Czechoslovak Mathematical Journal

Mario Petrich

Normal cryptogroups with an associate subgroup

Czechoslovak Mathematical Journal, Vol. 63 (2013), No. 2, 289–305

Persistent URL: `http://dml.cz/dmlcz/143313`

Czechoslovak Mathematical Journal, 63 (138) (2013), 289–305

NORMAL CRYPTOGROUPS WITH AN ASSOCIATE SUBGROUP

Mario Petrich, Bol

(Received November 17, 2010)

Abstract. Let S be a semigroup. For $a, x \in S$ such that $a = axa$, we say that x is an associate of a. A subgroup G of S which contains exactly one associate of each element of S is called an associate subgroup of S. It induces a unary operation in an obvious way, and we speak of a unary semigroup satisfying three simple axioms.

A normal cryptogroup S is a completely regular semigroup whose $\mathscr{H}$-relation is a congruence and $S/\mathscr{H}$ is a normal band. Using the representation of S as a strong semilattice of Rees matrix semigroups, in a previous communication we characterized those that have an associate subgroup.

In this paper, we use that result to find three more representations of this semigroup. The main one has a form akin to the one of semigroups in which the identity element of the associate subgroup is medial.

Keywords: semigroup, normal cryptogroup, associate subgroup, representation, strong semilattice of semigroups, Rees matrix semigroup

MSC 2010: 20M10

1. Introduction and summary

A normal cryptogroup S is a completely regular semigroup in which $\mathscr{H}$ is a congruence and $S/\mathscr{H}$ is a normal band (i.e., satisfies the identity $axya = ayxa$). It is faithfully represented as a strong semilattice Y of completely simple semigroups S_α, in notation $[Y; S_\alpha, \sigma_{\alpha,\beta}]$.

For elements a and x of an arbitrary semigroup S, x is an *associate* of a if $a = axa$. A subgroup G of S is an *associate subgroup* of S if every element s of S has a unique associate, say s^*, in G. This induces a unary operation $s \mapsto s^*$ on S which is governed by three simple axioms. By a sequence of generalizations, a structure theorem for a special class of such semigroups is proved in [1].

We combine the two concepts evoked above arriving at the semigroups in the title of the paper. Necessary and sufficient conditions on a strong semilattice of Rees matrix semigroups to contain an associate subgroup were established in [3]. They are viewed as unary semigroups in which the unary operation implicitly contains the concept of an associate subgroup. These conditions are simple enough: Y is a monoid and all structure homomorphisms restricted to maximal subgroups are isomorphisms. Since an associate subgroup is a maximal subgroup of the semigroup, all maximal subgroups are isomorphic to the associate subgroup. Also other properties of these semigroups can be found in [3].

The structure theorem in [1] arrives at a description in quite a different way. If the two constructions in [1] and [3] are to represent cases of a common generalization, it should be possible to bring their forms to sufficient similarity. Our aim here is to find an isomorphic copy of our description cited above in [3] to a form similar to the one in [1]. In the course of the needed argument, the most difficult part turned out to be showing that our form in [3] can be "normalized" in the sense that in the representation $[Y; S_\alpha, \sigma_{\alpha,\beta}]$ we may assume that all structure groups G_α may be set equal, and all the structure homomorphisms restricted to maximal subgroups may be set equal to the identity transformation. After this all is smooth sailing: essentially two successive changes of notation lead to the desired result.

In Section 2 we summarize the needed material, prove a theorem concerning completely simple semigroups, and briefly discuss the general case. In Section 3 we set up the first representation of normal cryptogroups with an associate subgroup and discuss some refinements of its parameters. Section 4 contains a long preparation for the second representation based on the first. A change of notation in the second representation in Section 5 leads to the third. Similarly, in Section 6, another change of notation results in the fourth representation, the main goal of the paper.

2. Background

Let S be a semigroup with a unary operation $s \mapsto s^*$ satisfying the following axioms.

(A1) $s = ss^*s \quad (s \in S)$.

(A5) $(s^*t^*)^{**} = s^*t^* \quad (s, t \in S)$.

(A6) $s = st^*s \Rightarrow s^* = t^* \quad (s, t \in S)$.

By [2, Lemma 3.5] we get $s^*s^{**} = t^{**}t^*$ for all $s, t \in S$. We denote this common value by z and call it the *zenith* of S.

We first state the precise relationship of the concepts of an associate subgroup and of a unary semigroup satisfying the above axioms.

Fact 2.1. *Let S be a semigroup. For every $s \in S$, let $A(s)$ be the set of all associates of s. If S has an associate subgroup G, for every $s \in S$ let us define the element s^* by*

$$A(s) \cap G = \{s^*\}. \tag{A}$$

Then the unary operation $s \mapsto s^*$ satisfies the above axioms. Conversely, if S has a unary operation $s \mapsto s^*$ satisfying the above axioms with zenith z, then H_z is an associate subgroup of S and (A) *holds.*

Proof. See [2, Theorem 3.1]. □

In view of this result, we will refer to the unary semigroup S satisfying axioms (A1), (A5) and (A6) simply as a *semigroup* and to its unary homomorphisms simply as *homomorphisms*.

For any regular semigroup S, we denote by $C(S)$ the *core* of S, that is the subsemigroup of S generated by the set $E(S)$ of its idempotents. Recall from [1] that an idempotent z of S is *medial* if $c = czc$ for all $c \in C(S)$. We now state a structure theorem for semigroups whose zenith is medial.

Fact 2.2. *Let C be an idempotent generated semigroup with a medial idempotent w. Let G be a group and ζ a homomorphism of G into the automorphism group $\mathscr{A}(wCw)$ of wCw, in notation $\zeta\colon\ g \mapsto \zeta_g$. On the set*

$$\{(x, g, a) \in Cw \times G \times wC;\ \zeta_g(aw) = wx\}$$

define a product by

$$(x, g, a)(y, h, b) = (x\zeta_g(ay), gh, \zeta_{h^{-1}}(ay)b)$$

and a unary operation by

$$(x, g, a)^* = (w, g^{-1}, w).$$

The algebra so obtained, denoted by $[C, G; w, \zeta]$, is a semigroup satisfying axioms (A1), (A5) *and* (A6) *whose zenith is medial. Conversely, every semigroup whose zenith is medial is isomorphic to some $[C, G; w, \zeta]$.*

Proof. See [1, Theorem 4]. □

In the proof of the converse of Fact 2.2, the isomorphism from a semigroup S onto a semigroup $[C, G; w, \zeta]$ is of the form

$$s \mapsto (ss^*, s^{**}, s^*s),$$

where the product of such triples has the second component equal to $s^{**}t^{**}$, see the multiplication in Fact 2.2. This is an essential feature of this product. As a bonus, $st(st)^*$ and $(st)^*st$ are computable in terms of ss^*, tt^*, s^*s and t^*t.

The structure of the general case remains open. It seems likely that a construction for the general case is feasable, based on the above mapping, and that it should represent a reasonable generalization of Fact 2.2. This is the principal motivation for the present study. We are still far from attacking the general case, so we will limit ourselves to the (very) special case of normal cryptogroups.

What we are aiming at is a case in which the first and the third entries multiply coordinatewise and the second does not, just the diametrically opposite case of that in Fact 2.2. We illustrate this on completely simple semigroups; in addition this represents a special case of our main study of normal cryptogroups, the topic of the present paper.

