ of A is congruent to 0 modulo p, then the induction hypothesis applied to the matrix obtained by omitting the first column immediately gives the result. Hence assume that A has a unit in the first column. We consider the unit in the first column with the least row index i_0. With the unit $a_{i_0,1}$ we annihilate all the other entries $\not\equiv_S 0$ in the i_0th row, which amounts to right multiplication by a product of upper triangular matrices. Note that above $a_{i_0,1}$ the entries are in $p\mathbb{Z}$. Deleting the first column we obtain a submatrix A' with fewer columns. So by the induction hypothesis we may assume that there are matrices U' and Y' such that $U'A'Y'$ has the required properties. The elementary transformations that involve U' and Y' can be applied to the full matrix A. The column transformations do not affect the first column at all, while the row transformations may be assumed not to change the row i_0, and they do not introduce units in the first column above i_0. Finally, by elementary row transformations the unit $a_{i_0,1}$ may be used to annihilate the elements $\not\equiv_S 0$ below, since the exponents k_i are decreasing, and without changing anything modulo S in the columns beyond the first. This shows (1) and (2).

In particular, if the ith row of A is 0 modulo p^k, then no column transformation changes this fact, and since in a 0-row modulo S there is nothing to annihilate, the elementary row transformations used do not change this 0-row, either. An analogous argument works for 0-columns. $\square$

According to Lemma 14 the matrix UAY is S-reduced; it is called an *S-reduced form* of A.

There are special configurations in a coordinate matrix that are important. Let $A = [a_{i,j}]$ be an integer matrix and S a structure matrix. The matrix A has a *cross at* (i_0, j_0) if $a_{i_0,j_0} \not\equiv_S 0$ and $a_{i_0,j} \equiv_S 0$, $a_{i,j_0} \equiv_S 0$ for all $i \neq i_0$ and $j \neq j_0$. We say that the cross is *located* in a sub-block of a matrix if the position (i_0, j_0) is in this sub-block.

An integer matrix $A = [a_{i,j}]$ has a (*horizontal*) double cross at $(i_0, j_1)|(i_0, j_2)$ where $j_1 \neq j_2$, if $a_{i_0,j_1} \not\equiv_S 0$, $a_{i_0,j_2} \not\equiv_S 0$, $a_{i_0,j} \equiv_S 0$ for all $j \notin \{j_1, j_2\}$, and $a_{i,j_1} \equiv_S 0$, $a_{i,j_2} \equiv_S 0$ for all $i \neq i_0$.

Similarly, we define a (*vertical*) double cross at $(i_1, j_0)|(i_2, j_0)$. Note that a matrix with a cross or a double cross is S-decomposable.

It is convenient to call an integer $r \times n$ matrix $D = [d_{i,j}]$ *p-diagonal* if all entries $d_{i,j} = 0$ for $i \neq j$ and the diagonal entries are p-powers or 0, i.e., $d_{i,i} = p^{s_i}$ for nonnegative integers s_i, or $d_{i,i} = 0$.

We continue this section with the well-known Smith Normal Form (in German: Elementarteilersatz) and a modification thereof that will be heavily in use later.

Smith Normal Form ([15, Chapter 3.7, Theorem 3.8 and 3.9])**.** Let H be a nonsingular, integer $k \times k$ matrix. Then there exist invertible integer matrices U, Y such that

$$UHY = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & d_k \end{bmatrix}$$

where d_i are positive integers and d_i divides d_{i+1} for $i = 1, \dots, k-1$. The numbers d_i are uniquely determined by H.

Lemma 15. *Let l, r, n, k be natural numbers where $1 \leqslant l \leqslant r$ and $n \geqslant 1$. For an $r \times n$ integer matrix H there are p-invertible matrices Y and U where $U = \begin{bmatrix} U_1 & pU_2 \\ U_3 & U_4 \end{bmatrix}$ and U_1 is a (p-invertible) $l \times l$ matrix such that UHY is congruent modulo p^k to $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ where A, B are p-diagonal with l and $r-l$ rows, respectively.*

If $l = r$, then B has no rows and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = [\,A \quad 0\,]$; if A has no columns, then $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 \\ B \end{bmatrix}$; if B has no lines then $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = [A]$.

P r o o f. (a) The matrices U and Y are obtained as products of elementary matrices. Any elementary column transformation is allowed but the special form of the matrix U restricts the row transformations that are allowed. Any multiple of the first l rows may be added to another row and any row transformation between the last $r-l$ rows is allowed, while only multiples of p-folds of the last $r-l$ rows may be added to one of the first l rows.

We also can change matrices modulo p^k. This has the effect that we can multiply a given matrix by diagonal matrices with determinants relatively prime to p either from the left or the right. In particular, any row or column may be multiplied by a unit modulo p^k. This will be used to obtain pure p-powers at certain places.

(b) If $l = r$, then $U = U_1$, arbitrary row and column transformations are allowed and we obtain the Smith Normal Form, i.e., there are integer matrices U, Y with determinant ± 1 such that $A = UHY$ is a matrix with nonzero entries only on the diagonal. Multiplying by a suitable p-invertible diagonal matrix we obtain a p-diagonal matrix of size $r \times n$ modulo p^k.

(c) Let $l < r$, let $h_1, h_2 < k$ be nonzero integers. Let the j_0th-column of the $r \times n$ matrix H have entries $a_{i_1,j_0} \in p^{h_1}\mathbb{Z} \setminus p^{h_1+1}\mathbb{Z}$ and $a_{i_2,j_0} \in p^{h_2}\mathbb{Z} \setminus p^{h_2+1}\mathbb{Z}$ with row indices $i_1 \leqslant l$ and $i_2 > l$. Then, modulo p^k, annihilation of either a_{i_1,j_0} or a_{i_2,j_0} is possible.

In particular, if there is no annihilation possible in a column of H, then either all entries with row index $\leqslant l$ are 0 modulo p^k or all entries with row index $> l$ are 0 modulo p^k.

(d) We use induction on $r+n$ and start with $r+n = 2$. Then $r = n = 1$ and H has the claimed form. We assume the statement to be correct for $r+n \leqslant m-1 \geqslant 3$. Now let $r+n = m$. If $l = r$, then we get the claimed form for H, by (b). If $1 \leqslant l < r$, then $H = \begin{bmatrix} H_u \\ H_d \end{bmatrix}$ where H_u is an $l \times n$ matrix, the upper part of H, and H_d is the lower part. There are two cases, either H_u has a column that allows to annihilate downward in this column of H_d, or not.

Case 1. If H_u has a column that allows to annihilate downward the elements of H_d in this column, then, by (b), there are p-invertible matrices U_u, Y such that U_uH_uY is p-diagonal, and we may additionally arrange rows and columns to get $U_uH_uY = \begin{bmatrix} H_u^1 & 0 & 0 \\ 0 & H_u^2 & 0 \end{bmatrix}$ where H_u^1, H_u^2 are p-diagonal and each column of H_u^1 allows to annihilate the column of H_d below, and no column of H_u^2 allows to annihilate the column of H_d below. Moreover, we may assume that H_u^1, H_u^2 have no 0-columns. The $r' \times n'$ matrix H_u^1 has at least one column, i.e., $n' \geqslant 1$. We annihilate with H_u^1 in H_d and get

$$\begin{bmatrix} U_u & 0 \\ 0 & I_{r-l} \end{bmatrix} \cdot H \cdot Y = \left[\begin{array}{ccc} H_u^1 & 0 & 0 \\ 0 & H_u^2 & 0 \\ \hline 0 & H_d^1 & H_d^2 \end{array}\right].$$

Note that H_u^2 may not be present. But this is a simplification that is covered by the following argument. If H_u^2 is present, then $\left[\begin{array}{cc} H_u^2 & 0 \\ \hline H_d^1 & H_d^2 \end{array}\right]$ is an $(r-r') \times (n-n')$ matrix and $n' \geqslant 1$. By induction hypothesis there are p-invertible matrices $U' = \begin{bmatrix} U_1' & pU_2' \\ U_3' & U_4' \end{bmatrix}$ and Y' such that $U' \cdot \left[\begin{array}{cc} H_u^2 & 0 \\ \hline H_d^1 & H_d^2 \end{array}\right] \cdot Y' = \left[\begin{array}{cc} H_u' & 0 \\ \hline 0 & H_d' \end{array}\right]$ where H_u', H_d' are p-diagonal. We want to express explicitly that all matrices that multiply from the left are in accordance with the hypothesis. In fact

$$\begin{bmatrix} I_{r'} & 0 & 0 \\ 0 & U_1' & pU_2' \\ 0 & U_3' & U_4' \end{bmatrix} \cdot \begin{bmatrix} U_u & 0 \\ 0 & I_{r-l} \end{bmatrix} \cdot H \cdot Y \cdot \begin{bmatrix} I_{n'} & 0 \\ 0 & Y' \end{bmatrix} = \left[\begin{array}{ccc} H_u^1 & 0 & 0 \\ 0 & H_u' & 0 \\ \hline 0 & 0 & H_d' \end{array}\right]$$

as desired.

Case 2. If H_u has no column that allows to annihilate downward the elements of H_d in this column, then by (c) this means that either there is a 0-matrix below the nonzero columns of H_u or there is a column of H_d that allows to annihilate upward the elements of H_u in this column. If there are only 0-columns on H_d below the nonzero columns in H_u then we continue as in Case 1 skipping the unnecessary annihilation in H_d. Otherwise we continue with H_d as before with H_u in Case 1 and obtain mutatis mutandis the same result. This finally proves the claim. $\square$

We verify next that certain row and column transformations of an S-decomposable matrix keep the matrix S-decomposable.

Lemma 16. *Let A be an integer matrix and S a structure matrix. Let U be a matrix describing a product of elementary row transformations such that each elementary factor annihilates an entry $\not\equiv_S 0$. Let Y be a matrix describing a product* *of elementary column transformations such that each elementary factor annihilates an entry* $\not\equiv_S 0$.

If A *is* S*-decomposable, i.e.,* $PAQ \equiv_{PSP^{-1}} \operatorname{diag}(W_i|i)$ *for permutation matrices* P, Q, *then*

$$P(UAY)Q \equiv_{PSP^{-1}} \operatorname{diag}(W_i' \mid i),$$

and for all i *the blocks* W_i, W_i' *have the same size. The* PSP^{-1}*-decomposition of* $P(UAY)Q$ *is a refinement of the* PSP^{-1}*-decomposition of* PAQ.

In particular, if A *is* S*-decomposable, then also* UAY *is* S*-decomposable. If* A *has a cross, then also* UAY *has a cross at the same location. If* A *has a double cross, then* UAY *has a double cross at the same line or a cross with location at the line of the double cross.*

P r o o f. We show the statement for a single elementary row transformation U. Using transposition we obtain the same result for elementary column transformations. Clearly, the result also holds for products of such elementary transformations.

Let $A = [a_{i,j}]$ and let $U = I + cE_{i,j}$ where $E_{i,j}$ is the usual matrix unit with 1 at location (i, j) and 0 elsewhere. So U adds the c-fold of the jth row of A to the ith row of A. Since U annihilates an entry $\not\equiv_S 0$ there is a column index j_0 with $a_{i,j_0} \not\equiv_S 0$ such that $ca_{j,j_0} + a_{i,j_0} \equiv_S 0$. Hence also $a_{j,j_0} \not\equiv_S 0$. Since A is S-decomposable, i.e., $PAQ \equiv_{PSP^{-1}} \operatorname{diag}(W_i|i)$, the nonzero entries a_{i,j_0}, a_{j,j_0} must appear in a column of one and the same block, say W_1. We may even assume that P permutes the jth row to position 1 and the ith row to position 2. So $PUP^{-1} = I + cE_{2,1}$, i.e., PUP^{-1} adds the c-fold of the first row of PAQ to the second row of PAQ. Thus we have

$$\begin{aligned} P(UA)Q \equiv_{PSP^{-1}} (PUP^{-1})(PAQ) &\equiv_{PSP^{-1}} (PUP^{-1}) \operatorname{diag}(W_i \mid i) \\ &\equiv_{PSP^{-1}} \operatorname{diag}(W_i' \mid i) \end{aligned}$$

where PUP^{-1} is an elementary row transformation in W_1 only, i.e., the block W_1 changes to the block W_1', and all other blocks are unchanged. Even for an S-indecomposable W_1 the matrix W_1' might now be S-decomposable. This shows the statement for UA, and hence for UAY, including that an S-decomposition of UAY is a refinement of an S-decomposition of A.

