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Mathematica Bohemica, Vol. 138 (2013), No. 1, 87–104

OSCILLATION OF THIRD-ORDER HALF-LINEAR NEUTRAL DIFFERENCE EQUATIONS

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(Received September 21, 2011)

Abstract. Some new criteria for the oscillation of third order nonlinear neutral difference equations of the form

$$\Delta(a_n(\Delta^2(x_n + b_n x_{n-\delta}))^\alpha) + q_n x^\alpha_{n+1-\tau} = 0$$

and

$$\Delta(a_n(\Delta^2(x_n - b_n x_{n-\delta}))^\alpha) + q_n x^\alpha_{n+1-\tau} = 0$$

are established. Some examples are presented to illustrate the main results.

Keywords: third order neutral difference equation, oscillation, nonoscillation

MSC 2010: 39A10

1. Introduction

Consider the third order neutral difference equations

$$\Delta(a_n(\Delta^2(x_n + b_n x_{n-\delta}))^\alpha) + q_n x^\alpha_{n+1-\tau} = 0 \tag{1.1}$$

and

$$\Delta(a_n(\Delta^2(x_n - b_n x_{n-\delta}))^\alpha) + q_n x^\alpha_{n+1-\tau} = 0 \tag{1.2}$$

where $n \in \mathbb{N} = \{n_0, n_0+1, n_0+2, \ldots\}$, n_0 is a nonnegative integer subject to

(i) α is a ratio of odd positive integers;

(ii) $\{a_n\}, \{b_n\}, \{q_n\}$ are positive sequences and a_n satisfies

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} = \infty; \tag{1.3}$$

(iii) $0 \leqslant b_n \leqslant b < 1$, δ and τ are positive integers.

Let $\theta = \max\{\delta, \tau\}$. By a solution of equation (1.1) ((1.2)) we mean a real sequence $\{x_n\}$ defined for all $n \geqslant n_0 - \theta$ and satisfying (1.1) ((1.2)) for $n \geqslant n_0$. A nontrivial solution $\{x_n\}$ is said to be nonoscillatory if it is either eventually positive or eventually negative, and it is oscillatory otherwise.

Recently, there has been increasing interest in studying the oscillatory and nonoscillatory behavior of solutions of neutral type difference equations, see for example [1], [2], [3] and the references cited therein. For example, the first order linear difference equations of neutral type

$$\Delta(x_n + p_n x_{n-k}) + q_n x_{n-l} = 0$$

and its special cases have been investigated in [7], [10], [11], [12] and the nonlinear case has been considered in [1], [3]. Compared to first order neutral difference equations, the study of higher order equations, especially third order neutral difference equations, has received considerably less attention, even though such equations arise in population dynamics [4], see for example [6], [8], [9], [13], [14], [15], [16], [17], [18], [19] and the references contained therein.

The purpose of this paper is to obtain some new sufficient conditions for the oscillation of all solutions of equations (1.1) and (1.2). The results obtained in this paper have been motivated by that of in [5]. In Section 2, we present sufficient conditions which ensure that all solutions of equation (1.1) are either oscillatory or converge to zero and we present similar results for (1.2) in Section 3. Examples are provided to illustrate the main results.

2. Oscillation of equation (1.1)

First we establish some new oscillatory criteria for equation (1.1). We begin with some useful lemmas, which we intend to use later. For each solution $\{x_n\}$ of equation (1.1), we define the corresponding sequence $\{z_n\}$ by

$$z_n = x_n + b_n x_{n-\delta}. \tag{2.1}$$

Lemma 2.1. *Let $\{x_n\}$ be a positive solution of equation (1.1). Then there are only the following two cases for z_n defined in (2.1):*

(i) $z_n > 0$, $\Delta z_n > 0$, $\Delta^2 z_n > 0$;
(ii) $z_n > 0$, $\Delta z_n < 0$, $\Delta^2 z_n > 0$

for $n \geqslant n_1 \in \mathbb{N}$, where n_1 is sufficiently large.