Theorem 2.3. *Let I be a left and Λ a right zero semigroup, respectively, let $B = I \times \Lambda$ be their direct product, G a group, and $p\colon \Lambda \times I \to G$ a function, in notation $(\lambda, i) \mapsto p_{\lambda i}$. We write the elements of the cartesian product $G \times B$ as $(g; i, \lambda)$, and on it we define a multiplication by*

$$(g; i, \lambda)(h; j, \mu) = (g p_{\lambda j} h; i, \mu). \tag{1}$$

We fix an element $1 \in \Lambda \cap I$ and assume that $p_{\lambda 1} = p_{1i} = e$, the identity element of G, for all $\lambda \in \Lambda$ and $i \in I$. Next we define a unary operation on $G \times B$ by

$$(g; i, \lambda)^* = (1, g^{-1}, 1).$$

The algebra so obtained, denoted by $[G, B; p]$, is a unary completely simple semigroup satisfying axioms (A1), (A5) *and* (A6).

Conversely, every such semigroup is isomorphic to some $[B, G; p]$.

P r o o f. *Direct.* This requires simple verification.

Converse. Given a completely simple unary semigroup S satisfying axioms (A1), (A5) and (A6), by the Rees theorem it is isomorphic to a Rees matrix semigroup, say $\varphi\colon S \to T$. If z is the zenith of S, its image $z\varphi$ can be used, in an obvious way, to define an operation on T making φ an isomorphism.

Let $T = \mathscr{M}(I, G, \Lambda; P)$ with a unary operation $t \mapsto t^*$. We may denote its zenith by $z = (1, p_{11}^{-1}, 1)$ where $1 \in I \cap \Lambda$. We will change the matrix P to a matrix Q according to [4, Lemma III.3.6], namely we let

$$q_{\lambda i} = p_{\lambda 1}^{-1} p_{\lambda i} p_{1i}^{-1} p_{11} \quad (i \in I,\ i \in \Lambda),$$

$Q = (q_{\lambda i})$ and $U = \mathscr{M}(I, G, \Lambda; Q)$. Also we define functions u and v by

$$u\colon\ i \mapsto u_i = p_{11}^{-1} p_{1i}, \quad v\colon\ \lambda \mapsto v_\lambda = p_{\lambda 1} \quad (i \in I,\ \lambda \in \Lambda).$$

By the above reference, Q is normalized at 1 and the mapping

$$\chi\colon\ (i, g, \lambda) \mapsto (i, u_i g v_\lambda, \lambda) \quad ((i, g, \lambda) \in T)$$

is an isomorphism of T onto U. Giving U the unary operation induced by $(1, e, 1)$ as a zenith, it is readily verified that χ preserves the unary operation. Therefore χ is an isomorphism of T onto U, and the composition $\varphi\chi$ is an isomorphism of S onto U.

We can easily convert U into the form $[B, G; p]$, and the assertion of the converse follows. □

Notice that the form of the multiplication in (1) is the often used notation for Rees matrix semigroups. Observe the similarities and the differences between $[B, G; p]$ and $[C, G; w, \zeta]$. We may consider the sandwich matrix of the former as a mapping $B^{\mathrm{opp}} \to G$, where B^{opp} is the semigroup defined on B with “opposite multiplication”, that is $a \circ b = ba$ for all $a, b \in B$. In Fact 2.2, we have a homomorphism $G \to \mathscr{A}(wCw)$, that is in the opposite direction. The basic difference is also between the form of their product. It seems likely that the form of the general case will incorporate the features of both of these examples.

3. First representation

We follow the book [4] for terminology and notation. For emphasis, we now state a few of these.

For a nonempty set X, ι_X denotes the identity map on X. Let S be a semigroup. Then $E(S)$ denotes the set of all idempotents in S. Further, S is completely regular if it is a union of (maximal pairwise disjoint) groups. Alternatively, it is a semilattice of completely simple semigroups, which we denote by $S = (Y; S_\alpha)$. The latter will be represented by Rees matrix semigroups $\mathscr{M}(I, G, \Lambda; P)$ where P may be assumed normalized. If Green's relation $\mathscr{H}$ is a congruence on a completely regular semigroup S, we call it a *cryptogroup*. If, in addition, $S/\mathscr{H}$ is a normal band (that is, it satisfies the identity $axya = ayxa$), then S is termed a *normal cryptogroup*. We will represent these as strong semilattices of completely simple semigroups, defined as follows.

Let Y be a semilattice. For every $\alpha \in Y$, let S_α be a semigroup and assume that $S_\alpha \cap S_\beta = \emptyset$ whenever $\alpha \neq \beta$. For any $\alpha, \beta \in Y$ such that $\alpha \geqslant \beta$, let $\sigma_{\alpha,\beta}\colon\ S_\alpha \to S_\beta$ be a homomorphism, and suppose that

$$\sigma_{\alpha,\alpha} = \iota_{S_\alpha} \quad (\alpha \in Y),$$
$$\sigma_{\alpha,\beta}\sigma_{\beta,\gamma} = \sigma_{\alpha,\gamma} \quad (\alpha \geqslant \beta \geqslant \gamma \text{ in } Y).$$

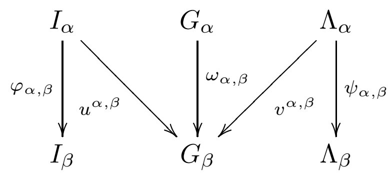
On $S = \bigcup_{\alpha \in Y} S_\alpha$ let us define a multiplication by: for $a \in S_\alpha$, $b \in S_\beta$,

$$a \circ b = (a\sigma_{\alpha,\alpha\beta})(b\sigma_{\beta,\alpha\beta}).$$

Then S is a semigroup said to be a *strong semilattice* Y of semigroups S_α, in notation $S = [Y; S_\alpha, \sigma_{\alpha,\beta}]$.

By [4, Theorem IV.1.6], a semigroup S is a normal cryptogroup if and only if S is isomorphic to a strong semilattice of completely simple semigroups. The latter will be represented as Rees matrix semigroups $\mathscr{M}(I, G, \Lambda; P)$.

Notation 3.1. Let Y be a semilattice. For all $\alpha \in Y$, let $S_\alpha = \mathscr{M}(I_\alpha, G_\alpha, \Lambda_\alpha; P^\alpha)$ and let $\alpha, \beta \in Y$ be such that $\alpha \geqslant \beta$. Let the following mappings be given:



where ω_α is a homomorphism, $u^{\alpha,\beta}\colon\ i \mapsto u_i^{\alpha,\beta}$, $v^{\alpha,\beta}\colon\ \lambda \mapsto v_\lambda^{\alpha,\beta}$, $\varphi_{\alpha,\alpha} = \iota_{I_\alpha}$, $u_i^{\alpha,\alpha} = v_\lambda^{\alpha,\alpha} = e_\alpha$ is the identity element of G_α, $\psi_{\alpha,\alpha} = \iota_{\Lambda_\alpha}$ for all $i \in I_\alpha$, $\lambda \in \Lambda_\alpha$. Define a function $\sigma_{\alpha,\beta}$ by

$$\sigma_{\alpha,\beta}\colon\ (i, g, \lambda) \mapsto (i\varphi_{\alpha,\beta}, u_i^{\alpha,\beta}(g\omega_{\alpha,\beta})v_\lambda^{\alpha,\beta}, \lambda\psi_{\alpha,\beta}) \quad ((i, g, \lambda) \in S_\alpha)$$

with notation

$$\sigma_{\alpha,\beta} = \chi(\varphi_{\alpha,\beta}, u^{\alpha,\beta}, \omega_{\alpha,\beta}, v^{\alpha,\beta}, \psi_{\alpha,\beta}).$$

The present paper is based on [3, Theorem 6.1] of which we now state the essential part.