In particular, a cross or a double cross leads to special S-decompositions of a matrix. A cross cannot be refined. But a double cross is possibly refined to a cross. $\square$

For an ordered structure matrix S the S-decomposability of an integer matrix is inherited by its S-reduced forms.

Corollary 17. *Let A be an integer matrix and S an ordered structure matrix. If A is S-decomposable, then the S-reduced forms are S-decomposable. More precisely, if for permutation matrices P, Q the matrix PAQ is S-decomposed, and if B is an S-reduced form of A, then PBQ is also S-decomposed, and this S-decomposition is possibly finer than the S-decomposition of PAQ.*

In particular, if the matrix A has a 0-line modulo S or a cross, then an S-reduced form of A has a 0-line modulo S or a cross at the same position, respectively. If the matrix A has a double cross, then its S-reduced form has a double cross or a cross at the same position.

Proof. Combining Lemmata 14 and 16, we obtain that the S-decomposability of A is inherited by its S-reduced forms and the S-decomposition of the S-reduced forms is possibly finer. □

A $(0,1)$-*matrix* has only the entries 0, 1. Let l, h be natural numbers. Let $M = [M_{i,j}]_{1\leqslant i\leqslant h, 1\leqslant j\leqslant l}$ be a block matrix over some set containing the symbol 0. The $(0,1)$-matrix $C(M) = [c_{i,j}]$ of size $h \times l$ is called the *connection matrix* of M if $c_{i,j} = 1$ provided the block $M_{i,j}$ has no 0-lines, and $c_{i,j} = 0$ otherwise. Note that if all blocks $M_{i,j}$ are 1×1 matrices, i.e., the entries of M, then the connection matrix $C(M)$ is the so called $(0,1)$-*pattern* of M. That is, the nonzero entries are replaced by 1. In particular, in this case M is decomposable if and only if $C(M)$ is decomposable.

Let $M = [M_{i,j}]$ be a block matrix. Let all blocks $M_{i,j} = \begin{bmatrix} X_{i,j} & 0 \\ 0 & Z_{i,j} \end{bmatrix}$ be decomposed, where the submatrices $X_{i,j}$, $Z_{i,j}$ both have rows and columns. The block matrix M is said to have a *compatible decomposition* if for all i, j all $X_{i,1}, X_{i,2}, \ldots$ have the same number of rows and all $X_{1,j}, X_{2,j}, \ldots$ have the same number of columns. Note that then automatically the submatrices $Z_{i,j}$ also have the same numbers of rows along a block row and the same number of columns along a block column.

Clearly, a block matrix with compatible decomposition is decomposable. Lemma 18 provides a partial converse.

Lemma 18. *Let l, h be natural numbers. Let $M = [M_{i,j}]_{1\leqslant i\leqslant h,\, 1\leqslant j\leqslant l}$ be a block matrix over some set containing the symbol 0, with all the blocks $M_{i,j}$ having at least two rows and two columns.*

Assume that the connection matrix $C(M)$ is indecomposable and that M is decomposable. Then there are permutation matrices $P_1, \ldots, P_h, Q_1, \ldots, Q_l$ such that

$$M' = [M'_{i,j}] = \operatorname{diag}(P_1, \ldots, P_h)[M_{i,j}]_{1\leqslant i\leqslant h,\, 1\leqslant j\leqslant l} \operatorname{diag}(Q_1, \ldots, Q_l)$$

where $M'_{i,j} = P_i M_{i,j} Q_j = \begin{bmatrix} X_{i,j} & 0 \\ 0 & Z_{i,j} \end{bmatrix}$, and this is a compatible decomposition of M'.

P r o o f. There are permutation matrices H, K such that $HMK = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is decomposed. The matrix H permutes the rows of M, i.e., the row of M with index i is moved to the new row with index i' of $\operatorname{diag}(A, B)$. Clearly, the entries of the i'th row of HM are then permuted by the permutation K of the columns. Briefly we use the term "the ith row of M belongs to $[A, 0]$" to indicate that H moves the ith row of M so that this is now a row of $[A, 0]$, after permutation of the entries by K.

We use the following assertion without explicit reference:

If $M = [m_{i,j}]$, then the entry $m_{i,j} = 0$ if the ith row belongs to $[A, 0]$ and the jth column belongs to $\begin{bmatrix} 0 \\ B \end{bmatrix}$. Similarly, $m_{i,j} = 0$ if the ith row belongs to $[0, B]$ and the jth column belongs to $\begin{bmatrix} A \\ 0 \end{bmatrix}$.

Let i_0 be fixed. By way of contradiction assume that all rows of the block row $[M_{i_0,1}, \dots, M_{i_0,l}]$ belong to $[A, 0]$. There is a column permutation T of M that permutes the columns of each $M_{i_0,j}$ separately, i.e., $T = \operatorname{diag}(T_j \mid j)$ such that $M_{i_0,j}T_j = [M'_{i_0,j} | M''_{i_0,j}]$ where the columns are reordered so that the columns of $M'_{i_0,j}$ belong to $\begin{bmatrix} A \\ 0 \end{bmatrix}$ and the columns of $M''_{i_0,j}$ belong to $\begin{bmatrix} 0 \\ B \end{bmatrix}$. But then $M''_{i_0,j} = 0$.

The connection matrix $C(M)$ is indecomposable, hence also the connection matrix $C(MT)$ is indecomposable, since T permutes only inside of the blocks. In particular, $C(TM)$ has no 0-line. Thus, there must be a j_0 such that the matrix M_{i_0,j_0} is up to line permutations completely contained in A. By permutations of block rows and block columns we may assume $i_0 = j_0 = 1$. Moreover, we may assume that the matrices $[M_{1,1}, \dots, M_{1,s}]$ completely belong to A (both rows and columns) and the remaining $M_{1,j}$, $j > s$, have at least one column in B. Clearly, this changes M but we still use the letter M. Observe that this forces the corresponding first row of the connection matrix $C(M)$ to be $[\underbrace{*, \dots, *}_{s}, 0, \dots, 0]$.

There may be more block rows $[M_{i,1}, \dots, M_{i,l}]$ that completely belong to $[A, 0]$, say those starting with $M_{1,1}, \dots, M_{t,1}$, and all other block rows have at least one row in $[0, B]$. Hence each block $M_{i,j}$ has at least one 0-line if either $i \leqslant t$ and $j > s$, or $i > t$ and $j \leqslant s$. Translated to the connection matrix this means $C(M) = \begin{bmatrix} D & 0 \\ 0 & E \end{bmatrix}$ where D is of size $t \times s$. This is a decomposition of $C(M)$, a contradiction. Thus, all rows of a block row $[M_{i,1}, \dots, M_{i,l}]$ never belong to $[A, 0]$, and by symmetry all rows never belong to $[0, B]$. This conclusion holds also for columns. In other words, each block $M_{i,j}$ has at least one row that belongs to $[A, 0]$ and at least one row that belongs to $[0, B]$, analogously for the columns.

For each i there is a permutation of the rows of the block row $[M_{i,1}, \dots, M_{i,l}]$ such that the upper rows belong to $[A, 0]$, and the lower rows belong to $[0, B]$. For each j

there is a permutation of the columns of the block column $\begin{bmatrix} M_{1,j} \\ \vdots \\ M_{h,j} \end{bmatrix}$ such that the left columns belong to $\begin{bmatrix} A \\ 0 \end{bmatrix}$, and the right columns belong to $\begin{bmatrix} 0 \\ B \end{bmatrix}$. Hence, we get a block matrix $M' = [M'_{i,j}]$ such that each block $M'_{i,j}$ has the same rows and the same columns as $M_{i,j}$. This amounts to the transformation

$$M' = [M'_{i,j}] = \operatorname{diag}(P_1, \ldots, P_h)[M_{i,j}]_{1\leqslant i\leqslant h, 1\leqslant j\leqslant l} \operatorname{diag}(Q_1, \ldots, Q_l)$$

with permutation matrices $P_1, \ldots, P_h, Q_1, \ldots, Q_l$, since we have always permuted whole lines of the matrix M. This decomposition $\operatorname{diag}(A, B)$ induces decompositions of all blocks $M'_{i,j}$ in the form $M'_{i,j} = P_i M_{i,j} Q_j = \begin{bmatrix} X_{i,j} & 0 \\ 0 & Z_{i,j} \end{bmatrix}$. All blocks $M_{i,j}$ share rows and columns with both A and B, consequently $X_{i,j}$, $Z_{i,j}$ both have rows and columns. Moreover, the numbers of rows of $X_{i,j}$ along a block row are equal, and along a block column the numbers of columns of $X_{i,j}$ are equal. So, this is a compatible decomposition. □

5. Direct decomposition and coordinate matrices

We are mainly interested in direct decompositions of our groups. Lemma 19 clarifies how the decomposability of an almost completely decomposable group appears in coordinate matrices.

A group G is *decomposable* if $G = G_1 \oplus G_2$ for some $G_1 \neq 0 \neq G_2$ and *indecomposable* otherwise. A group is *clipped* if it has no completely decomposable direct summands.

Lemma 19. *A clipped, p-reduced, p-local almost completely decomposable group G with regulating regulator R is directly decomposable if and only if it has a properly decomposable coordinate matrix.*

Proof. Suppose that $G = G_1 \oplus G_2$ is a decomposition with $G_1 \neq 0 \neq G_2$. The regulating regulator R of G is of the form $R = R_1 \oplus R_2$ where R_1 is the regulating regulator of G_1 and R_2 is the regulating regulator of G_2. Also $G/R = (G_1 + R)/R \oplus (G_2 + R)/R \cong (G_1/R_1) \oplus (G_2/R_2)$ and it is clear that we can choose a basis $(g_1, \ldots, g_r)$ of G modulo R such that $g_1, \ldots, g_{r'} \in G_1$ and $g_{r'+1}, \ldots, g_r \in G_2$, and a p-basis $(x_1, \ldots, x_n)$ such that $(x_1, \ldots, x_{n'})$ is a p-basis of R_1 and $(x_{n'+1}, \ldots, x_n)$ is a p-basis of R_2. The coordinate matrix obtained as in Lemma 6 from these bases is of the form $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, i.e., properly decomposed.

Conversely, suppose that the $r \times n$ coordinate matrix α of G is properly decomposable. Then there are permutation matrices P, Q such that $\alpha' = P\alpha Q = \operatorname{diag}(A, B)$ where A has size $r_1 \times n_1$ with $1 \leqslant r_1 < r$ and $1 \leqslant n_1 < n$. The coordinate matrix α comes with a p-basis $(x_1, \ldots, x_n)$ of R and a basis $(g_1, \ldots, g_r)$ of G modulo R, and these data define the structure matrix $S = \operatorname{diag}(\operatorname{ord}(g_1 + R), \ldots, \operatorname{ord}(g_r + R))$. The definition of the coordinate matrix in matrix form is

$$S \begin{bmatrix} g_1 \\ \vdots \\ g_r \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

hence

$$(PSP^{-1}) \left(P \begin{bmatrix} g_1 \\ \vdots \\ g_r \end{bmatrix} \right) = (P\alpha Q) \left(Q^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \left(Q^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right).$$

We have new bases $P \begin{bmatrix} g_1 \\ \vdots \\ g_r \end{bmatrix}$ and $Q^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix}$ that are just rearrangements of the original ones. We observe that PSP^{-1} is the structure matrix that belongs to the new basis $P \begin{bmatrix} g_1 \\ \vdots \\ g_r \end{bmatrix}$ of G modulo R, which is easy because it suffices to check what happens if P is a two-cycle. We now conclude that $\alpha' = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is the coordinate matrix with respect to the new bases, and it follows immediately that $G = \langle x'_1, \ldots, x'_{n_1} \rangle_*^G \oplus \langle x'_{n_1+1}, \ldots, x'_n \rangle_*^G$. $\square$

Note that, if the coordinate matrix of a group G with structure matrix S is S-decomposable, then replacing the entries that are $\equiv_S 0$ by 0 we obtain again a coordinate matrix of G which is now decomposable.