P r o o f. Assume that $\{x_n\}$ is a positive solution of equation (1.1) for all $n \geqslant n_0$. We see that $z_n > x_n > 0$, and

$$\Delta(a_n(\Delta^2 z_n)^\alpha) = -q_n x^\alpha_{n+1-\tau} < 0. \tag{2.2}$$

Thus $a_n(\Delta^2 z_n)^\alpha$ is nonincreasing and of one sign. Therefore, $\Delta^2 z_n$ is also of one sign and so we have two possibilities: $\Delta^2 z_n < 0$ or $\Delta^2 z_n > 0$ for $n \geqslant n_1$ by (2.2). If $\Delta^2 z_n < 0$, then there is a constant $M > 0$ such that $a_n(\Delta^2 z_n)^\alpha \leqslant -M < 0$. Summing from n_1 to $n-1$, we obtain $\Delta z_n \leqslant \Delta z_{n_1} - M^{1/\alpha} \sum\limits_{s=n_1}^{n-1} 1/a_s^{1/\alpha}$. Letting $n \to \infty$ and using (1.3), we see that $\Delta z_n \to -\infty$. Thus, $\Delta z_n < 0$ eventually. But $\Delta^2 z_n < 0$ and $\Delta z_n < 0$ eventually imply $z_n < 0$ for all $n \geqslant n_1$; a contradiction. This proves that $\Delta^2 z_n > 0$ and we have only two cases (i) and (ii) for z_n. The proof is now complete. □

Lemma 2.2. *Let $\{x_n\}$ be a positive solution of equation* (1.1), *and let the corresponding z_n satisfy* (ii). *If*

$$\sum_{n=n_0}^{\infty} \sum_{s=n}^{\infty} \left[\frac{1}{a_s} \sum_{t=s}^{\infty} q_t\right]^{1/\alpha} = \infty, \tag{2.3}$$

then $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = 0$.

P r o o f. Let $\{x_n\}$ be a positive solution of (1.1). Since $z_n > 0$ and $\Delta z_n < 0$, there is a finite limit, $\lim\limits_{n\to\infty} z_n = l$. We shall prove that $l = 0$. Assume that $l > 0$. Then for any $\varepsilon > 0$ we have $l + \varepsilon > z_n > l$ eventually. Choose $0 < \varepsilon < l(1-b)/b$. It is easy to verify that $x_n = z_n - b_n x_{n-\delta} > l - b z_{n-\delta} > l - b(l+\varepsilon) > k z_n$, where $k = (l - b(l+\varepsilon))/(l+\varepsilon) > 0$. From the last inequality and (2.2) we have

$$\Delta(a_n(\Delta^2 z_n)^\alpha) \leqslant -q_n k^\alpha z^\alpha_{n+1-\tau}.$$

Summing the last inquality from n to ∞, we obtain

$$a_n(\Delta^2 z_n)^\alpha \geqslant k^\alpha \sum_{s=n}^{\infty} q_s z^\alpha_{s+1-\tau}.$$

Using $z^\alpha_{n+1-\tau} \geqslant l^\alpha$, we get

$$\Delta^2 z_n \geqslant kl\left[\frac{1}{a_n}\sum_{s=n}^{\infty} q_s\right]^{1/\alpha}.$$

Summing again from n to ∞, we have

$$-\Delta z_n \geqslant kl \sum_{s=n}^{\infty} \left[\frac{1}{a_s} \sum_{t=s}^{\infty} q_t \right]^{1/\alpha}.$$

Summing the last inquality from n_1 to ∞, we obtain

$$z_{n_1} \geqslant kl \sum_{n=n_1}^{\infty} \sum_{s=n}^{\infty} \left[\frac{1}{a_s} \sum_{t=s}^{\infty} q_t \right]^{1/\alpha}.$$

This contradicts (2.3). Thus $l = 0$. Moreover, the inequality $0 < x_n \leqslant z_n$ implies that $\lim\limits_{n\to\infty} x_n = 0$. The proof is now complete. □

Lemma 2.3. *Assume that $u_n > 0$, $\Delta u_n \geqslant 0$, $\Delta^2 u_n \leqslant 0$ for all $n \geqslant n_0$. Then for each $l \in (0,1)$ there exists an integer $N \geqslant n_0$ such that $u_{n-\tau}/(n-\tau) \geqslant l u_n/n$ for $n \geqslant N$.*