Fact 3.2. *Let S be a normal cryptogroup. Then S has an associate subgroup if and only if $S \cong [Y; S_\alpha, \sigma_{\alpha,\beta}]$ for some parameters, where Y is a monoid and all $\omega_{\alpha,\beta}$ in $\sigma_{\alpha,\beta} = \chi(\varphi_{\alpha,\beta}, u^{\alpha,\beta}, \omega_{\alpha,\beta}, v^{\alpha,\beta}, \psi_{\alpha,\beta})$ are isomorphisms.*

If these conditions are satisfied, let ε be the identity element of Y and $z = (k, p_{\nu\kappa}^{-1}, \nu) \in S_\varepsilon$. Then H_z is an associate subgroup of $[Y; S_\alpha, \sigma_{\alpha,\beta}]$ and for the corresponding unary operation, for $a = (i, g, \lambda) \in S_\alpha$,

$$\text{(U)} \qquad a^* = (k, (v_\nu^{\varepsilon,\alpha} p^\alpha_{\nu\psi_{\varepsilon,\alpha},i} g p^\alpha_{\lambda,k\psi_{\varepsilon,\alpha}} u_k^{\varepsilon,\alpha})^{-1}\omega_{\varepsilon,\alpha}^{-1}, \nu).$$

P r o o f. In the light of [4, Theorem IV.1.6], this forms part of [3, Theorem 6.1]. □

In view of this result, we may set

$$S = [Y, S_\alpha, \sigma_{\alpha,\beta}] \text{ with zenith } z = (k, p^{-1}_{\nu\kappa}, \nu).$$

The cited [3, Theorem 6.1] includes other characterizations of the class of normal cryptogroups with an associate subgroup. In particular, it suffices to assume only that all $\omega_{\varepsilon,\alpha}$ be isomorphisms. In addition, a group G is an associate subgroup of S if and only if G is a maximal subgroup of S_ε. In view of the example in [3, Section 4] this does not mean that the unary semigroups resulting from two associate subgroups must be isomorphic.

Corollary 3.3. *In a normal cryptogroup, the zenith uniquely determines the unary operation.*

In view of this, for normal cryptogroups, it suffices to give the idempotent which will serve as the zenith z, for then a^* is the unique solution of the equation $a = axa$ where x is in the group H_z. As a consequence we have

Corollary 3.4. *Let S and T be normal cryptogroups. Then any multiplicative homomorphism of S into T which preserves the zeniths is a (unary) homomorphism.*

The converse of Corollary 3.4 holds trivially. The semigroup $[Y; S_\alpha, \sigma_{\alpha,\beta}]$ is our *first representation.*

We will need the following notation. Tacitly we continue with the notation already introduced, but for the remainder of this section ignore the unary operation.

Notation 3.5. For any $\alpha, \beta, \gamma \in Y$ such that $\alpha \geqslant \beta \geqslant \gamma$, let $c_{\alpha,\beta,\gamma}$ be an element of G_γ in the following conditions.

(H) $$p^\alpha_{\lambda i}\omega_{\alpha,\beta} = v^{\alpha,\beta}_\lambda p^\beta_{\lambda\psi_{\alpha,\beta}, i\varphi_{\alpha,\beta}} u^{\alpha,\beta}_i \quad (\lambda \in \Lambda_\alpha,\ i \in I_\alpha).$$

(B) $$\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}, \quad \psi_{\alpha,\beta}\psi_{\beta,\gamma} = \psi_{\alpha,\gamma}.$$

(L) $$(u^{\beta,\gamma}_{i\varphi_{\alpha,\beta}})(u^{\alpha,\beta}_i \omega_{\beta,\gamma})c^{-1}_{\alpha,\beta,\gamma} = u^{\alpha,\gamma}_i \quad (i \in I_\alpha).$$

(M) $$g\omega_{\alpha,\beta}\omega_{\beta,\gamma} = c^{-1}_{\alpha,\beta,\gamma}(g\omega_{\alpha,\gamma})c_{\alpha,\beta,\gamma} \quad (g \in G_\alpha).$$

(R) $$c_{\alpha,\beta,\gamma}(v^{\alpha,\beta}_\lambda \omega_{\beta,\gamma})(v^{\beta,\gamma}_{\lambda\psi_{\alpha,\beta}}) = v^{\alpha,\gamma}_\lambda \quad (\lambda \in \Lambda_\alpha).$$

(H, B, L, M and R stand for a homomorphism, band, left, middle and right, respectively.)

The next result combines some of the notation introducing certain refinements; it will be used in the next section.

Theorem 3.6. *Let Y be a semilattice. For all $\alpha \in Y$, let $S_\alpha = \mathscr{M}(I_\alpha, G_\alpha, \Lambda_\alpha; P^\alpha)$ and assume that $S_\alpha \cap S_\beta = \emptyset$ whenever $\alpha \neq \beta$. For any $\alpha \geqslant \beta$, let*

$$\sigma_{\alpha,\beta} = \chi(\varphi_{\alpha,\beta}, u^{\alpha,\beta}, \omega_{\alpha,\beta}, v^{\alpha,\beta}, \psi_{\alpha,\beta}).$$

(i) *For any $\alpha \in Y$, $\sigma_{\alpha,\alpha} = \iota_{S_\alpha}$.*
(ii) *For any $\alpha, \beta \in Y$ such that $\alpha \geqslant \beta$, $\sigma_{\alpha,\beta}$ is a homomorphism if and only if condition* (H) *holds.*
(iii) *For any $\alpha, \beta, \gamma \in Y$ such that $\alpha \geqslant \beta \geqslant \gamma$, $\sigma_{\alpha,\beta}\sigma_{\beta,\gamma} = \sigma_{\alpha,\gamma}$ if and only if conditions* (B), (L), (M) *and* (R) *hold.*
(iv) *If these conditions are satisfied, then $[Y; S_\alpha, \sigma_{\alpha,\beta}]$ is a normal cryptogroup. Conversely, if S is a normal cryptogroup, then $S \cong [Y; S_\alpha, \sigma_{\alpha,\beta}]$ for some parameters.*

Proof. (i) This is trivial.

(ii) This is readily verified.