Combining Theorem 12 and Lemma 19 we obtain the following corollary.

Corollary 20. *A p-reduced, p-local almost completely decomposable group G with regulating regulator R, coordinate matrix α, and the corresponding structure matrix S, is decomposable if and only if there is a first component U of an S-pair and a conforming matrix Y such that $U\alpha Y$ is S-decomposable.*

Proof. Suppose that G is decomposable. Then G has a decomposable coordinate matrix α' by Lemma 19, and by Theorem 12 there exist a first component U of an S-pair and a conforming matrix Y such that $\alpha' \equiv_S U\alpha Y$, i.e., it is S-decomposable.

Conversely, suppose that $\alpha' = U\alpha Y$ is S-decomposable. Then Theorem 12 implies that α' is the coordinate matrix of a group G' near-isomorphic to G. By Lemma 10 we may assume that G' has a coordinate matrix $\alpha'' \equiv_S \alpha'$ where α'' is decomposable. So the group G' is decomposable by Lemma 19, and by Arnold's Theorem, G itself is decomposable. $\square$

In the sequel the left multiplication of a coordinate matrix by the first component U of an S-pair is realized by a sequence of row transformations and the right multiplication by a conforming matrix Y is realized by a sequence of column transformations. However, due to the required structure of the matrices U and Y that are allowed as multipliers, only certain special row and column transformations are allowed as follows.

Lemma 21. *Suppose that the basis elements $(\gamma_1, \dots, \gamma_r)$ of G/R have orders $\operatorname{ord}(\gamma_i) = p^{k_i}$ with $k_1 \geqslant k_2 \geqslant \dots \geqslant k_r \geqslant 1$. Then the following row operations on a coordinate matrix are permitted:*

(1) *Any multiple of a row may be added to any row below it.*
(2) *Any multiple of the $p^{k_{i_1}-k_{i_2}}$-fold of row i_2 may be added to a row $i_1 < i_2$.*
(3) *Any row may be multiplied by an integer relatively prime to p.*

The permitted column transformations on a coordinate matrix depend on the poset of critical types and will be described later in the special cases that we consider.

6. (1,3)-groups

A $(1,3)$-*group* G is a p-local, p-reduced almost completely decomposable group with critical typeset $T_{\mathrm{cr}}(G) = \{\tau_0, \tau_1, \tau_2, \tau_3\}$ where τ_0 is incomparable with τ_1, τ_2, τ_3 and $\tau_1 < \tau_2 < \tau_3$. As $T_{\mathrm{cr}}(G)$ is $\vee$-free, any $(1,3)$-group has a regulating regulator.

Standard setting for (1,3)-groups. Let G be a $(1,3)$-group with regulator $R = R_0 \oplus R_1 \oplus R_2 \oplus R_3$ where R_i is homogeneous completely decomposable of rank $r_i \geqslant 1$ and type τ_i. In particular, $n = \operatorname{rank} G = r_0 + r_1 + r_2 + r_3$.

Let $\alpha = [\alpha_{i,j}]$ be the coordinate matrix of G. We may assume that a p-basis $(x_1, \dots, x_n)$ of R is so chosen that $(x_1, \dots, x_{r_0})$ is a p-basis of R_0, $(x_{r_0+1}, \dots, x_{r_0+r_1})$ is a p-basis of R_1, $(x_{r_0+r_1+1}, \dots, x_{r_0+r_1+r_2})$ is a p-basis of R_2, and $(x_{r_0+r_1+r_2+1}, \dots, x_{r_0+r_1+r_2+r_3})$ is a p-basis of R_3. This divides the coordinate matrix in four blocks α_0, β_1, β_2, β_3 of sizes $r \times r_i$, $i = 0, 1, 2, 3$ and we have $\alpha = [\alpha_0 \mid \beta_1 \mid \beta_2 \mid \beta_3]$. The matrix $\beta = [\beta_1 \mid \beta_2 \mid \beta_3]$ is called the β-*part* of the coordinate matrix.

It is usually convenient and at places crucial that also the generators of G/R of equal orders are grouped together. Let $G/R \cong \bigoplus_{h=1}^{f} (\mathbb{Z}_{p^{k_h}})^{l_h}$ where $l_h \geqslant 1$ and

$k = k_1 > k_2 > \ldots > k_f \geqslant 1$. Then $S = \operatorname{diag}(p^{k_1} I_{l_1}, \ldots, p^{k_f} I_{l_f})$ is the (ordered) structure matrix of size $r = \sum_{h=1}^{f} l_h$.

We will use throughout these orderings of the p-basis of R and the basis of G/R. The ordering of the basis of G/R has effects for S-pairs. Let (U, U') be an S-pair. The integers l_h define a block structure on U, namely, $U = [U_{h,m}]_{1 \leqslant h,m \leqslant f}$, where the block $U_{h,m}$ is an $l_h \times l_m$ matrix. A p-invertible $r \times r$ block matrix $U = [U_{h,m}]_{1 \leqslant h,m \leqslant f}$ is the first component of an S-pair (U, U') if and only if all entries of the block $U_{h,m}$ are divisible by $p^{k_h - k_m}$ for $h \leqslant m$. In particular, all p-invertible lower block triangular matrices U, i.e., $U_{h,m} = 0$ for all $h < m$, serve as first components of S-pairs (U, U').

Recall that $T_{\mathrm{cr}}(G) = (\tau_0, \tau_1 < \tau_2 < \tau_3)$. The ordering of the columns of a coordinate matrix corresponding to the ordering of the p-basis of R has effects for conforming matrices. An integer $n \times n$ block matrix $Y = [Y_{i,j}]$ is conforming with G (see Example 5) if and only if it has the form

$$Y = \begin{bmatrix} Y_{0,0} & 0 & 0 & 0 \\ 0 & Y_{1,1} & Y_{1,2} & Y_{1,3} \\ 0 & 0 & Y_{2,2} & Y_{2,3} \\ 0 & 0 & 0 & Y_{3,3} \end{bmatrix} \tag{4}$$

where $Y_{i,j}$ is an $r_i \times r_j$ integer matrix and the diagonal blocks $Y_{i,i}$ are p-invertible.

According to the block structure of coordinate matrices, induced by the ordering of the types and the ordering of the basis of the regulator quotient, elementary row transformations may be performed with whole blocks following the same rules that apply to single rows.

In the following "group" means $(1,3)$-group and we tacitly assume the conventions of the standard setting. Specifically, we assume that $S = \operatorname{diag}(p^{k_1} I_{l_1}, \ldots, p^{k_f} I_{l_f})$ is the ordered structure matrix as block matrix with $k_1 > \ldots > k_f \geqslant 1$, and the conforming matrices Y are upper block triangular matrices as above. We use the term *standard coordinate matrix* $\alpha = [\alpha_0 \mid \beta_1 \mid \beta_2 \mid \beta_3]$ to correspond to the ordering of the p-basis of R.

We state the Regulator Criterion, Lemma 13, in the special case of $(1,3)$-groups.

Lemma 22. *Let G be a $(1,3)$-group. Then G has a regulating regulator. The completely decomposable subgroup R of finite index in G is the regulator of G if and only if R_0 and $R_1 \oplus R_2 \oplus R_3$ are pure in G, and this holds if and only if α_0 and the β-part of a coordinate matrix $\alpha = [\alpha_0 \mid \beta_1 \mid \beta_2 \mid \beta_3]$, relative to some p-basis of R (ordered as above), both have p-rank r.*

Corollary 23. *Let G be a $(1,3)$-group with regulator R and $G/R \cong \bigoplus\limits_{h=1}^{f} (\mathbb{Z}_{p^{k_h}})^{l_h}$ where $l_h \geqslant 1$, i.e., $r = \operatorname{rank}(G/R) = \sum\limits_{h=1}^{f} l_h$. Then $\operatorname{rank}(G) \geqslant \max\{4, 2r\}$.*

Proof. Since the critical typeset of a $(1,3)$-group has cardinality 4 we get $\operatorname{rank}(G) \geqslant 4$. Furthermore, a coordinate matrix of a group G has the size $r \times \operatorname{rank}(G)$. By Lemma 22 the coordinate matrix α of G consists of two disjoint sections α_0 and the β-part, both of rank r. So $\operatorname{rank}(G) \geqslant 2r$. □

The integers l_h determine blocks of rows of sizes l_h on the coordinate matrix $\alpha = [\alpha_0 \mid \beta_1 \mid \beta_2 \mid \beta_3]$: The first l_1 rows form the first block, then the next l_2 rows form the second block and so on. The row blocks intersected with the column blocks of the coordinate matrix α determine submatrices $M_{i,j}$ of sizes $l_h \times r_j$ such that $[\alpha_0 \mid \beta_1 \mid \beta_2 \mid \beta_3] = [M_{i,j}]$. We are *allowed* to perform the following column operations on $\alpha = [\alpha_0 \mid \beta_1 \mid \beta_2 \mid \beta_3]$ without leaving the near-isomorphism class of G. These column operations correspond to conforming elementary matrices.

Lemma 24.

(1) *Any multiple of a column of α_0 may be added to any other column of α_0.*
(2) *Any multiple of a column of β_i may be added to another column of β_j provided that $j \geqslant i$.*
(3) *Any column may be multiplied by an integer relatively prime to p.*

Modulo S-congruence, the column operations (1) through (3) allow getting the reduced column-echelon form for α_0, β_1, β_2, and β_3. If it happens that, while annihilating an entry, other entries that were zero modulo S change to nonzero entries, then those entries are called *fill-ins*.

Lemma 25. *Let $[\alpha_0 \mid \beta_1 \mid \beta_2 \mid \beta_3]$ be a standard coordinate matrix of a clipped $(1,3)$-group G. Then the following statements hold:*

(1) *α_0 and the β-part $\beta = [\beta_1 \mid \beta_2 \mid \beta_3]$ both have p-rank equal to r.*
(2) *We are allowed column transformations that transform α_0 to I_r modulo S without changing β.*
(3) *We are allowed row and column transformations that transform $[\beta_1 \mid \beta_2]$ into an S-reduced form $[\tilde\beta_1 \mid \tilde\beta_2]$. Let s be the number of units in $[\tilde\beta_1 \mid \tilde\beta_2]$. Then $\tilde\beta_3$ is an $r \times (r-s)$ matrix. If $\tilde\beta = [\tilde\beta_1 \mid \tilde\beta_2 \mid \tilde\beta_3]$ has rank r, then $\tilde\beta$ is a coordinate matrix of G.*
(4) *We are allowed row and column transformations that turn the first l_1 rows of β_1 into a p-diagonal matrix.*

(5) *If $k_1 = k_2 + 1$, then the first $l_1 + l_2$ rows of β_1 can be transformed into the form* $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ *where A, B are p-diagonal matrices (possibly without columns) with l_1 and l_2 rows, respectively.*

(6) *For a fixed regulator and fixed regulator quotient, a clipped $(1,3)$-group is, up to near isomorphism, uniquely determined by $[\beta_1 \mid \beta_2]$.*

Proof. (1) Lemma 22.

(2) Since α_0 has p-rank r and, being clipped, has no 0-column, it is p-invertible. So the reduced column-echelon form modulo S must have r pivots that are units and hence must be $\equiv_S I_r$. The reduced column-echelon form is achieved by column transformations in α_0 that do not change the β-part of the coordinate matrix.

(3) By Lemma 14 we are allowed row and column transformations that change even $[\beta_1 \mid \beta_2 \mid \beta_3]$ into an S-reduced form, say $\tilde{\beta} = [\tilde{\beta}_1 \mid \tilde{\beta}_2 \mid \tilde{\beta}_3]$. Each unit in $[\tilde{\beta}_1 \mid \tilde{\beta}_2]$ creates a 0-row in $\tilde{\beta}_3$. Thus the rank of $\tilde{\beta}_3$, i.e., the number of columns is $\geqslant r - s$. Since $\tilde{\beta}_3$ can be transformed to the reduced column echelon form by the allowed transformations and since G is clipped the number of columns of $\tilde{\beta}_3$ is $r - s$. Only the allowed transformations were done so $\tilde{\beta}$ is a coordinate matrix of G.