P r o o f. From the monotonicity property of $\{\Delta u_n\}$, we have

$$u_n - u_{n-\tau} = \sum_{s=n-\tau}^{n-1} \Delta u_s \leqslant (\Delta u_{n-\tau})\tau$$

or

$$\frac{u_n}{u_{n-\tau}} \leqslant 1 + \tau \frac{\Delta u_{n-\tau}}{u_{n-\tau}}. \tag{2.4}$$

Also,

$$u_{n-\tau} \geqslant u_{n-\tau} - u_{n_0} \geqslant \Delta u_{n-\tau}(n-\tau-n_0).$$

So, for each $l \in (0,1)$, there is an integer $N \geqslant n_0$ such that

$$\frac{u_{n-\tau}}{\Delta u_{n-\tau}} \geqslant l(n-\tau), \ \ n \geqslant N. \tag{2.5}$$

Combining (2.4) with (2.5), we obtain

$$\frac{u_n}{u_{n-\tau}} \leqslant 1 + \frac{\tau}{l(n-\tau)} \leqslant \frac{n}{l(n-\tau)}$$

and the proof is complete. □

Lemma 2.4. *Assume that $z_n > 0$, $\Delta z_n > 0$, $\Delta^2 z_n > 0$, $\Delta^3 z_n \leqslant 0$ for all $n \geqslant N$. Then $z_{n+1}/\Delta z_n \geqslant (n-N)/2$ for $n \geqslant N$.*

P r o o f. From the monotonicity property of $\{\Delta^2 z_n\}$, we have

$$\Delta z_n = \Delta z_N + \sum_{s=N}^{n-1} \Delta^2 z_s \geqslant (n-N)\Delta^2 z_n.$$

Summing from N to $n-1$, we obtain

$$z_{n+1} \geqslant z_n \geqslant z_N + \sum_{s=N}^{n-1} (s-N)\Delta^2 z_s = z_N + (n-N)\Delta z_n - z_{n+1} + z_N.$$

Hence, $z_{n+1} \geqslant \frac{1}{2}(n-N)\Delta z_n$, $n \geqslant N$. This completes the proof. □

Lemma 2.5. *Assume that* $\Delta z_n > 0$, $\Delta^2 z_n > 0$, $\Delta^3 z_n \leqslant 0$ *for all* $n \geqslant N$. *Then* $(n-N)\Delta^2 z_n/\Delta z_n \leqslant 1$ *for* $n \geqslant N$.

P r o o f. The result follows from the inequality

$$\Delta z_n \geqslant \sum_{s=N}^{n-1} \Delta^2 z_s \geqslant (n-N)\Delta^2 z_n.$$

Now, we present the oscillation results. For simplicity we introduce the following notation:

$$P = \liminf_{n\to\infty} \frac{n^\alpha}{a_n} \sum_{s=n}^{\infty} P_l(s), Q = \limsup_{n\to\infty} \frac{1}{n} \sum_{s=n_0}^{n-1} \frac{s^{\alpha+1}}{a_s} P_l(s) \tag{2.6}$$

where $P_l(s) = l^\alpha(1-b)^\alpha q_s\big((s-\tau)/s\big)^\alpha \big((s-\tau-N)/2\big)^\alpha$ with $l \in (0,1)$ arbitrarily chosen and N large enough. Moreover, for z_n satisfying the case (i), we define

$$w_n = a_n\Big(\frac{\Delta^2 z_n}{\Delta z_n}\Big)^\alpha \tag{2.7}$$

and

$$r = \liminf_{n\to\infty} \frac{n^\alpha w_{n+1}}{a_{n+1}} \quad \text{and} \quad R = \limsup_{n\to\infty} \frac{n^\alpha w_n}{a_n}. \tag{2.8}$$

□

Lemma 2.6. *Assume that* $\{a_n\}$ *is nondecreasing. Let* $\{x_n\}$ *be a positive solution of equation* (1.1).

(I) *Let* $P < \infty$ *and* $Q < \infty$. *Suppose that the corresponding* z_n *satisfies case* (i) *of Lemma* 2.1. *Then*

$$P \leqslant r - r^{1+1/\alpha} \quad \textit{and} \quad P + Q \leqslant 1. \tag{2.9}$$

(II) *If* $P = \infty$ *or* $Q = \infty$, *then* $\{z_n\}$ *does not belong under the case* (i) *of Lemma* 2.1.