(iii) We let $(i, g, \lambda) \in S_\alpha$ and calculate

$$(1) \qquad (i,g,\lambda)\sigma_{\alpha,\beta}\sigma_{\beta,\gamma} = (i\varphi_{\alpha,\gamma}, u_i^{\alpha,\gamma}(g\omega_{\alpha,\beta})v_\lambda^{\alpha,\beta}, \lambda\psi_{\alpha,\beta})\sigma_{\beta,\gamma}$$

$$(2) \qquad = (i\varphi_{\alpha,\beta}\varphi_{\beta,\gamma}, u^{\beta,\gamma}_{i\varphi_{\alpha,\beta}}(u_i^{\alpha,\beta}\omega_{\beta,\gamma})(g\omega_{\alpha,\beta}\omega_{\beta,\gamma}) \times (v_\lambda^{\alpha,\beta}\omega_{\beta,\gamma})v^{\beta,\gamma}_{\lambda\psi_{\alpha,\beta}}, \lambda\psi_{\alpha,\beta}\psi_{\beta,\gamma}),$$

$$(3) \qquad (i,g,\lambda)\sigma_{\alpha,\gamma} = (i\varphi_{\alpha,\gamma}, u_i^{\alpha,\gamma}(g\omega_{\alpha,\gamma})v_\lambda^{\alpha,\gamma}, \lambda\psi_{\alpha,\gamma}).$$

Direct. By hypothesis, the expressions in (2) and (3) are equal. Hence condition (B) holds and

$$(4) \qquad u^{\beta,\gamma}_{i\varphi_{\alpha,\beta}}(u_i^{\alpha,\beta}\omega_{\beta,\gamma})(g\omega_{\alpha,\beta}\omega_{\beta,\gamma})(v_\lambda^{\alpha,\beta}\omega_{\beta,\gamma})v^{\beta,\gamma}_{\lambda\psi_{\alpha,\beta}} = u_i^{\alpha,\gamma}(g\omega_{\alpha,\gamma})v_\lambda^{\alpha,\gamma}.$$

If $g = 1$, this yields

$$(u^{\beta,\gamma}_{i\varphi_{\alpha,\beta}})(u_i^{\alpha,\beta}\omega_{\beta,\gamma})(v_\lambda^{\alpha,\beta}\omega_{\beta,\gamma})(v^{\beta,\gamma}_{\lambda\psi_{\alpha,\beta}}) = u_i^{\alpha,\gamma}v_\lambda^{\alpha,\gamma}$$

which implies

$$(u_i^{\alpha,\gamma})^{-1}(u^{\beta,\gamma}_{i\varphi_{\alpha,\beta}})(u_i^{\alpha,\beta}\omega_{\beta,\gamma}) = v_\lambda^{\alpha,\gamma}(v^{\beta,\gamma}_{\lambda\psi_{\alpha,\beta}})^{-1}(v_\lambda^{\alpha,\beta}\omega_{\beta,\gamma})^{-1},$$

where the left hand side depends only on i and the right hand side only on λ. We may thus put the common value equal to a constant $c_{\alpha,\beta,\gamma}$. This implies (L) and (R). Using these two equalities in (4) and cancelling yields condition (M).

Converse. Using conditions (L), (M) and (R) we get the equality in (4). This together with the rest of the hypotheses gives the desired equality.

(iv) This follows from parts (i)–(iii) and [4, Theorem IV.1.6]. □

The second representation is a special case of the first: in it we require in addition that $G_\alpha = G_\varepsilon$ for all $\alpha \in Y$ and that all $\omega_{\alpha,\beta}$ be equal to ι_{G_ε}. The isomorphism of the first onto the second representation is surprisingly simple but the procedure establishing the second representation and proving all the details is relatively long. Indeed, several lemmas interspersed with voluminous notation precede the proof of the only theorem in this section.

Notation 4.1. We continue with the notation of the first representation.

For $\alpha \geqslant \beta$ and $i \in I_\alpha, \lambda \in \Lambda_\alpha$, set

$$s_i^{\alpha,\beta} = (u_i^{\alpha,\beta} c_{\varepsilon,\alpha,\beta}^{-1})\omega_{\varepsilon,\beta}^{-1}, \quad t_\lambda^{\alpha,\beta} = (c_{\varepsilon,\alpha,\beta} v_\lambda^{\alpha,\beta})\omega_{\varepsilon,\beta}^{-1}.$$

For $\alpha \geqslant \beta \geqslant \gamma$, let

$$d_{\alpha,\beta,\gamma} = c_{\alpha,\beta,\gamma}\omega_{\varepsilon,\gamma}^{-1}, \quad e_{\alpha,\beta,\gamma} = d_{\varepsilon,\alpha,\gamma} d_{\alpha,\beta,\gamma} d_{\varepsilon,\beta,\gamma}^{-1} d_{\varepsilon,\alpha,\beta}^{-1}.$$

Lemma 4.2. *Let $\alpha \geqslant \beta$ and $g \in G$. Then*

$$g\omega_{\alpha,\beta}\omega_{\varepsilon,\beta}^{-1} = d_{\varepsilon,\alpha,\beta}^{-1}(g\omega_{\varepsilon,\alpha}^{-1})d_{\varepsilon,\alpha,\beta}.$$

P r o o f. For any $h \in G_\beta$, by condition (M) we have

$$h\omega_{\varepsilon,\alpha}\omega_{\alpha,\beta} = c_{\varepsilon,\alpha,\beta}^{-1}(h\omega_{\varepsilon,\beta})c_{\varepsilon,\alpha,\beta},$$

so for $g = h\omega_{\varepsilon,\alpha}$ we get

$$g\omega_{\alpha,\beta}\omega_{\varepsilon,\beta}^{-1} = [c_{\varepsilon,\alpha,\beta}^{-1}(g\omega_{\varepsilon,\alpha}^{-1}\omega_{\varepsilon,\beta})c_{\varepsilon,\alpha,\beta}]\omega_{\varepsilon,\beta}^{-1} = d_{\varepsilon,\alpha,\beta}^{-1}(g\omega_{\varepsilon,\alpha}^{-1})d_{\varepsilon,\alpha,\beta}.$$

□

We start with the first component.

Lemma 4.3. *For $\alpha \geqslant \beta \geqslant \gamma$ and $i \in I_\alpha$ we have*

$$s_{i\varphi_{\alpha,\beta}}^{\beta,\gamma} s_i^{\alpha,\beta} = s_i^{\alpha,\gamma} d_{\alpha,\beta,\gamma}.$$

P r o o f. From Notation 4.1 we get $u_i^{\alpha,\beta} = (s_i^{\alpha,\beta}\omega_{\varepsilon,\beta})c_{\varepsilon,\alpha,\beta}$. We substitute this into (4) getting successively

$$(s_{i\varphi_{\alpha,\beta}}^{\beta,\gamma}\omega_{\varepsilon,\beta})c_{\varepsilon,\beta,\gamma}[(s_i^{\alpha,\beta}\omega_{\varepsilon,\beta})c_{\varepsilon,\alpha,\beta}]\omega_{\beta,\gamma}c_{\alpha,\beta,\gamma}^{-1} = (s_i^{\alpha,\gamma}\omega_{\varepsilon,\gamma})c_{\varepsilon,\alpha,\gamma},$$

$$(s_{i\varphi_{\alpha,\beta}}^{\beta,\gamma}\omega_{\varepsilon,\beta})c_{\varepsilon,\beta,\gamma}(s_i^{\alpha,\beta}\omega_{\varepsilon,\beta}\omega_{\beta,\gamma})(c_{\varepsilon,\alpha,\beta}\omega_{\beta,\gamma})c_{\alpha,\beta,\gamma}^{-1} = (s_i^{\alpha,\gamma}\omega_{\varepsilon,\gamma})c_{\varepsilon,\alpha,\gamma},$$

$$(s_{i\varphi_{\alpha,\beta}}^{\beta,\gamma}\omega_{\varepsilon,\beta})c_{\varepsilon,\beta,\gamma}c_{\varepsilon,\beta,\gamma}^{-1}(s_i^{\alpha,\beta}\omega_{\varepsilon,\gamma})c_{\varepsilon,\beta,\gamma}(c_{\varepsilon,\alpha,\beta}\omega_{\beta,\gamma})c_{\alpha,\beta,\gamma}^{-1} = (s_i^{\alpha,\gamma}\omega_{\varepsilon,\gamma})c_{\varepsilon,\alpha,\gamma},$$

and applying $\omega_{\varepsilon,\gamma}^{-1}$, we obtain

$$s^{\beta,\gamma}_{i\varphi_{\alpha,\beta}} s_i^{\alpha,\beta} d_{\varepsilon,\beta,\gamma}(c_{\varepsilon,\alpha,\beta}\omega_{\beta,\gamma}\omega_{\varepsilon,\gamma}^{-1})d_{\alpha,\beta,\gamma}^{-1} = s_i^{\alpha,\gamma} d_{\varepsilon,\alpha,\gamma}.$$