(4) In the first l_1 rows of β_1 any row and column transformation is allowed.

(5) Lemma 15.

(6) Suppose that the groups G and G' have coordinate matrices $[\alpha_0 \mid \beta_1 \mid \beta_2 \mid \beta_3]$ and $[\alpha_0' \mid \beta_1 \mid \beta_2 \mid \beta_3']$, respectively, i.e., they have the same part $[\beta_1 \mid \beta_2]$. By (3) we may assume that $\tilde{\beta}_3$ and $\tilde{\beta}_3'$ have the same 0-rows and have the same reduced column echelon form, i.e., they are equal. Finally, as in (2) we may assume that $\alpha_0 \equiv_S \alpha_0' \equiv_S I_r$. This means that G and G' are both near-isomorphic to the same group and therefore near-isomorphic to one another. □

Certain features of the coordinate matrix of a $(1,3)$-group signal the existence of direct summands of small ranks.

Corollary 26. *Let $\alpha = [\alpha_0 \mid \beta_1 \mid \beta_2 \mid \beta_3]$ be a standard coordinate matrix of a $(1,3)$-group G. Then the following statements hold:*

(1) *If $[\beta_1 \mid \beta_2]$ contains a 0-column, then G has a direct summand of rank 1.*

(2) *If $[\beta_1 \mid \beta_2]$ contains a 0-row, then G has a direct summand of rank 2.*

(3) *If $[\beta_1 \mid \beta_2]$ contains a cross, then G has a summand of rank 2 or 3. Rank 3 happens only if the entry $\neq 0$ is not a unit.*

(4) *If $[\beta_1 \mid \beta_2]$ contains a horizontal double cross, then G has a summand of rank 3 or 4. Rank 4 occurs only if none of the entries $\neq 0$ is a unit.*

(5) *If $[\beta_1 \mid \beta_2]$ contains a vertical double cross, then G has a summand of rank $5 - s$ where s is the number of units in the column of the vertical double cross, i.e., the possible ranks are $5 - s = 3, 4, 5$.*

(6) *If β_1 has a unit in the first l_1 rows, then G has a summand of rank 2.*
(7) *If $k_1 = k_2 + 1$ and β_1 has a unit in the first $l_1 + l_2$ rows, then G has a summand of rank 2.*

Proof. (1) is obvious.

The claims (2) to (6) follow easily by using Lemma 25 (2) and (3).

For example, consider (5). Let $(i_1, j_0)|(i_2, j_0)$ be the location of the vertical double cross. The double cross links the rows i_1 and i_2. There will be two ranks coming from $\alpha_0 \equiv_S I_r$, one rank from the cross column j_0 and up to two ranks from β_3 depending on whether the rows i_1 and i_2 are zero-rows of β_3 or not.

(7) If $k_1 = k_2+1$ and β_1 has a unit in a row with index between l_1 and l_1+l_2, then, by Lemma 25 (5), there is a cross in the first $l_1 + l_2$ rows of β_1. Since the nonzero entry of this cross is a unit we may extend this cross to a cross of the whole β-part by allowed row and column transformations. Thus G has a summand of rank 2. □

7. Indecomposable groups with regulator quotient of exponent $\leqslant p^3$

Lemma 19 can be sharpened for $(1,3)$-groups.

Proposition 27. *If the part $[\beta_1 \mid \beta_2]$ of a standard coordinate matrix $[\alpha_0 \mid \beta_1 \mid \beta_2 \mid \beta_3]$ of a $(1,3)$-group G is S-decomposable, then G is decomposable. Conversely, if G is decomposable without direct summands of rank $\leqslant 2$, then it has a standard coordinate matrix with decomposable submatrix $[\beta_1 \mid \beta_2]$.*

Proof. Let $[\beta_1 \mid \beta_2]$ be S-decomposable. By Lemma 14 there is an S-reduced form $[\tilde{\beta}_1 \mid \tilde{\beta}_2] = U[\beta_1 \mid \beta_2]Y'$ where U is a lower triangular matrix and Y' is an upper triangular matrix. The two matrices U and $Y = \operatorname{diag}(I_r, Y', I_{r_3})$ are allowed row and column transformations for the coordinate matrix $[\alpha_0 \mid \beta_1 \mid \beta_2 \mid \beta_3]$, and $U[\alpha_0 \mid \beta_1 \mid \beta_2 \mid \beta_3]Y = [U\alpha_0 \mid \tilde{\beta}_1 \mid \tilde{\beta}_2 \mid U\beta_3]$. By Lemma 25 (2) the part $U\alpha_0$ can be changed to the identity matrix I_r modulo S. By Lemma 25 (3) the part $U\beta_3$, can be changed to $\tilde{\beta}_3$ modulo S, where $\tilde{\beta}_3$ is the identity matrix enlarged by some 0-rows. These transformations do not affect $[\tilde{\beta}_1 \mid \tilde{\beta}_2]$. Now, since $[\beta_1 \mid \beta_2]$ is S-decomposable, also $[\tilde{\beta}_1 \mid \tilde{\beta}_2] = U[\beta_1 \mid \beta_2]Y'$ is S-decomposable by Corollary 17. Hence the new coordinate matrix is $\tilde{\alpha} \equiv_S [I_r \mid [\tilde{\beta}_1 \mid \tilde{\beta}_2 \mid \tilde{\beta}_3]$. So the S-decomposability of $[\beta_1 \mid \beta_2]$ is inherited by $\tilde{\alpha}$ and by Corollary 20 the group G is decomposable.

Conversely, let G be decomposable without direct summands of rank $\leqslant 2$. Then, by Lemma 19 , our group G has a decomposable coordinate matrix. By permutations of the rows and of the columns we get a coordinate matrix for G as in the standard setting. Clearly, this coordinate matrix is decomposable. Since G has no direct summand of rank $\leqslant 2$ the coordinate matrix has no 0-column. Moreover, the part $[\beta_1 \,|\, \beta_2]$ has no 0-row, since otherwise, by Lemma 14, an S-reduced form of $[\beta_1 \,|\, \beta_2]$ would have a 0-row, and G would have a direct summand of rank 2 by Corollary 26 (2). Thus $[\beta_1 \mid \beta_2]$ has no 0-lines. But then the decomposability of the coordinate matrix $[\alpha_0 \mid \beta_1 \mid \beta_2 \mid \beta_3]$ implies decomposability of $[\beta_1 \mid \beta_2]$. $\square$

For the convenience of the reader we collect techniques, language conventions and standard conclusions in a preamble. Moreover, we list standard conclusions to avoid frequent repetitions in the proofs.

Preamble. The method of finding all near-isomorphism types of indecomposable $(1,3)$-groups is to start with a general coordinate matrix $\alpha = [\alpha_0 \,|\, \beta_1 \,|\, \beta_2 \,|\, \beta_3]$. We may restrict ourselves to the part $[\beta_1 \mid \beta_2]$ by Lemma 25 (3) and (6) and Proposition 27.

We specialize the coordinate matrix using several techniques.

(1) We use a matrix that is S-congruent to the coordinate matrix. By Lemma 10, this does not change the group. So we may replace entries that are $\equiv_S 0$ by 0.

(2) We use multiplications of rows and columns by units to create pure p-powers. These elementary transformations are allowed and do not change the group.

(3) We use the property "indecomposable", Corollary 26 together with Lemma 25 (4) and (5), to exclude direct summands of rank < 4 since by Corollary 23 such groups cannot be equal to a $(1,3)$-group. This is the case if the part $[\beta_1 \mid \beta_2]$ has a 0-line, a cross, a horizontal double cross that has at least one unit as an entry, or a vertical double cross with the upper entry a unit. In particular, this allows to simplify β_1 drastically.

(4) We use allowed elementary row and column transformations, Lemmata 21 and 24, to annihilate entries in β_2. But we wish to keep β_1 unchanged. Clearly, elementary row transformations will create fill-ins in β_1. So we have to make sure that we can reestablish β_1 in the original form after such an elementary row transformation by column transformations in β_1 only.

Language agreements. There are submatrices that change when other submatrices are transformed but whose actual values are irrelevant. In such cases we retain the name of the submatrix and call it a "*place holder*".

By *"An entry x leads to a cross in $[\beta_1 \,|\, \beta_2]$"* we mean that this entry x can be used as a pivot in its row and its column to generate a cross in $[\beta_1 \mid \beta_2]$ and precisely at this location. Clearly, we use only the allowed line transformations as in Lemmata 21 and 24. This cross displays a direct summand of rank 2 or 3, by Corollary 26 (3). So this is a contradiction by Corollary 23. We express ourselves similarly, if a double cross can be obtained that also displays a direct summand of an impossible rank, cf. Corollary 26.

Mostly we want to change certain submatrices either to a 0-matrix or to a matrix of the form $p^h I$, $h \geqslant 0$. In doing matrix transformations to this effect previous zero entries may become non-zero entries (fill-ins). By "*The fill-ins can be annihilated*" we mean that there are transformations that turn the fill-ins to zero without changing the newly achieved form. Of course, we only use the allowed line transformations as in Lemmata 21 and 24.

By "*The matrix B can be reestablished*" we mean that after some allowed transformation of another submatrix A that also changes B there are other allowed transformations that change B back to its original form without changing A. There may be a series of matrices that have to be reestablished, namely if the reestablishing of B causes changes of another submatrix C that in turn has to be reestablished etc.

By "*We transform a matrix A to its Smith Normal Form*" we mean first that this is an allowed transformation, i.e., there are p-invertible matrices U, Y such that UAY is a p-diagonal matrix. We mean secondly that it is possible to reestablish submatrices affected by these transformations. This may require a number of steps. We always want to reestablish all submatrices that were originally either 0 or $p^h I$, $h \geqslant 0$.

Since the last technique is crucial, we describe it in all detail in Example 29. By "Smith Normal Form" we mean the following straightforward extension of the usual Smith Normal Form formulated as the special case "$r = l$" of Lemma 15.

Lemma 28. *Let H be an integer matrix. Then there are p-invertible matrices U, Y such that $UHY \equiv \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix}$ modulo p^k where N is p-diagonal with all diagonal entries $\neq 0$.*

Example 29. We give an explicit example in which one submatrix is changed to Smith Normal Form and affected submatrices are reestablished in steps. Let us consider $\left[\begin{array}{c|c|c} A & I & p^2 I \\ \hline pI & B & 0 \end{array}\right]$. We transform B to its Smith Normal Form, say $UBY = \begin{bmatrix} I & 0 & 0 \\ 0 & pI & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus we obtain $\left[\begin{array}{c|c|c} A & Y & p^2 I \\ \hline pU & UBY & 0 \end{array}\right]$. We reestablish I above the original B by multiplying the first block row by Y^{-1} from the left and we reestablish pI to the left of the original B by multiplying the first block column by U^{-1} from the right. In fact also A changed to $Y^{-1}AU^{-1}$. But A was only a place holder so we do not change the name A. Doing all this changes $p^2 I$ changes to $p^2 Y^{-1}$. Now we reestablish $p^2 I$ by multiplying the third block column from the right by Y. There is another very important phenomenon, namely the splitting of block rows and block columns. Taking into account the splitting of the B-row and the B-column we

obtain
$$\left[\begin{array}{ccc|ccc|ccc} A_1 & A_2 & A_3 & I & 0 & 0 & p^2I & 0 & 0 \\ A_4 & A_5 & A_6 & 0 & I & 0 & 0 & p^2I & 0 \\ A_7 & A_8 & A_9 & 0 & 0 & I & 0 & 0 & p^2I \\ \hline pI & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & pI & 0 & 0 & pI & 0 & 0 & 0 & 0 \\ 0 & 0 & pI & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$
Note that the place holder A also splits. Moreover, if pI in the Smith Normal Form of B is not present, then the place holders A_2, A_4, A_5, A_6, A_8 are not present either. Conversely, if we assume, for instance, that there are entries in A_2, then A_1 is present, too.