P r o o f. Part (I): Assume that $\{x_n\}$ is a positive solution of equation (1.1) and the corresponding z_n satisfies (i). First note that

$$x_n = z_n - b_n x_{n-\delta} > z_n - b_n z_{n-\delta} \geqslant (1-b_n) z_n \geqslant (1-b) z_n.$$

Using the last inequality in equation (1.1), we obtain

$$\Delta(a_n(\Delta^2 z_n)^\alpha) \leqslant -(1-b)^\alpha q_n z_{n+1-\tau}^\alpha \leqslant 0. \tag{2.10}$$

The last inequality together with $\Delta a_n \geqslant 0$ gives $\Delta^3 z_n \leqslant 0$. So, there exists an integer $N \geqslant n_0$ such that z_n satisfies $z_{n-\tau} > 0$, $\Delta z_n > 0$, $\Delta^2 z_n > 0$, $\Delta^3 z_n \leqslant 0$ for $n \geqslant N$.

From the definition of w_n and (2.10), we see that $w_n > 0$ and satisfies

$$\begin{aligned} \Delta w_n &= \frac{\Delta(a_n(\Delta^2 z_n)^\alpha)}{(\Delta z_n)^\alpha} - \frac{a_{n+1}(\Delta^2 z_{n+1})^\alpha \Delta((\Delta z_n)^\alpha)}{(\Delta z_n)^\alpha (\Delta z_{n+1})^\alpha} \\ &\leqslant -q_n(1-b)^\alpha \Big(\frac{z_{n+1-\tau}}{\Delta z_n}\Big)^\alpha - \frac{\alpha}{a_{n+1}^{1/\alpha}} w_{n+1}^{1+1/\alpha}. \end{aligned} \tag{2.11}$$

From Lemma 2.3 with $u_n = \Delta z_n$, we have for l the same as in $P_l(s)$

$$\frac{1}{\Delta z_n} \geqslant \frac{l(n-\tau)}{n} \frac{1}{\Delta z_{n-\tau}}, \ n \geqslant N,$$

which with (2.11) gives

$$\Delta w_n \leqslant -l^\alpha q_n \Big(\frac{n-\tau}{n}\Big)^\alpha \Big(\frac{z_{n+1-\tau}}{\Delta z_{n-\tau}}\Big)^\alpha (1-b)^\alpha - \frac{\alpha}{a_{n+1}^{1/\alpha}} w_{n+1}^{1+1/\alpha}.$$

Using the fact from Lemma 2.4 that $z_{n+1} \geqslant \frac{1}{2}(n-N)\Delta z_n$, we have

$$\Delta w_n + P_l(n) + \frac{\alpha}{a_{n+1}^{1/\alpha}} w_{n+1}^{(\alpha+1)/\alpha} \leqslant 0. \tag{2.12}$$

Since $P_l(n) > 0$ and $w_n > 0$ for $n \geqslant N$, we have from (2.12) that $\Delta w_n \leqslant 0$ and

$$-\frac{\Delta w_n}{\alpha w_{n+1}^{(\alpha+1)/\alpha}} \geqslant \frac{1}{a_{n+1}^{1/\alpha}} \text{for } n \geqslant N.$$

Summing the last inequality from N to $n-1$ and using the fact that w_n is decreasing we obtain

$$\frac{-w_n + w_N}{\alpha w_n^{(\alpha+1)/\alpha}} \geqslant \sum_{s=N}^{n-1} \frac{1}{a_{s+1}^{1/\alpha}}$$

or

$$w_n \leqslant w_N \left(\alpha \sum_{s=N}^{n-1} a_{s+1}^{-1/\alpha} \right)^{-\alpha/(\alpha+1)}, \tag{2.13}$$

which in view of (1.3) implies that $\lim_{n\to\infty} w_n = 0$. On the other hand, from the definition of w_n, and Lemma 2.5, we see that

$$0 \leqslant r \leqslant R \leqslant 1. \tag{2.14}$$

Now, we prove that the first inequality in (2.9) holds. Let $\varepsilon > 0$. Then due to the definition of P and r, we can choose an integer $n_2 \geqslant N$ sufficiently large that $n^\alpha a_n^{-1} \sum_{s=n}^{\infty} P_l(s) \geqslant P - \varepsilon$ and $n^\alpha w_{n+1}/a_{n+1} \geqslant r - \varepsilon$ for all $n \geqslant n_2$.