Using Lemma 4.2, this becomes

$$s^{\beta,\gamma}_{i\varphi_{\alpha,\beta}} s_i^{\alpha,\beta} d_{\varepsilon,\beta,\gamma}(d_{\varepsilon,\beta,\gamma}^{-1} d_{\varepsilon,\alpha,\gamma} d_{\varepsilon,\beta,\gamma})d_{\alpha,\beta,\gamma}^{-1} = s_i^{\alpha,\gamma} d_{\varepsilon,\alpha,\gamma}$$

and the assertion follows. □

Next we treat the middle component.

Notation 4.4. For every $\alpha \in Y$, $i \in I_\alpha$, $\lambda \in \Lambda_\alpha$, let

$$q^\alpha_{\lambda i} = p^\alpha_{\lambda i}\omega_{\varepsilon,\alpha}^{-1}.$$

Lemma 4.5. *For any $\alpha \geqslant \beta$, $i \in I_\alpha$, $\lambda \in \Lambda_\alpha$, we have*

$$q^\alpha_{\lambda i} = t_\lambda^{\alpha,\beta} q^\beta_{\lambda\psi_{\alpha,\beta}, i\varphi_{\alpha,\beta}} s_i^{\alpha,\beta}.$$

Proof. Using Notations 4.1 and 4.4, we get

$$\begin{aligned}
t_\lambda^{\alpha,\beta} q^\beta_{\lambda\psi_{\alpha,\beta}, i\varphi_{\alpha,\beta}} s_i^{\alpha,\beta} &= (c_{\varepsilon,\alpha,\beta} v_\lambda^{\alpha,\beta})\omega_{\varepsilon,\beta}^{-1}(p^\beta_{\lambda\psi_{\alpha,\beta}, i\varphi_{\alpha,\beta}}\omega_{\varepsilon,\beta}^{-1})(u_i^{\alpha,\beta} c_{\varepsilon,\alpha,\beta}^{-1})\omega_{\varepsilon,\beta}^{-1} \\
&= [c_{\varepsilon,\alpha,\beta}(v_\lambda^{\alpha,\beta} p^\beta_{\lambda\psi_{\alpha,\beta}, i\varphi_{\alpha,\beta}} u_i^{\alpha,\beta})c_{\varepsilon,\alpha,\beta}^{-1}]\omega_{\varepsilon,\beta}^{-1} \\
&= [c_{\varepsilon,\alpha,\beta}(p^\alpha_{\lambda i}\omega_{\alpha,\beta})c_{\varepsilon,\alpha,\beta}^{-1}]\omega_{\varepsilon,\beta}^{-1} \quad \text{by Theorem 3.6 (ii)} \\
&= d_{\varepsilon,\alpha,\beta}(p^\alpha_{\lambda i}\omega_{\alpha,\beta}\omega_{\varepsilon,\beta}^{-1})d_{\varepsilon,\alpha,\beta}^{-1} \\
&= d_{\varepsilon,\alpha,\beta}[d_{\varepsilon,\alpha,\beta}^{-1}(p^\alpha_{\lambda i}\omega_{\varepsilon,\alpha}^{-1})d_{\varepsilon,\alpha,\beta}]d_{\varepsilon,\alpha,\beta}^{-1} \quad \text{by Lemma 4.2} \\
&= p^\alpha_{\lambda i}\omega_{\varepsilon,\alpha}^{-1} = q^\alpha_{\lambda i}.
\end{aligned}$$

□

We finally come to the third component.

Lemma 4.6. *For $\alpha \geqslant \beta \geqslant \gamma$ and $\lambda \in \Lambda_\alpha$ we have*

$$t_\lambda^{\alpha,\beta} t^{\beta,\gamma}_{\lambda\psi_{\alpha,\beta}} = d_{\alpha,\beta,\gamma}^{-1} t_\lambda^{\alpha,\gamma}.$$

Proof. From Notation 4.1 we get $v_\lambda^{\alpha,\beta} = c_{\varepsilon,\alpha,\beta}^{-1}(t_\lambda^{\alpha,\beta}\omega_{\varepsilon,\beta})$. We substitute this into (4) getting successively

$$c_{\alpha,\beta,\gamma}\{[c_{\varepsilon,\alpha,\beta}^{-1}(t_\lambda^{\alpha,\beta}\omega_{\varepsilon,\beta})]\omega_{\beta,\gamma}\}c_{\varepsilon,\beta,\gamma}^{-1}(t^{\beta,\gamma}_{\lambda\psi_{\alpha,\beta}}\omega_{\varepsilon,\gamma}) = c_{\varepsilon,\alpha,\gamma}^{-1}(t_\lambda^{\alpha,\gamma}\omega_{\varepsilon,\gamma}),$$

$$c_{\alpha,\beta,\gamma}(c_{\varepsilon,\alpha,\beta}^{-1}\omega_{\beta,\gamma})(t_\lambda^{\alpha,\beta}\omega_{\varepsilon,\beta}\omega_{\beta,\gamma})c_{\varepsilon,\beta,\gamma}^{-1}(t^{\beta,\gamma}_{\lambda\psi_{\alpha,\beta}}\omega_{\varepsilon,\gamma}) = c_{\varepsilon,\alpha,\gamma}^{-1}(t_\lambda^{\alpha,\gamma}\omega_{\varepsilon,\gamma}),$$

$$c_{\alpha,\beta,\gamma}(c_{\varepsilon,\alpha,\beta}^{-1}\omega_{\beta,\gamma})c_{\varepsilon,\beta,\gamma}^{-1}(t_\lambda^{\alpha,\beta}\omega_{\varepsilon,\gamma})c_{\varepsilon,\beta,\gamma}c_{\varepsilon,\beta,\gamma}^{-1}(t^{\beta,\gamma}_{\lambda\psi_{\alpha,\beta}}\omega_{\varepsilon,\gamma}) = c_{\varepsilon,\alpha,\gamma}^{-1}(t_\lambda^{\alpha,\gamma}\omega_{\varepsilon,\gamma}),$$

and applying $\omega^{-1}_{\varepsilon,\gamma}$, we obtain

$$d_{\alpha,\beta,\gamma}(c^{-1}_{\varepsilon,\alpha,\beta}\omega_{\beta,\gamma}\omega^{-1}_{\varepsilon,\gamma})d^{-1}_{\varepsilon,\beta,\gamma}t^{\alpha,\beta}_{\lambda}t^{\beta,\gamma}_{\lambda\psi_{\alpha,\beta}} = d^{-1}_{\varepsilon,\alpha,\gamma}t^{\alpha,\gamma}_{\lambda}$$

and using Lemma 4.2, this becomes

$$d_{\alpha,\beta,\gamma}d^{-1}_{\varepsilon,\beta,\gamma}d^{-1}_{\varepsilon,\alpha,\beta}d_{\varepsilon,\beta,\gamma}t^{\alpha,\beta}_{\lambda}t^{\beta,\gamma}_{\lambda\psi_{\alpha,\beta}} = d^{-1}_{\varepsilon,\alpha,\gamma}t^{\alpha,\gamma}_{\lambda}$$

whence the assertion. □

By virtue of Theorem 3.6 and Lemmas 4.3, 4.5 and 4.6 we are now able to introduce