Our first main result says that sometimes there are no indecomposable groups.

Theorem 30. *$(1,3)$-groups with regulator quotient of exponent $\leqslant p^2$ are decomposable.*

P r o o f. By way of contradiction let G be an indecomposable $(1,3)$-group with regulator quotient of exponent p. The structure matrix is $S = pI_r$ and all entries of β_1 are 0 by Lemma 25 (4), so G is not even clipped.

Now let G be an indecomposable $(1,3)$-group with $\exp(G/R) = p^2$. In this case we may assume that $S = \operatorname{diag}(p^2 I_{l_1}, pI_{l_2})$ is the structure matrix, where $l_1 \geqslant 1$ and $l_2 \geqslant 0$. By Lemma 25 (5) we have that $\beta_1 = \begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix}$ where X, Z are p-diagonal with l_1 and l_2 rows, respectively. Then, again by Corollary 26 (1), (6) and (7), β_1 has no units and no 0-column. Thus
$$[\beta_1 \mid \beta_2] = \begin{array}{cc} \underbrace{\left[\begin{array}{c|c} pI & A \\ \hline 0 & B \\ \hline 0 & C \end{array}\right]}_{\beta_1\ \ \beta_2} & \begin{array}{c} p^2 \\ \underline{p^2} \\ p \end{array} \quad l_1 \end{array}$$

Due to pI on the left, A has only zeros or units. A unit in B leads to a cross, so B has no units. But then A has no unit to avoid a horizontal double cross in $[\beta_1 \mid \beta_2]$ that displays a direct summand of rank 3, cf. Corollary 26 (4). Hence $A = 0$ and G is decomposable by Proposition 27. □

We next produce examples of indecomposable $(1,3)$-groups that later turn out to present all near-isomorphism types of indecomposable $(1,3)$-groups with regulator quotient of exponent p^3.

Proposition 31. *The following six $(1,3)$-groups G with regulator quotient of exponent p^3 given by the isomorphism types of their regulator with fixed types, their regulator quotient and their coordinate matrix $\alpha = [\alpha_0 \,|\, \beta_1 \,|\, \beta_2 \,|\, \beta_3]$ are indecomposable and pairwise not near-isomorphic.*

(1) $\alpha = [1 \mid p^2 \mid p \mid 1]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^3}$ *and* $\operatorname{rank} G = 4$.

(2) $\alpha = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \middle| \begin{smallmatrix} p \\ 0 \end{smallmatrix} \middle| \begin{smallmatrix} 1 \\ p \end{smallmatrix} \middle| \begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ *with regulator quotient isomorphic to* $(\mathbb{Z}_{p^3})^2$ *and* $\operatorname{rank} G = 5$.

(3) $\alpha = \left[\begin{smallmatrix} 1 & 0 \\ \hline 0 & 1 \end{smallmatrix} \middle| \begin{smallmatrix} 0 \\ \hline p \end{smallmatrix} \middle| \begin{smallmatrix} p \\ \hline 1 \end{smallmatrix} \middle| \begin{smallmatrix} 1 \\ \hline 0 \end{smallmatrix}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2}$ *and* $\operatorname{rank} G = 5$.

(4) $\alpha = \left[\begin{smallmatrix} 1 & 0 \\ \hline 0 & 1 \end{smallmatrix} \middle| \begin{smallmatrix} p \\ \hline p \end{smallmatrix} \middle| \begin{smallmatrix} 1 \\ \hline 0 \end{smallmatrix} \middle| \begin{smallmatrix} 0 \\ \hline 1 \end{smallmatrix}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2}$ *and* $\operatorname{rank} G = 5$.

(5) $\alpha = \left[\begin{smallmatrix} 1 & 0 \\ \hline 0 & 1 \end{smallmatrix} \middle| \begin{smallmatrix} p^2 \\ \hline 0 \end{smallmatrix} \middle| \begin{smallmatrix} p \\ \hline 1 \end{smallmatrix} \middle| \begin{smallmatrix} 1 \\ \hline 0 \end{smallmatrix}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p$ *and* $\operatorname{rank} G = 5$.

(6) $\alpha = \left[\begin{smallmatrix} 1 & 0 \\ \hline 0 & 1 \end{smallmatrix} \middle| \begin{smallmatrix} p \\ \hline 1 \end{smallmatrix} \middle| \begin{smallmatrix} p \\ \hline 0 \end{smallmatrix} \middle| \begin{smallmatrix} 1 \\ \hline 0 \end{smallmatrix}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p$ *and* $\operatorname{rank} G = 5$.

P r o o f. The claims on ranks are clear and the regulator property and the structure of the regulator quotient are easily verified.

By Proposition 27 a group without direct summands of rank $\leqslant 2$ and with a coordinate matrix $[\alpha_0 \mid \beta_1 \mid \beta_2 \mid \beta_3]$ is indecomposable if and only if $[\beta_1 \mid \beta_2]$ is S-indecomposable and, by Theorem 12, this is the case if and only if $U[\beta_1 \mid \beta_2]Y_\beta$ is not S-decomposable where U is the first component of an S-pair (in particular p-invertible) and $Y_\beta = \begin{bmatrix} Y_{1,1} & Y_{1,2} \\ 0 & Y_{2,2} \end{bmatrix}$ is the relevant submatrix of a conforming matrix Y as in Equation (4). In our examples the $Y_{i,j}$ are integers and we may assume that $Y_\beta = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ since multiplication by a p-invertible diagonal matrix from the right-hand side will not change a decomposition but allows to get entries 1 on the diagonal.

For (1) it is enough to observe that modulo p^3 there are no 0-entries in the row

$$[p^2 \mid p] \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \equiv [p^2 \mid p^2 a + p] \bmod p^3.$$

For (2) we recall that the 2×2 matrix $U = [u_{i,j}]$ is p-invertible, so by the same argument with the diagonal matrix as above, but multiplying by a diagonal matrix from the left-hand side, we may assume that either $u_{1,1} = u_{2,2} = 1$ or $u_{1,2} = u_{2,1} = 1$. We deal only with the first case, the second case being similar. It is enough to note that the following matrix is not decomposable modulo p^3:

$$\begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix} \begin{bmatrix} p & 1 \\ 0 & p \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \equiv \left[\begin{array}{c|c} p & ap + bp + 1 \\ cp & cap + p + c \end{array}\right] \bmod p^3.$$

Since both the entries in the first row are not 0 modulo p^3, the only possibility for a decomposition is $c \equiv 0$ modulo p^2. But then the other entry in the second row is not 0 modulo p^3.

For the remaining cases we use the argument with the diagonal matrix multiplying from the left-hand side to obtain that the diagonal entries of U are 1.

For (3) it is enough to state that the following matrix has no 0-line modulo $S = \mathrm{diag}(p^3, p^2)$ and is not S-decomposable:

$$(5) \qquad \begin{bmatrix} 1 & pb \\ c & 1 \end{bmatrix} \begin{bmatrix} 0 & p \\ p & 1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \equiv_S \left[\begin{array}{c|c} bp^2 & bap^2 + p + bp \\ \hline p & ap + cp + 1 \end{array}\right].$$

Since both the entries in the second row are not 0 modulo p^2, the only possibility for a decomposition is $b \equiv 0$ modulo p. But then the other entry in the first row is not 0 modulo p^3.

For (4) it is enough to verify that the following matrix has no 0-line modulo $S = \mathrm{diag}(p^3, p^2)$ and is not S-decomposable:

$$\begin{bmatrix} 1 & pb \\ c & 1 \end{bmatrix} \begin{bmatrix} p & 1 \\ p & 0 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \equiv_S \left[\begin{array}{c|c} p + bp^2 & ap + abp^2 + 1 \\ \hline cp + p & acp + ap + c \end{array}\right].$$

Since both the entries in the first row are not 0 modulo p^3, the only possibility for a decomposition is $c \equiv 0$ modulo p. But then the other entry in the second row is not 0 modulo p^2.

For (5) it is obvious that the following matrix has no 0-line and is not decomposable modulo $S = \mathrm{diag}(p^3, p)$:

$$(6) \qquad \begin{bmatrix} 1 & p^2b \\ c & 1 \end{bmatrix} \begin{bmatrix} p^2 & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \equiv_S \left[\begin{array}{c|c} p^2 & ap^2 + p + bp^2 \\ \hline 0 & 1 \end{array}\right].$$

For (6) it is obvious that the first column of the following matrix has no 0-entries

$$\begin{bmatrix} 1 & p^2b \\ c & 1 \end{bmatrix} \begin{bmatrix} p & p \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \equiv_S \left[\begin{array}{c|c} p + p^2b & a(p + p^2b) + p \\ \hline cp + 1 & a(cp + 1) + cp \end{array}\right].$$

So to be decomposable the second column must be 0. For this it is necessary that $a \equiv 0 \bmod p$. But then the entry at the position $(1, 2)$ is $\not\equiv_S 0$. So the given matrix is not decomposable modulo $S = \mathrm{diag}(p^3, p)$.

It remains to show that the six groups above are pairwise not near-isomorphic. Since the isomorphism types of the regulator and the regulator quotient are near-isomorphism invariants, it is enough to prove that the groups under (3) and (4) are not near-isomorphic, and that the groups under (5) and (6) are not near-isomorphic.

Let $[\beta_1|\beta_2]$ and $[\beta_1'|\beta_2']$ be the parts of groups G, G' as in (3) and in (4), respectively. By Theorem 12, if the groups G and G' are nearly isomorphic, then there are U, Y_β as above such that $U[\beta_1 \mid \beta_2]Y_\beta \equiv_S [\beta_1' \mid \beta_2']$. By Equation (5) we have to show that the following two matrices are not diagonal equivalent:

$$\left[\begin{array}{c|c} bp^2 & bap^2+p+bp \\ \hline p & ap+cp+1 \end{array}\right] \quad \text{and} \quad \begin{bmatrix} p & 1 \\ p & 0 \end{bmatrix}.$$

This is obvious.

By Equation (6) we have to show that for the two groups as in (5) and in (6) the following two matrices are not diagonal equivalent:

$$\left[\begin{array}{c|c} p^2 & ap^2+p+bp^2 \\ \hline 0 & 1 \end{array}\right] \quad \text{and} \quad \begin{bmatrix} p & p \\ 1 & 0 \end{bmatrix}.$$

This is obvious. So the six groups above belong to different near-isomorphism types. $\square$

By Proposition 31 we know that there are at least six near-isomorphism types of indecomposable $(1,3)$-groups with regulator quotient of exponent p^3. In the next theorem we show that these are all. For the techniques of the proof we recommend to read the preamble again.

Theorem 32. *For a given isomorphism type of the regulator, there are six near isomorphism types as in Proposition* 31 *of indecomposable* $(1,3)$*-groups with regulator quotient of exponent* $\leqslant p^3$.

Proof. Let G be a $(1,3)$-group with regulator R and $\exp(G/R) \leqslant p^3$ given by a coordinate matrix $[\alpha_0 \mid \beta_1 \mid \beta_2 \mid \beta_3]$. We will find all indecomposable $(1,3)$-groups that are direct summands of G. By Theorem 30 we may assume that $\exp(G/R) = p^3$. It is easy to see that every indecomposable $(1,3)$-group of rank 4 is of type Proposition 31 (1). Therefore we further assume without loss of generality that G has no direct summand of rank $\leqslant 4$.

The assumption means that the coordinate matrices of G with a part $[\beta_1 \mid \beta_2]$ that has 0-lines, crosses or double crosses, cf. Collorary 26 (1)–(4), are excluded.