Summing (2.12) from n to ∞ and using $\lim_{n\to\infty} w_n = 0$, we have

$$w_n \geqslant \sum_{s=n}^{\infty} P_l(s) + \alpha \sum_{s=n}^{\infty} \frac{w_{s+1}^{1+1/\alpha}}{a_{s+1}^{1/\alpha}},\ n \geqslant n_2. \tag{2.15}$$

Using the fact that $\Delta a_n \geqslant 0$, it follows from (2.15) that

$$\begin{aligned} \frac{n^\alpha w_n}{a_n} &\geqslant \frac{n^\alpha}{a_n} \sum_{s=n}^{\infty} P_l(s) + \frac{\alpha n^\alpha}{a_n} \sum_{s=n}^{\infty} \frac{s^{\alpha+1} a_{s+1} w_{s+1}^{1+1/\alpha}}{s^{\alpha+1} a_{s+1}^{1+1/\alpha}} \\ &\geqslant (P-\varepsilon) + n^\alpha \frac{(r-\varepsilon)^{1+1/\alpha}}{a_n} \sum_{s=n}^{\infty} \frac{\alpha a_{s+1}}{s^{\alpha+1}} \\ &\geqslant (P-\varepsilon) + n^\alpha \frac{(r-\varepsilon)^{1+1/\alpha}}{a_n} \sum_{s=n}^{\infty} \frac{\alpha a_{s+1}}{s^{\alpha+1}} \\ &\geqslant (P-\varepsilon) + n^\alpha (r-\varepsilon)^{1+1/\alpha} \sum_{s=n}^{\infty} \frac{\alpha}{s^{\alpha+1}} \end{aligned}$$

and so

$$\frac{n^\alpha w_n}{a_n} \geqslant (P-\varepsilon) + (r-\varepsilon)^{1+1/\alpha} n^\alpha \sum_{s=n}^{\infty} \frac{\alpha}{s^{\alpha+1}}. \tag{2.16}$$

From (2.16) and $\sum_{s=n}^{\infty} \alpha/s^{\alpha+1} \geqslant \alpha \int_n^\infty \mathrm{d}s/s^{\alpha+1}$, we have

$$\frac{n^\alpha w_n}{a_n} \geqslant (P-\varepsilon) + (r-\varepsilon)^{1+1/\alpha}.$$

Taking lim inf on both sides as $n \to \infty$, we obtain that

$$r \geqslant (P-\varepsilon) + (r-\varepsilon)^{1+1/\alpha}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the desired result:

$$P \leqslant r - r^{1+1/\alpha}. \tag{2.17}$$

To complete the proof of part (I) it remains to prove the second inequality in (2.9). Multiplying the inequality (2.12) by $n^{\alpha+1}/a_n$ and summing from n_2 to $n-1$, we obtain

$$\sum_{s=n_2}^{n-1} \frac{s^{\alpha+1}\Delta w_s}{a_s} \leqslant -\sum_{s=n_2}^{n-1} \frac{s^{\alpha+1}}{a_s} P_l(s) - \alpha \sum_{s=n_2}^{n-1} \Big(\frac{s^\alpha w_{s+1}}{a_{s+1}}\Big)^{(\alpha+1)/\alpha}. \tag{2.18}$$

By summation by parts, we obtain

$$\begin{aligned}\frac{n^{\alpha+1} w_n}{a_n} \leqslant \frac{n_2^{\alpha+1} w_{n_2}}{a_{n_2}} &- \sum_{s=n_2}^{n-1} \frac{s^{\alpha+1} P_l(s)}{a_s} \\ &- \alpha \sum_{s=n_2}^{n-1} \Big(\frac{s^\alpha w_{s+1}}{a_{s+1}}\Big)^{(\alpha+1)/\alpha} + \sum_{s=n_2}^{n-1} w_{s+1} \Delta\Big(\frac{s^{\alpha+1}}{a_s}\Big).\end{aligned}$$