Notation 4.7. For every $\alpha \in Y$, let $Q^{\alpha} = (q^{\alpha}_{\lambda i})$ and

$$T_{\alpha} = \mathscr{M}(I_{\alpha}, G_{\varepsilon}, \Lambda_{\alpha}; Q^{\alpha}).$$

For any $\alpha, \beta \in Y$ such that $\alpha \geqslant \beta$, let

$$\tau_{\alpha,\beta} = \chi(\varphi_{\alpha,\beta}, s^{\alpha,\beta}, \iota_{G_{\varepsilon}}, t^{\alpha,\beta}, \psi_{\alpha,\beta}).$$

Finally, let

$$T = [Y; T_{\alpha}, \tau_{\alpha,\beta}],$$

with $z = (k, p^{-1}_{\nu\kappa}, \nu)$ as the zenith of its unary operation.

Lemma 4.8. *For every $\alpha \in Y$, let ξ_{α} be an isomorphism of S_{α} onto T_{α}, and assume that for any $\alpha \geqslant \beta$, the diagram*

$$\begin{array}{ccc} S_{\alpha} & \xrightarrow{\ \xi_{\alpha}\ } & T_{\alpha} \\ {\scriptstyle \sigma_{\alpha,\beta}}\downarrow & & \downarrow{\scriptstyle \tau_{\alpha,\beta}} \\ S_{\beta} & \xrightarrow{\ \xi_{\beta}\ } & T_{\beta} \end{array}$$

commutes. Then $\xi = \bigcup_{\alpha \in Y} \xi_{\alpha}$ is an isomorphism of S onto T.

P r o o f. Straightforward verification. □

We are finally ready for the principal result of this section.

Theorem 4.9. *The mapping*

$$\xi\colon\ (i,g,\lambda)\mapsto(i,g\omega_{\varepsilon,\alpha}^{-1},\lambda)\quad((i,g,\lambda)\in S_\alpha,\alpha\in Y)$$

is an isomorphism of S onto T.

P r o o f. For every $\alpha\in Y$, let $\xi_\alpha=\xi|_{S_\alpha}$ so that ξ_α is a bijection of S_α onto T_α. For $a=(i,g,\lambda)$, $b=(j,h,\mu)\in S_\alpha$ we have

$$\begin{aligned}(a\xi_\alpha)(b\xi_\alpha)&=(i,g\omega_{\varepsilon,\alpha}^{-1},\lambda)(j,h\omega_{\varepsilon,\alpha}^{-1},\mu)=(i,(g\omega_{\varepsilon,\alpha}^{-1})q_{\lambda j}(h\omega_{\varepsilon,\alpha}^{-1}),\mu)\\&=(i,(gp_{\lambda j}h)\omega_{\varepsilon,\alpha}^{-1},\mu)=(ab)\xi_\alpha\end{aligned}$$

and ξ_α is an isomorphism of S_α onto T_α.

In view of Lemma 4.8 it remains to show that for $\alpha\geqslant\beta$ its diagram commutes. Indeed, for $a=(i,g,\lambda)\in S_\alpha$ we have

$$a\xi_\alpha\tau_{\alpha,\beta}=(i\varphi_{\alpha,\beta},s_i^{\alpha,\beta}(g\omega_{\varepsilon,\alpha}^{-1})t_\lambda^{\alpha,\beta},\lambda\psi_{\alpha,\beta}),\tag{5}$$

$$a\sigma_{\alpha,\beta}\xi_\beta=(i\varphi_{\alpha,\beta},[u_i^{\alpha,\beta}(g\omega_{\alpha,\beta})v_\lambda^{\alpha,\beta}]\omega_{\varepsilon,\beta}^{-1},\lambda\psi_{\alpha,\beta})\tag{6}$$

and using Lemma 4.2 we get

$$\begin{aligned}s_i^{\alpha,\beta}(g\omega_{\varepsilon,\alpha}^{-1})t_\lambda^{\alpha,\beta}&=(u_i^{\alpha,\beta}c_{\varepsilon,\alpha,\beta}^{-1})\omega_{\varepsilon,\beta}^{-1}(g\omega_{\varepsilon,\alpha}^{-1})(c_{\varepsilon,\alpha,\beta}v_\lambda^{\alpha,\beta})\omega_{\varepsilon,\beta}^{-1}\\&=(u_i^{\alpha,\beta}\omega_{\varepsilon,\beta})d_{\varepsilon,\alpha,\beta}^{-1}(g\omega_{\varepsilon,\alpha}^{-1})d_{\varepsilon,\alpha,\beta}(v_\lambda^{\alpha,\beta}\omega_{\varepsilon,\beta}^{-1})\\&=(u_i^{\alpha,\beta}\omega_{\varepsilon,\beta}^{-1})(g\omega_{\alpha,\beta}\omega_{\varepsilon,\beta}^{-1})(v_\lambda^{\alpha,\beta}\omega_{\varepsilon,\beta}^{-1})=[u_i^{\alpha,\beta}(g\omega_{\alpha,\beta})v_\lambda^{\alpha,\beta}]\omega_{\varepsilon,\beta}^{-1}\end{aligned}$$

which implies the equality of (5) and (6).

Obviously ξ fixes z, which in view of Corollary 3.4 implies that ξ is a homomorphism, and is thus an isomorphism. □

The semigroup $[Y;T_\alpha,\tau_{\alpha,\beta}]$ is our *second representation.*

5. Third representation

In the third representation we abandon the homomorphisms $\tau_{\alpha,\beta}$ in the second and write the product directly from the triples. In this procedure, we extend the sandwich matrices of individual completely simple components to all the relevant indices. We thus arrive at four functions from different domains into a group. This prepares the ground for a new construction, the third representation.

Notation 5.1. Let T be the second representation with all accompanying notation. For each $\alpha \in Y$, provide I_α and Λ_α with the structure of a left and a right zero semigroup, respectively. Set

$$I = [Y; I_\alpha, \varphi_{\alpha,\beta}], \quad \Lambda = [Y; \Lambda_\alpha, \psi_{\alpha,\beta}]$$

so that I is a left and Λ is a right normal band, respectively.