Let $[\alpha_0 \| \beta_1 \mid \beta_2 \mid \beta_3]$ be a coordinate matrix of G, and assume $S = \operatorname{diag}(p^3 I_{l_1}, p^2 I_{l_2}, pI_{l_3})$ to be the structure matrix where $l_1 \geqslant 1$ and it is left open whether l_2, l_3 are zero or not. By Lemma 25 (5) the first $l_1 + l_2$ rows of β_1 may be assumed to equal $\operatorname{diag}(X, Z)$ with p-diagonal matrices X, Z that do not contain units by Corollary 26 (7). It is easy to see that 0-columns in X or Z lead to crosses or 0-columns in the $[\beta_1 \mid \beta_2]$ part of the coordinate matrix and contradict the hypothesis. Therefore we have without loss of generality

$$[\beta_1 \mid \beta_2] = \begin{array}{ll}
\left[\begin{array}{cc|c||c}
p^2I & 0 & 0 & A \\
0 & pI & 0 & C \\
0 & 0 & 0 & D \\
\hline
0 & 0 & pI & E \\
0 & 0 & 0 & F \\
\hline
H_1 & H_2 & H_3 & H
\end{array}\right]
&
\begin{array}{ll}
p^3 & \\
p^3 & \\
p^3 & l_1 \\
\hline
p^2 & \\
p^2 & l_1 + l_2 \\
\hline
p &
\end{array}
\\
\quad\underbrace{\hphantom{p^2I \quad 0 \quad\; 0 \quad}}_{\beta_1}\underbrace{\hphantom{A\;}}_{\beta_2} &
\end{array} \tag{7}$$

where the letter I denotes an identity matrix of some size and 0 denotes some 0-matrix. The part β_2 is the block column labeled by the blocks $A, \ldots, H$. Note that the matrices $A, \ldots, H$ have columns since β_2 has columns.

In the sequel we will frequently and tacitly use the following trivial observation.

If a SQUARE matrix has no rows, then it has no columns and if it has no columns, then it has no rows. Hence if a square block X appears in some matrix, then the block row and the block column defined by X either are both present or both absent.

Recall if we form Smith Normal Forms we always mean that all affected blocks can be reestablished, in particular, 0-blocks and those of the form $p^h I$.

We determine the entries of the row H. As the computation in the row H is modulo p the entries in this block row may be assumed to be either 0 or units.

The block H_1 is absent if and only if the A-row is absent, and in this case nothing is done. So suppose H_1 and with it the A-row are present. A unit in H_1 allows to annihilate all entries in its row in $[\beta_1 \,|\, \beta_2]$. This changes p^2I, but p^2I can be reestablished by row transformations alone. If the columns H_2 or H_3 are present, then there are fill-ins to the right of p^2I that are in $p^2\mathbb{Z}$. These can be annihilated by the pI's in columns H_2 and H_3. If the columns H_2 or H_3 are absent, there are no fill-ins to consider. Next, using the unit in H_1 as a pivot we annihilate by elementary row transformations the entry in p^2I, cf. Lemma 21 (2), and obtain a cross located at the unit. Thus $H_1 = 0$.

The block H_3 is absent if and only if the E-row is absent, and if this is so, then nothing is done. Suppose that H_3 is present and with it the row E. A unit in H_3 allows to create zeros in its row. If the H_2-column appears, then also the row C is present and the fill-ins in the column H_2 can be removed using pI in the row C. Then the unit can be used to create a cross located at the unit, a contradiction. Thus $H_3 = 0$.

The block H_2 is absent if and only if the C-row is absent, and if this is so, then nothing is done. Suppose that H_2 is present. Then also the row C is present. It is easy to see that $H_2 \neq 0$. We create the Smith Normal Form of H_2. The submatrix pI in the H_2-column can be reestablished by row transformations alone. H_2 cannot be 0 to avoid a horizontal double cross in the row C. Thus the Smith Normal Form of H_2 is $\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$ or $[\, I \quad 0 \,]$ or $\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$ or $[\, I \,]$. In the general case the Smith Normal Form splits the matrix pI in the row C and we get $\left[\begin{smallmatrix} pI & 0 \\ 0 & pI \end{smallmatrix}\right]$ instead, as in (8). Moreover, the H-row splits in two block rows, too, labeled as shown in (8).

The matrix (8) incorporates all possibilities where block rows as well as block columns may be absent. In fact, the absence of H_1, H_2 and H_3 is covered by the absence of A, C and E, respectively, and different Smith Normal Forms of H_2 are obtained by the absence of the row C or the row H or both of them.

$$(8) \qquad [\beta_1 \mid \beta_2] = \underbrace{\left[\begin{array}{ccc|c||c} p^2I & 0 & 0 & 0 & A \\ 0 & pI & 0 & 0 & B \\ 0 & 0 & pI & 0 & C \\ 0 & 0 & 0 & 0 & D \\ \hline 0 & 0 & 0 & pI & E \\ 0 & 0 & 0 & 0 & F \\ \hline 0 & I & 0 & 0 & L \\ 0 & 0 & 0 & 0 & H \end{array}\right]}_{\beta_1 \qquad\qquad \beta_2} \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ \hline p^2 \\ p^2 \\ \hline p \\ p \end{array} \qquad \begin{array}{c} \\ \\ \\ l_1 \\ \\ l_1 + l_2 \\ \\ \\ \end{array}$$

The B-row and the L-row are linked. They are either both present or both absent. Our final goal is to obtain a block form for β_2 such that all blocks are 0 or p^hI, $h \geqslant 0$. This is done by forming the Smith Normal Form in parts of β_2. In the process the submatrices of β_2 are broken up into smaller blocks and the block structure of β_1 has to be refined correspondingly. Establishing the Smith Normal Form for a sub-block of β_2 is accomplished by row and column transformations in $[\beta_1 \mid \beta_2]$ that affect various other parts of $[\beta_1 \mid \beta_2]$. Blocks of the form 0 or p^hI that are changed by the transformations must be reestablished by other allowed transformations in order to achieve the goal of having nothing but blocks of the form 0 or p^hI. The identities $Y^{-1}((p^hI)Y) = p^hI$ and $(U(p^hI))U^{-1} = p^hI$ show that a row or column transformation of a matrix p^hI can be reversed by the inverse column or row transformation. This fact will be used frequently below and has been used before. If certain rows or columns are absent, then the issue of fill-ins disappears altogether, and we will not mention these special cases every time.

By Lemma 25 (3), given $[\beta_1 \mid \beta_2]$, the part β_3 is arbitrary except that it must guarantee that the rank of $[\beta_1 \mid \beta_2 \mid \beta_3]$ is r and that its reduced column echelon form has no 0-column. Having obtained the matrix $[\beta_1 \mid \beta_2]$ it is easy to supplement β_3 and to read off the types of groups listed in Proposition 31 and to exclude others.

(a) C, D, E, F, H have no 0-rows. $L = 0$, $A, B, D \equiv 0 \bmod p$. Write pA, pB, pD instead of A, B, D.

By Corollary 26 (2) D, F, H have no 0-rows. A 0-row in C, E displays a cross in $[\beta_1 \mid \beta_2]$ which cannot be. Suppose that the row D is present. A unit in D leads to a cross, hence we write pD.

Suppose that the row A occurs. A unit in A allows to annihilate all other entries in its column. Doing so the matrix p^2I is changed but can be reestablished. If rows are absent, then there are no fill-ins. The fill-ins below p^2I are either $\equiv_S 0$, as in the D-row and below, or, in the B- and C-rows, can be annihilated by column transformations in β_1. We obtain horizontal double crosses in $[\beta_1 \,|\, \beta_2]$ located in A. This results in indecomposable summands of rank 4 with the coordinate matrix $[1 \,|\, p^2 \,|\, p \,|\, 1]$ that are of type in Proposition 31 (1). By hypothesis this case is excluded, hence $A \equiv_p 0$.

If the row L is present, then we can annihilate L by means of I on the left and get $L = 0$. By way of contradiction we assume that B has a unit. Then we can annihilate all other entries in this column with this unit. The fill-ins in the A-row can be annihilated by p^2I on the left. The fill-ins in the C- and E-rows can be annihilated by the respective pI's on the right. The fill-ins in the D- and F-rows can be annihilated by means of I below, since $L = 0$. The fill-ins in the H-row are $\equiv_S 0$. Thus, without loss of generality,

$$(9) \qquad [\beta_1 \mid \beta_2] = \begin{array}{ccc}
\left[\begin{array}{ccc|c||c}
p^2I & 0 & 0 & 0 & pA \\
0 & pI & 0 & 0 & pB \\
0 & 0 & pI & 0 & C \\
0 & 0 & 0 & 0 & pD \\
\hline
0 & 0 & 0 & pI & E \\
0 & 0 & 0 & 0 & F \\
\hline
0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & H
\end{array}\right]
&
\begin{array}{l}
A, p^3 \\
B, p^3 \\
C, p^3 \\
D, p^3 \\
\hline
E, p^2 \\
F, p^2 \\
\hline
L, p \\
H, p
\end{array}
&
\begin{array}{l}
\\ \\ \\
l_1 \\
\\
l_1 + l_2 \\
\\ \\
\end{array}
\\
\underbrace{\hphantom{p^2I\;0\;0\;0\;0\;0\;0\;0}}_{\beta_1}\underbrace{\hphantom{pA}}_{\beta_2} & &
\end{array}$$

(b) Only the A-row is present.

In this case $pA \neq 0$, since G is clipped. The Smith Normal Form of pA is pI since there are no 0-lines. So G has an indecomposable summand of rank 4 with the coordinate matrix $[1 \mid p^2 \mid p \mid 1]$ that is of type in Proposition 31 (1).

(c) One of the rows B through H is present.

We show that we can establish Smith Normal Forms for C, E, H simultaneously.

$$(10) \qquad [\beta_1 \mid \beta_2] = \begin{array}{ccc}
\left[\begin{array}{ccc|c||c|c|c|c}
p^2I & 0 & 0 & 0 & pA_1 & pA_2 & pA_3 & pA_4 \\
0 & pI & 0 & 0 & pB_1 & pB_2 & pB_3 & pB_4 \\
0 & 0 & pI & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & pD_1 & pD_2 & pD_3 & pD_4 \\
\hline
0 & 0 & 0 & pI & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & F_1 & F_2 & F_3 & F_4 \\
\hline
0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0
\end{array}\right]
&
\begin{array}{l}
A, p^3 \\
B, p^3 \\
C, p^3 \\
D, p^3 \\
\hline
E, p^2 \\
F, p^2 \\
\hline
L, p \\
H, p
\end{array}
&
\begin{array}{l}
\\ \\ \\
l_1 \\
\\
l_1 + l_2 \\
\\ \\
\end{array}
\\
\underbrace{\hphantom{p^2I\;0\;0\;0\;0}}_{\beta_1}\underbrace{\hphantom{pA_1\;pA_2\;pA_3\;pA_4}}_{\beta_2} & &
\end{array}$$

To obtain (10) we use that C, E, H have entries that are either 0 or units due to the blocks pI in β_1 or due to computation modulo p in case of H, and that they have no 0-rows.

We first produce the Smith Normal Form of C which is $[I\,|\,0]$ or $[I]$. In the process the matrix pI changes due to row transformations but can be reestablished by means of column transformations. Accordingly, β_2 splits into two columns; in (10) it is the first column of β_2 and the remaining three columns combined. We now create zeros below I of the row C in the rows E and H. Fill-ins can be removed. If the Smith Normal Form is $[I]$, then only the first column of β_2 is present. This possibility is not lost. If C is not present, then nothing is done. This case is contained in (10) because the absence of row C means the first block column of β_2 is not present, i.e., pA_1, pB_1, pD_1, F_1 are not present.

Next, the Smith Normal Form of E below the 0-block of the row C is formed and it is $[I\,|\,0]$ or $[I]$. Again the pI in the row E, changed by row transformations, can be reestablished by means of column transformations. This causes a further split of the columns of β_2, and we have the first two columns of (10) and the last two columns combined. Below the I in the row E we produce zeros in the row H which creates no fill-ins. If the row E is not present or if the Smith Normal Form is $[I]$ the suitable deletions in (10) will cover these cases.

Finally, changing H to the Smith Normal Form creates no fill-ins and splits the third column of β_2 resulting into the four columns shown in (10) and no special cases are lost.

(d) $pA_1 - pA_2 = pA_4 = 0$, $pA_3 = pI$, $pB_1 = pB_2 - pB_3 = 0$, $pD_3 = 0$, $pD_4 =_{p^2} 0$, $F_2 = F_3 = 0$; $F_4 \equiv_p 0$. Write pF_4, p^2D_4 in place of F_4, pD_4.