Define a function $q\colon\ (\lambda, i) \mapsto q_{\lambda i}$ by

$$q_{\lambda i} = t_\lambda^{\alpha,\alpha\beta} q^{\alpha\beta}_{\lambda\psi_{\alpha,\alpha\beta}, i\varphi_{\beta,\alpha\beta}} s_i^{\beta,\alpha\beta} \quad (\lambda \in \Lambda_\alpha,\ i \in I_\beta).$$

We also write

$$s_{i,\beta} = s_i^{\alpha,\alpha\beta}, \quad t_{\alpha,\lambda} = t_\lambda^{\beta,\alpha\beta} \quad (i \in I_\beta,\ \lambda \in \Lambda_\alpha).$$

For $\alpha = \beta$, the new $q_{\lambda i}$ coincides with q^α_λ so that the above represents an extension of $q^\alpha_{\lambda i}$ for $\lambda \in \Lambda_\alpha$ and $i \in I_\alpha$ to all $\alpha \in Y$. The new notation $s_{i,\beta}$ and $t_{\alpha,\lambda}$ is less precise but it will prove adequate.

The multiplication in T becomes: for $a = (i, g, \lambda) \in T_\alpha$ and $b = (j, h, \mu) \in T_\beta$,

$$\begin{aligned}
ab &= (a\tau_{\alpha,\alpha\beta})(b\tau_{\beta,\alpha\beta}) \\
&= (i\varphi_{\alpha,\alpha\beta}, s_i^{\alpha,\alpha\beta} g t_\lambda^{\alpha,\alpha\beta}, \lambda\psi_{\alpha,\alpha\beta})(j\varphi_{\beta,\alpha\beta}, s_j^{\beta,\alpha\beta} h t_\mu^{\beta,\alpha\beta}, \mu\psi_{\beta,\alpha\beta}) \\
&= (ij, s_i^{\alpha,\alpha\beta} g [t_\lambda^{\alpha,\alpha\beta} q^{\alpha\beta}_{\lambda\psi_{\alpha,\alpha\beta}, j\varphi_{\beta,\alpha\beta}} s_j^{\beta,\alpha\beta}] h t_\mu^{\beta,\alpha\beta}, \lambda\mu) \\
&= (ij, s_i^{\alpha,\alpha\beta} g q_{\lambda j} h t_\mu^{\beta,\alpha\beta}, \lambda\mu) = (ij, s_{i,\beta} g q_{\lambda j} h t_{\alpha,\mu}, \lambda\mu),
\end{aligned}$$

and the unary operation is

$$\begin{aligned}
a^* &= (k, [(t_\nu^{\varepsilon,\alpha} q^\alpha_{\nu\psi_{\varepsilon,\alpha}, i} s_i^{\alpha,\alpha}) g (t_\lambda^{\alpha,\alpha} q^\alpha_{\lambda, k\varphi_{\varepsilon,\alpha}} s_k^{\varepsilon,\alpha})]^{-1}, \nu) \\
&= (k, (q_{\nu i} g q_{\lambda k})^{-1}, \nu) = (k, q_{\lambda k}^{-1} g^{-1} q_{\nu i}^{-1}, \nu).
\end{aligned}$$

Observe that these formulas are generalizations of those for a Rees matrix semigroup. Abstractly we proceed as follows.

Construction 5.2. *Let*

$$I = [Y; I_\alpha, \varphi_{\alpha,\beta}], \quad \Lambda = [Y; \Lambda_\alpha, \psi_{\alpha,\beta}]$$

be a left and a right normal band, respectively, let G be a group with identity element 1, and the functions

$$
\begin{array}{ccccc}
 & & \{(\alpha,\beta,\gamma) \in Y^3;\ \alpha \geqslant \beta \geqslant \gamma\} & & \\
 & & \downarrow{\scriptstyle e} & & \\
I \times Y & \xrightarrow{\ s\ } & G & \xleftarrow{\ t\ } & Y \times \Lambda \\
 & & \uparrow{\scriptstyle q} & & \\
 & & \Lambda \times I & &
\end{array}
$$

in notation

$$(i,\alpha) \mapsto s_{i,\alpha}, \quad (\alpha,\beta,\gamma) \mapsto d_{\alpha,\beta,\gamma}, \quad (\alpha,\lambda) \mapsto t_{\alpha,\lambda}, \quad (\lambda,\iota) \mapsto q_{\lambda i}.$$

Assume that the following conditions are satisfied:

$$
\begin{aligned}
s_{i,\alpha} = t_{\alpha,\lambda} &= 1 \quad \text{if } i \in I_\alpha,\ \lambda \in \Lambda_\alpha, \\
s_{i\varphi_{\alpha,\beta},\gamma} s_{i,\beta} &= s_{i,\gamma} d_{\alpha,\beta,\gamma} \quad \text{if } i \in I_\alpha,\ \alpha \geqslant \beta \geqslant \gamma, \\
q_{\lambda i} &= t_{\beta,\lambda} q_{\lambda\psi_{\alpha,\alpha\beta}, i\varphi_{\beta,\alpha\beta}} s_{i,\alpha} \quad \text{if } \lambda \in \Lambda_\alpha,\ i \in I_\beta, \\
t_{\beta,\lambda} t_{\gamma,\lambda\psi_{\alpha,\beta}} &= d^{-1}_{\alpha,\beta,\gamma} t_{\gamma,\lambda} \quad \text{if } \lambda \in \Lambda_\alpha,\ \alpha \geqslant \beta \geqslant \gamma.
\end{aligned}
$$

Let ε be the identity element of Y and fix $k \in I_\varepsilon$, $\nu \in \Lambda_\varepsilon$. On the set

$$U = \{(i,g,\lambda) \in I_\alpha \times G \times \Lambda_\alpha;\ \alpha \in Y\}$$

define a multiplication by: for $i \in I_\alpha$, $j \in I_\beta$,

$$(i,g,\lambda)(j,h,\mu) = (ij, s_{i,\beta} g q_{\lambda j} h t_{\alpha,\mu}, \lambda\mu)$$

and a unary operation by

$$(i,g,\lambda)^* = (k, q^{-1}_{\lambda k} g^{-1} q^{-1}_{\nu i}, \nu).$$

Denote the resulting algebra by $[I,\Lambda,G;s,d,t,q]$.

With the notation established and by virtue of the above discussion, we conclude

Lemma 5.3. *The identity mapping is an isomorphism of* $T = [Y; T_\alpha, \tau_{\alpha,\beta}]$ *onto* $U = [I, \Lambda, G; s, d, t, q]$.

As a consequence, we have that U is a normal cryptogroup with an associate subgroup. We could verify this directly from the conditions imposed upon the parameters.

The relationship of $[Y; T_\alpha, \tau_{\alpha,\beta}]$ and $[I, \Lambda, G; s, d, t, q]$ discussed above is sufficiently transparent so that, by starting with the latter, by essentially reversing the steps we can easily construct T_α's and $\tau_{\alpha,\beta}$'s to obtain the relationship in the above lemma. From Fact 3.2, Theorem 4.9 and Lemma 5.3, we derive

Theorem 5.4. *The algebra* $[I, \Lambda, G; s, d, t, q]$ *is a normal cryptogroup with an associate subgroup. Conversely, every such semigroup is isomorphic to some* $[I, \Lambda, G; s, d, t, q]$.

The semigroup $[I, \Lambda, G; s, d, t, q]$ is our *third representation.*

We now discuss briefly the possibility of normalization in this representation.

Lemma 5.5. *Let* $S = [Y; S_\alpha, \sigma_{\alpha,\beta}]$ *where* $S_\alpha = \mathscr{M}(I_\alpha, G_\alpha, \Lambda_\alpha; P^\alpha)$ *for all* $\alpha \in Y$. *There exists a normal cryptogroup* $Z = [Y; Z_\alpha, \zeta_{\alpha,\beta}]$ *where* $Z_\alpha = \mathscr{M}(I_\alpha, G_\alpha, \Lambda_\alpha; R^\alpha)$ *for every* $\alpha \in Y$ *is such that* $S \cong Z$ *and each* R^α *is normalized.*

Proof. For each $\alpha \in Y$, the semigroup Z_α and the isomorphism $\xi_\alpha \colon S_\alpha \to Z_\alpha$ needed in Lemma 4.8 are provided by [4, Lemma III.3.6]. For any $\alpha \geqslant \beta$, we define $\zeta_{\alpha,\beta} = \xi_\alpha^{-1}\sigma_{\alpha,\beta}\xi_\beta$. Simple chasing of diagrams completes the proof. □

In view of this lemma, we may start in Section 3 with sandwich matrices P^α normalized at 1_α, say. The transition in Section 4 to Q^α, see Notation 4.4, retains the normalization at 1_α. However, in the extension of $q_{\lambda i}$ to all $\lambda \in \Lambda$ and $i \in I$, see Notation 5.1, we do not have normalization.

It follows that in the completely simple case we do have possibility of normalization. This is what happens in Theorem 2.3, for there P is normalized at 1 and $z = (1, e, 1)$ is used as the zenith which made it possible to have a simple expression for a^*. But for the general normal cryptogroup that may not be possible.

Theorem 2.3 may now be interpreted as a special case of Theorem 5.4.

6. Fourth representation

From the left and the right normal bands in the third representation we form their spined product and use it as an "underlying" semigroup for the fourth representation.

Notation 6.1. With the notation of the preceding section, for every $\alpha \in Y$ let $B_\alpha = I_\alpha \times \Lambda_\alpha$, the direct product of the left zero semigroup I_α and the right zero semigroup Λ_α. For any $\alpha \geqslant \beta$ define

$$\eta_{\alpha,\beta}\colon\ (i,\lambda) \mapsto (i\varphi_{\alpha,\beta}, \lambda\psi_{\alpha,\beta})$$

and set

$$B = [Y; B_\alpha, \eta_{\alpha,\beta}],$$

that is, B is a spined product of I and Λ.

We also introduce

$$\begin{aligned} w &= (k,\nu) \in B_\varepsilon \quad \text{where } \varepsilon \text{ is the identity element of } Y, \\ m_{x,\beta} &= s_{i,\beta} \quad \text{if } i \in I_\alpha,\ \alpha \geqslant \beta,\ x = (i, \nu\psi_{\varepsilon,\alpha}), \\ r_{a,x} &= q_{\lambda i} \quad \text{if } a = (k\varphi_{\varepsilon,\alpha}, \lambda),\ x = (i, \nu\psi_{\varepsilon,\beta}), \\ n_{\beta,a} &= t_{\beta,\lambda} \quad \text{if } \lambda \in \Lambda_\alpha,\ \alpha \geqslant \beta,\ a = (k\varphi_{\varepsilon,\alpha}, \lambda). \end{aligned}$$

Abstractly we proceed as follows.

Construction 6.2. *Let $B = [Y; B_\alpha, \eta_{\alpha,\beta}]$ be a normal band, let ε be the identity element of Y, $w = (k,\nu) \in B_\varepsilon$, let G be a group with identity element 1 and the functions*

$$\begin{array}{ccccc} & & \{(\alpha,\beta,\gamma) \in Y^3;\ \alpha \geqslant \beta \geqslant \gamma\} & & \\ & & \downarrow d & & \\ Bw \times Y & \xrightarrow{\ m\ } & G & \xleftarrow{\ n\ } & Y \times wB \\ & & \uparrow r & & \\ & & wB \times Bw & & \end{array}$$

in notation

$$(x,\alpha) \mapsto m_{x,\alpha}, \quad (\alpha,\beta,\gamma) \mapsto d_{\alpha,\beta,\gamma}, \quad (\alpha,a) \mapsto n_{\alpha,a}, \quad (a,x) \mapsto r_{a,x}.$$

Assume that the following conditions are satisfied:

$$\begin{aligned} m_{x,\alpha} = n_{\alpha,a} &= 1 \quad \text{if } x \in B_\alpha w,\ a \in wB_\alpha, \\ m_{x\eta_{\alpha,\beta},\gamma} m_{x,\beta} &= m_{x,\gamma} d_{\alpha,\beta,\gamma} \quad \text{if } x \in B_\alpha w,\ \alpha \geqslant \beta \geqslant \gamma, \\ r_{a,x} &= n_{\beta,a} r_{a\eta_{\alpha,\alpha\beta}, x\eta_{\beta,\alpha\beta}} m_{x,\alpha} \quad \text{if } a \in wB_\alpha,\ x \in B_\alpha w, \\ n_{\beta,a} n_{\gamma, a\eta_{\alpha,\beta}} &= d^{-1}_{\alpha,\beta,\gamma} n_{\gamma,a} \quad \text{if } a \in wB_\alpha,\ \alpha \geqslant \beta \geqslant \gamma. \end{aligned}$$

On the set

$$V = \{(x, g, a) \in Bw \times G \times wB;\ aw = wx\}$$

define a multiplication by: for $x \in B_\alpha$, $y \in B_\beta$,

$$(x, g, a)(y, h, b) = (xy, m_{x,\beta} g r_{a,y} h n_{\alpha,\beta}, ab)$$

and a unary operation by

$$(x, g, a)^* = (w, r_{a,w}^{-1} g^{-1} r_{w,x}^{-1}, w).$$

Denote the resulting algebra by $[B, G; m, d, n, r]$.

With the above notation and in view of the relevant discussion, we derive

Lemma 6.3. *The identity mapping is an isomorphism of* $U = [I, \Lambda, G; s, d, t, q]$ *onto* $V = [B, G; m, d, n, r]$.

It then follows that V is a normal cryptogroup with an associate subgroup. This of course can be verified directly from the conditions imposed upon the parameters.

Given $[B, G; m, d, n, r]$ with the accompanying conditions, we can easily reverse the above discussion and construct the corresponding $[I, \Lambda, G; s, d, t, q]$. Now Theorem 5.4 and Lemma 6.3 imply

Theorem 6.4. *The algebra* $[B, G; m, d, n, r]$ *is a normal cryptogroup with an associate subgroup. Conversely, every such semigroup is isomorphic to some* $[B, G; m, d, n, r]$.

Note that for $V = [B, G; m, d, n, r]$ we have $V/\mathscr{H} \cong B$ and any maximal subgroup of V is isomorphic to G.

The semigroup $[B, G; m, d, n, r]$ is our *fourth representation*. It is interesting to compare this representation with Fact 2.2 for semigroups with a medial zenith.

References

[1] *T. S. Blyth, P. M. Martins*: On associate subgroups of regular semigroups. Commun. Algebra *25* (1997), 2147–2156.

[2] *P. M. Martins, M. Petrich*: Unary semigroups with an associate subgroup. Commun. Algebra *36* (2008), 1999–2013.

[3] *M. Petrich*: The existence of an associate subgroup in normal cryptogroups. Publ. Math. Debrecen *73* (2008), 281–298.

[4] *M. Petrich, N. R. Reilly*: Completely Regular Semigroups. Canadian Mathematical Society Series of Monographs and Advanced Texts 23, John Wiley & Sons, New York, 1999.

Author's address: Mario Petrich, 21420 Bol, Brač, Croatia.