We show, starting with (10), how to obtain (11). Note that the statements are not proved in the order they are listed above. In fact, it is necessary to follow a certain sequence in this proof. For the convenience of the reader we always indicate which part of the listed claims is dealt with.

$$
(11) \qquad [\beta_1 \mid \beta_2] = \begin{array}{c}
\left[\begin{array}{ccc|c||c|c|c|c}
p^2I & 0 & 0 & 0 & 0 & 0 & pI & 0 \\
0 & pI & 0 & 0 & 0 & 0 & 0 & pB_4 \\
0 & 0 & pI & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & pD_1 & pD_2 & 0 & p^2D_4 \\
\hline
0 & 0 & 0 & pI & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & F_1 & 0 & 0 & pF_4 \\
\hline
0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0
\end{array}\right]
\\
\underbrace{\hphantom{p^2I\quad 0\quad 0\quad pI}}_{\beta_1}\underbrace{\hphantom{pD_1\quad pD_2\quad 0\quad p^2D_4}}_{\beta_2}
\end{array}
\begin{array}{l}
A, p^3 \\
B, p^3 \\
C, p^3 \\
\underline{D, p^3} \\
E, p^2 \\
\underline{F, p^2} \\
L, p \\
H, p
\end{array}
\begin{array}{l}
\\ \\ \\
l_1 \\
\\
l_1 + l_2 \\
\\ \\
\end{array}
$$

(d1) Block column 4.

Suppose that F_4 contains a unit. Then zeros can be created first in its row and then in its column resulting in a cross. So $F_4 \equiv_p 0$ and we rename F_4 to pF_4.

Suppose that pD_4 contains a p. Then a cross results at this place. Hence $pD_4 \equiv_{p^2} 0$ and we rename pD_4 to p^2D_4.

Suppose that pA_4 is present and $\not\equiv 0 \bmod p^3$. Then p^2I is present and pA_4 contains an entry p. This p can be used to create 0 in its row in β_2 and in its column in the row A. The fill-ins in p^2I caused by row transformations can be undone by column transformations. Next, 0 can be created in pB_4 below p. Fill-ins in the row B coming from p^2I of the row A can be eliminated by means of the block pI in the row B. Entries in p^2D_4 below p can be made 0 with no fill-ins because $p(p^2I) \equiv 0 \bmod p^3$. Finally, the entries in pF_4 below p can be made 0 with no fill-ins because $p^2I \equiv 0 \bmod p^2$. We have obtained a horizontal double cross in the row A resulting in a summand of rank 4, contrary to assumption. So we get $pA_4 = 0$.

(d2) F-row.

Note that, if F_2 is present, then the E-row is present, and if F_3 is present, then the H-row is present. The entries of the matrices F_2, F_3 are either units or 0 by the identity matrices above and below in the E- and H-row, respectively, since annihilating with those I's creates fill-ins in the F-row that are $\equiv_{p^2} 0$. A unit in F_2 or in F_3 leads to a cross. Thus $F_2 = F_3 = 0$.

(d3) D-row.

Note that pD_3 is present if and only if the row H is present. The entries in pD_3 are either 0 or in $p\mathbb{Z} \setminus p^2\mathbb{Z}$ due to I in the H-row. Suppose that there is a p in pD_3. With it we make zeros in its row. Annihilation in p^2D_4 creates fill-ins in the row H but these are 0 modulo p. In addition fill-ins in the row A appear. These are in $p^2\mathbb{Z}$ and can be removed with p^2I from β_1 in the row A. Annihilation in pD_2 creates fill-ins in the row H. The fill-ins in the H-row can be removed by means of I above it in row E. In the process new fill-ins appear in row H in β_1, but these are $0 \bmod p$. Annihilation in pD_1 again creates fill-ins in the row H. They can be removed by means of I in the row C. Now p alone is not zero in its row of $[\beta_1 \mid \beta_2]$. Therefore all entries above and below p can be removed except for those in the row H. But then the group G has a direct summand that is not clipped with partial coordinate matrix $[\beta_1 \mid \beta_2] = \begin{bmatrix} 0 & p \\ 0 & 1 \end{bmatrix}$. Hence $pD_3 = 0$.

(d4) A- and B-row.

Note that if pA_1, pB_1 are present, then the C-row is present, and if pA_2, pB_2 are present, then the E-row is present. The blocks pA_1, pA_2, pB_1, pB_2 can be annihilated by the respective identity matrices in the rows C and E. The fill-ins in the A- and B-rows are in $p^2\mathbb{Z}$ and can be annihilated by p^2I and pI, respectively. This in turn creates fill-ins in the L-row that are $\equiv_p 0$. The matrix pB_3 can be annihilated by means of pI to the left. This creates fill-ins in the L-row to the right of I that can be annihilated by means of I in the H-row. So $pB_3 = 0$. Now pA_3 has no 0-line to avoid crosses. But then its Smith Normal Form is pI, and when changing to the Smith Normal Form, p^2I to the left and I below can be reestablished. Hence we arrive at (11).

(e) $pF_4 = 0$ and $F_1 = [I \mid 0]$.

Assume that pF_4 is present. Then the F-row and the fourth block column are present. We will show that $pF_4 = 0$.

Note the following consequences if additional blocks are present, and some properties of the entries of pB_4, F_1.

(1) If pB_4 is present, then the rows B, L and the fourth block column are present. Moreover, the entries of pB_4 are either 0 or in $p\mathbb{Z} \setminus p^2\mathbb{Z}$ since the entries in $p^2\mathbb{Z}$ can be annihilated by pI to the left in β_1. The fill-ins in the L-row below pB_4 are 0 modulo p. There is no 0-row in pB_4 to avoid a vertical double cross located in β_1.

(2) If F_1 is present, then the rows C and the block column 1 is present. Moreover, the entries in F_1 are either 0 or units, since the entries of F_1 in $p\mathbb{Z}$ can be annihilated by the I above in the C-row. The fill-ins in the F-row are 0 modulo p^2.

There are four cases, depending on whether pB_4, F_1 are present or not.

(1) If both pB_4, F_1 are absent, then this leads to a cross located in pF_4. So we may assume that either pB_4 or F_1 is present or both.

(2) Assume that pB_4 is present and F_1 is not. The Smith Normal Form of pB_4 is $[pI \mid 0]$ since there is no 0-row. We annihilate in pF_4 and get $[0 \mid pF_4']$. There are fill-ins below pI in the F-row of β_1 that can be annihilated by I in the L-row. An entry p in pF_4' allows to annihilate in p^2D_4 if this block is present at all. Now this leads to a cross located at this p. Thus pF_4=0 in this case.

(3) Assume that F_1 is present and pB_4 is not. The Smith Normal Form of F_1 is $\left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right]$ or a specialization thereof. We annihilate in pF_4 and get $\begin{bmatrix} 0 \\ pF_4' \end{bmatrix}$. There are fill-ins right of I in the C-row above pF_4 that can be annihilated by pI in the C-row. An entry p in pF_4' allows to annihilate in p^2D_4 if this block is present at all. Now this leads to a cross located at this p. Thus $pF_4 = 0$ in this case.

(4) Assume that both pB_4, F_1 are present. Then the Smith Normal Forms of pB_4, F_1 are as in (2) and (3). We annihilate with both, pI in the Smith Normal Form of pB_4, and with I in the Smith Normal Form of F_1. This can be done independently

and the fill-ins can be annihilated as in (2) and (3). We get $\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & pF_4'' \end{array}\right]$. Now this leads to a cross located at this p. Thus $pF_4 = 0$ also in this last case.

Hence $pF_4 = 0$. But then the Smith Normal Form of F_1 is $[I \mid 0]$ since there are no 0-rows.

(f) $p^2 D_4 = 0$, $pB_4 = pI$, $pD_1 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & pI \end{array}\right]$, $pD_2 = \begin{bmatrix} pI \\ 0 \end{bmatrix}$.

Assume that $p^2 D_4$ is present. Then the D-row and the fourth block column are present. We will show that $p^2 D_4 = 0$.

Note the following consequences if additional blocks are present, and some properties of the entries of pB_4, F_1, pD_1, pD_2.

(1) If pB_4 is present, then the rows B, L are present. Moreover, the entries of pB_4 are either 0 or in $p\mathbb{Z} \setminus p^2\mathbb{Z}$ since the entries in $p^2\mathbb{Z}$ can be annihilated by pI to the left in β_1. The fill-ins in the L-row below pB_4 are 0 modulo p. There is no 0-row in pB_4 to avoid a vertical double cross located in β_1.

(2) If F_1 is present, then the rows F, C, the block column 1 and pD_1 are present. Recall that the Smith Normal Form of F_1 is $[I \mid 0]$.

(3) If pD_1 is present, then the row C and the block column 1 are present. Moreover, the entries of pD_1 are either 0 or in $p\mathbb{Z} \setminus p^2\mathbb{Z}$ since the entries in $p^2\mathbb{Z}$ can be annihilated by I above in the row C. The fill-ins in β_1 are 0 modulo p^3.

(4) If pD_2 is present then the row E and the block column 2 are present. Moreover, the entries of pD_2 are either 0 or in $p\mathbb{Z} \setminus p^2\mathbb{Z}$ since the entries in $p^2\mathbb{Z}$ can be annihilated by I below in the row E. The fill-ins in β_1 are 0 modulo p^3.

(f1) Presence and absence of pB_4.

If pB_4 is present, we establish its Smith Normal Form which is $[pI \,|\, 0]$ because there is no 0-row. We annihilate in $p^2 D_4$ and get $[0 \mid p^2 D_4']$. The fill-ins below pI in the D-row of β_1 are all in $p^2\mathbb{Z}$ and can be annihilated by I below in the L-row. If pB_4 is not present, then $p^2 D_4$ is not changed. All present blocks above and below $p^2 D_4'$ or $p^2 D_4$, respectively, are 0. It remains to show that $p^2 D_4'$ or $p^2 D_4$, respectively, are 0. These two cases can be dealt with together.

(f2) Presence and absence of F_1.

If F_1 is present, we establish its Smith Normal Form which is $[I \,|\, 0]$. We annihilate in pD_1 and get $[0 \mid pD_1']$. There are no fill-ins. If F_1 is not present, then pD_1 is not changed. Since we continue to produce blocks that are either 0 or of the form $p^h I$ these two cases can be dealt with together. Note that if $F_1 \neq I$, then the first block column splits.

(f3) Presence and absence of the block columns 1 and 2 of β_2.

(1) If none of the two block columns 1 or 2 is present and $p^2D_4 \neq 0$, then this leads to a cross located in p^2D_4 regardless of whether pB_4 is present or not. So $p^2D_4 = 0$ in this case.

(2) If the second block column is present and the first block column is not present, i.e., if pD_2 is present and pD_1 not, then the Smith Normal Form of pD_2 is pI, since there is no 0-line. A 0-row leads to a cross located in p^2D_4, a 0-column leads to a horizontal double cross located in the E-row.

We use the Smith Normal Form of pD_2 to annihilate p^2D_4. There are fill-ins below p^2D_4 in the E-row that can be annihilated by pI to the left in β_1. Thus $p^2D_4 = 0$ in this case.

(3) If the first block column is present and the second block column is not present, i.e., if pD_1 is present and pD_2 not, then we create the Smith Normal Form of pD_1 in the case that F_1 is not present, and the Smith Normal Form of pD'_1 in the case that F_1 is present and $pD_1 = [0 \mid pD'_1]$. There are no 0-lines in the respective Smith Normal Forms, since a 0-row leads to a cross located in p^2D_4, a 0-column leads to a horizontal double cross located in the C-row. So the Smith Normal Forms of pD_1 and of pD'_1 both are pI.

We use the Smith Normal Form either of pD_1 or of pD'_1 to annihilate p^2D_4. There are fill-ins above p^2D_4 in the C-row that can be annihilated by pI to the left in β_1. Thus $p^2D_4 = 0$ in this case.

(4) If the first and the second block columns are present, then pD_1 and pD_2 both are present. The Smith Normal Form of pD_2 is $pD_2 = \left[\begin{array}{c} pI \\ \hline 0 \end{array}\right] f$ since there is no 0-column.

We use the pI in the Smith Normal Form of pD_2 to annihilate in pD_1. Depending on the presence of F_1 we obtain either

$$\begin{bmatrix} 0 & 0 \\ 0 & pD''_1 \end{bmatrix} f \quad \text{or} \quad \begin{bmatrix} 0 \\ pD'_1 \end{bmatrix} f.$$

Now again depending on the presence of F_1 we form the Smith Normal Form of pD''_1 or of pD'_1 which is in both cases pI since those Smith Normal Forms have no 0-lines, since a 0-row in pD''_1 or in pD'_1 leads to a cross located in p^2D_4 and a 0-column in pD''_1 or in pD'_1 leads to horizontal double crosses located either in the row C or the row E.

We annihilate in p^2D_4 with pI in the respective Smith Normal Forms of pD_2, and of pD''_1 or pD'_1. Hence we obtain for p^2D_4 either

$$\left[\begin{array}{c} 0 \\ 0 \\ \hline p^2D'_4 \end{array}\right] \quad \text{or} \quad \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ \hline 0 & p^2D''_4 \end{array}\right],$$

depending on the presence of pB_4. There are fill-ins above and below p^2D_4 in the rows C and D, respectively. But those can be annihilated as in (2) and (3).

If p^2D_4'' or p^2D_4' are not 0, then this leads to a cross located in p^2D_4. This last contradiction shows that $p^2D_4 = 0$.

A consequence of $p^2D_4 = 0$ is that the Smith Normal Form of pB_4 is pI, to avoid a 0-column.

Furthermore, we have obtained that pD_1, pD_2 are of the form $\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & pI \end{array}\right]$ and $\begin{bmatrix} pI \\ 0 \end{bmatrix}$, respectively.

(g) Final Coordinate Matrices.

The coordinate matrices of indecomposable $(1,3)$-groups of rank 4 have only the A-row, and this displays the group as in Proposition 31 (1).

All coordinate matrices of indecomposable $(1,3)$-groups that have no summand of rank $\leqslant 4$ can be transformed to a matrix of the form as in (12). Note that not all block lines in this matrix must be present and that block rows and block columns that intersect in a (square) block of the form p^hI either are both present or both absent.

$$[\beta_1 \mid \beta_2] = \left[\begin{array}{cccc|c||cc|c|c|c}
p^2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 \\
0 & pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & pI \\
0 & 0 & pI & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & pI & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & pI & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & pI & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ \hline
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0
\end{array}\right]
\begin{array}{rl}
A,\ p^3 & \\
B,\ p^3 & \\
C^1,\ p^3 & \\
C^2,\ p^3 & \\
D^1,\ p^3 & \\
D^2,\ p^3 & l_1 \\ \hline
E,\ p^2 & \\
F,\ p^2 & l_1 + l_2 \\ \hline
L,\ p & \\
H,\ p &
\end{array} \tag{12}$$

By (f) the D-row and the D_1-column split, so in turn also the C-row splits and we end up with $[\beta_1 \mid \beta_2]$ as above. Now we can read off the $[\beta_1 \mid \beta_2]$-part of the indecomposable groups. There are the following row constellations: (A, H), (B, L), (C^1, F), (C^2, D^2), (D^1, E). The corresponding types of groups following the list in Proposition 31 are (5), (6), (4), (2), (3). $\square$

Theorem 33. *There are indecomposable* $(1,3)$*-groups with regulator quotient of exponent* p^4 *of arbitrary large rank.*

P r o o f. Let A be a square integer matrix that considered over $\mathbb{Z}_p$ has a characteristic polynomial equal to the minimum polynomial that is a power of some irreducible polynomial modulo p. Then A and any matrix similar to A modulo p is indecomposable modulo p. We prove that the $(1,3)$-group G of rank $5n$ with regulator quotient isomorphic to $(\mathbb{Z}_{p^4})^n \oplus (\mathbb{Z}_{p^2})^n$ and such that the coordinate matrix

$$[\alpha_0 \mid \beta_1 \mid \beta_2 \mid \beta_3] = \begin{array}{c} \left[\begin{array}{cc|c|c|c} I_n & 0 & p^2 I_n & pI_n & I_n \\ \hline 0 & I_n & pA & I_n & 0 \end{array}\right] \end{array} \begin{array}{c} p^4 \\ p^2 \end{array}$$

is indecomposable. We have to show that

$$\begin{bmatrix} p^2 M_{1,1} & pM_{1,2} \\ pM_{2,1} & M_{2,2} \end{bmatrix} = U[\beta_1 \mid \beta_2]Y$$

$$= \left[\begin{array}{c|c} (p^2U_{1,1} + p^3U_{1,2}A)Y_{1,1} & (p^2U_{1,1} + p^3U_{1,2}A)Y_{1,2} + (pU_{1,1} + p^2U_{1,2})Y_{2,2} \\ \hline pU_{2,2}AY_{1,1} & pU_{2,2}AY_{1,2} + (pU_{2,1} + U_{2,2})Y_{2,2} \end{array}\right]$$

is S-indecomposable where $U = \begin{bmatrix} U_{1,1} & p^2U_{1,2} \\ U_{2,1} & U_{2,2} \end{bmatrix}$ is the first component of an S-pair and $Y = \begin{bmatrix} Y_{1,1} & Y_{1,2} \\ 0 & Y_{2,2} \end{bmatrix}$ is p-invertible. The set of pairs (U, Y) is a group acting on the set of matrices $\begin{bmatrix} p^2 M_{1,1} & pM_{1,2} \\ pM_{2,1} & M_{2,2} \end{bmatrix}$. We will switch to an isomorphic group action that allows a simpler treatment of the decomposition problem. As

$$M = \begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} = \begin{bmatrix} p^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p^2 M_{1,1} & pM_{1,2} \\ pM_{2,1} & M_{2,2} \end{bmatrix} \begin{bmatrix} p^{-1} & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} U_{1,1} & pU_{1,2} \\ pU_{2,1} & U_{2,2} \end{bmatrix} = \begin{bmatrix} p^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_{1,1} & p^2U_{1,2} \\ U_{2,1} & U_{2,2} \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} Y_{1,1} & pY_{1,2} \\ 0 & Y_{2,2} \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_{1,1} & Y_{1,2} \\ 0 & Y_{2,2} \end{bmatrix} \begin{bmatrix} p^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

the pairs $\left(\begin{bmatrix} U_{1,1} & pU_{1,2} \\ pU_{2,1} & U_{2,2} \end{bmatrix}, \begin{bmatrix} Y_{1,1} & pY_{1,2} \\ 0 & Y_{2,2} \end{bmatrix} \right)$ form a group isomorphic to the original group of operators acting on the set of matrices $M = [M_{i,j}]$, because

$$\begin{bmatrix} p^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_{1,1} & p^2U_{1,2} \\ U_{2,1} & U_{2,2} \end{bmatrix} \begin{bmatrix} p^2 M_{1,1} & pM_{1,2} \\ pM_{2,1} & M_{2,2} \end{bmatrix} \begin{bmatrix} Y_{1,1} & Y_{1,2} \\ 0 & Y_{2,2} \end{bmatrix} \begin{bmatrix} p^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} U_{1,1} & pU_{1,2} \\ pU_{2,1} & U_{2,2} \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \begin{bmatrix} Y_{1,1} & pY_{1,2} \\ 0 & Y_{2,2} \end{bmatrix}.$$

In particular, the pairs $\left(\begin{bmatrix} U_{1,1} & 0 \\ 0 & U_{2,2} \end{bmatrix}, \begin{bmatrix} Y_{1,1} & 0 \\ 0 & Y_{2,2} \end{bmatrix}\right)$ are in both operating groups and describe corresponding operations. A necessary condition for the S-decomposability of $\begin{bmatrix} p^2M_{1,1} & pM_{1,2} \\ pM_{2,1} & M_{2,2} \end{bmatrix}$ is that modulo p

$$M = \begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \equiv \begin{bmatrix} U_{1,1}Y_{1,1} & U_{1,1}Y_{2,2} \\ U_{2,2}AY_{1,1} & U_{2,2}Y_{2,2} \end{bmatrix}$$

is decomposable. Since $U_{i,i}, Y_{j,j}$ are p-invertible, so, possibly with the exception of $M_{2,1}$, the matrices $M_{i,j}$ are p-invertible.

By way of contradiction we assume that M is decomposable modulo p. From now on all congruences are modulo p. For M the hypothesis of Lemma 18 is satisfied, thus there are permutation matrices P_1, P_2, Q_1, Q_2, all of size n such that

$$M' = \begin{bmatrix} M'_{1,1} & M'_{1,2} \\ M'_{2,1} & M'_{2,2} \end{bmatrix} \equiv \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$$

where $M'_{i,j} \equiv \begin{bmatrix} X_{i,j} & 0 \\ 0 & Z_{i,j} \end{bmatrix}$ with $X_{i,j}$, $Z_{i,j}$ that have rows and columns for all (i,j), i.e., M' has a compatible decomposition. Note that all $X_{i,j}$, $Z_{i,j}$ are p-invertible for $(i,j) \neq (2,1)$, since the matrices $M_{i,j}$ are p-invertible for $(i,j) \neq (2,1)$. All $X_{i,j}$ are of the same size for all i, j, and the same holds for the $Z_{i,j}$.

We choose p-invertible matrices U'_i, Y'_i for $i = 1,2,3,4$ where U'_1, U'_3, Y'_1, Y'_3 have the size of $X_{i,j}$ and the other matrices have the size of $Z_{i,j}$. It is easy to see that we may choose U'_i, Y'_i even so that

$$\operatorname{diag}(U'_1, U'_2, U'_3, U'_4) M' \operatorname{diag}(Y'_1, Y'_2, Y'_3, Y'_4) \equiv \begin{bmatrix} I_n & I_n \\ C & I_n \end{bmatrix} \tag{13}$$

where the block C is decomposed as the blocks $M'_{i,j}$.

This shows that if there are $U' = \begin{bmatrix} U'_{1,1} & pU'_{1,2} \\ pU'_{2,1} & U'_{2,2} \end{bmatrix}$ and $Y' = \begin{bmatrix} Y'_{1,1} & pY'_{1,2} \\ 0 & Y'_{2,2} \end{bmatrix}$ such that $U' \begin{bmatrix} I_n & I_n \\ A & I_n \end{bmatrix} Y'$ is decomposable modulo p, then there are also matrices $U = \begin{bmatrix} U_{1,1} & pU_{1,2} \\ pU_{2,1} & U_{2,2} \end{bmatrix}$ and $Y = \begin{bmatrix} Y_{1,1} & pY_{1,2} \\ 0 & Y_{2,2} \end{bmatrix}$ such that

$$\begin{aligned} U \begin{bmatrix} I_n & I_n \\ A & I_n \end{bmatrix} Y &= \begin{bmatrix} U_{1,1} & pU_{1,2} \\ pU_{2,1} & U_{2,2} \end{bmatrix} \begin{bmatrix} I_n & I_n \\ A & I_n \end{bmatrix} \begin{bmatrix} Y_{1,1} & pY_{1,2} \\ 0 & Y_{2,2} \end{bmatrix} \\ &\equiv \begin{bmatrix} U_{1,1}Y_{1,1} & U_{1,1}Y_{2,2} \\ U_{2,2}AY_{1,1} & U_{2,2}Y_{2,2} \end{bmatrix} \equiv \begin{bmatrix} I_n & I_n \\ U_{2,2}AY_{1,1} & I_n \end{bmatrix} \end{aligned}$$

with $U_{2,2}AY_{1,1}$ properly decomposed modulo p.

Hence $U_{1,1} \equiv U_{2,2} \equiv Y_{1,1}^{-1} \equiv Y_{2,2}^{-1} \bmod p$. But then also $U_{2,2}AY_{1,1} \equiv Y_{1,1}^{-1}AY_{1,1} \bmod p$ is decomposed, contradicting the hypothesis on A. This shows that the groups above with the indicated coordinate matrices are indecomposable. □

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