Since $\Delta a_n \geqslant 0$, we have

$$\Delta\Big(\frac{s^{\alpha+1}}{a_s}\Big) = \frac{\Delta(s^{\alpha+1})}{a_{s+1}} - \frac{s^{\alpha+1}\Delta a_s}{a_s a_{s+1}} \leqslant \frac{(\alpha+1)(s+1)^\alpha}{a_{s+1}}.$$

Hence,

$$\begin{aligned}\frac{n^{\alpha+1} w_n}{a_n} \leqslant \frac{n_2^{\alpha+1} w_{n_2}}{a_{n_2}} &- \sum_{s=n_2}^{n-1} \frac{s^{\alpha+1} P_l(s)}{a_s} \\ &+ \sum_{s=n_2}^{n-1} \Big[(\alpha+1)(s+1)^\alpha \frac{w_{s+1}}{a_{s+1}} - \alpha\Big(\frac{s^\alpha w_{s+1}}{a_{s+1}}\Big)^{(\alpha+1)/\alpha}\Big].\end{aligned}$$

Using the inequality

$$Bu - Au^{1+1/\alpha} \leqslant \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}$$

with $u = s^\alpha w_{s+1}/a_{s+1} > 0$, $A = \alpha$ and $B = (\alpha+1)((s+1)/s)^\alpha$, we obtain

$$n^{\alpha+1} \frac{w_n}{a_n} \leqslant \frac{n_2^{\alpha+1} w_{n_2}}{a_{n_2}} - \sum_{s=n_2}^{n-1} \frac{s^{\alpha+1} P_l(s)}{a_s} + \sum_{s=n_2}^{n-1} \Big(\frac{s+1}{s}\Big)^{\alpha(\alpha+1)}.$$

It follows that

$$n^{\alpha}\frac{w_n}{a_n} \leqslant \frac{1}{n}\frac{n_2^{\alpha+1}w_{n_2}}{a_{n_2}} - \frac{1}{n}\sum_{s=n_2}^{n-1}\frac{s^{\alpha+1}P_l(s)}{a_s} + \frac{1}{n}\sum_{s=n_2}^{n-1}\Big(\frac{s+1}{s}\Big)^{\alpha(\alpha+1)}. \tag{2.19}$$

Taking lim sup on both sides as $n \to \infty$, we obtain

$$R \leqslant -Q + 1.$$

Combining this with the inequalities in (2.17) and (2.14) we have

$$P \leqslant r - r^{1+\frac{1}{\alpha}} \leqslant r \leqslant R \leqslant -Q + 1,$$

which gives the desired second inequality in (2.9). The proof of part (I) is complete.

Part (II): Assume that $\{x_n\}$ is a positive solution of equation (1.1). We shall show that $\{z_n\}$ does not belong to case (i) of Lemma 2.1. Assume the contrary. First we assume $P = \infty$. Then exactly as in the proof of the first part, we obtain (2.15). Then

$$\frac{n^{\alpha}w_n}{a_n} \geqslant \frac{n^{\alpha}}{a_n}\sum_{s=n}^{\infty}P_l(s).$$

Taking the lim inf on both sides as $n \to \infty$, we obtain in view of (2.14) that

$$1 \geqslant r \geqslant \infty.$$

This is a contradiction. Next, we assume that $Q = \infty$. Then taking lim inf and lim sup on the left and right hand side of (2.19), respectively, we obtain

$$0 \leqslant R \leqslant -\infty.$$

This contradiction completes the proof.

Now we are ready to present the following oscillation criterion for equation (1.1). □

Theorem 2.7. *Assume that condition* (2.3) *holds and* $\{a_n\}$ *is nondecreasing. Let* $\{x_n\}$ *be a solution of* (1.1). *If*

$$P = \lim\inf_{n\to\infty}\frac{n^{\alpha}}{a_n}\sum_{s=n}^{\infty}P_l(s) > \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}, \tag{2.20}$$

then $\{x_n\}$ *is oscillatory or* $x_n \to 0$ *as* $n \to \infty$.

P r o o f. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $\{x_n\}$ is a positive solution (since the proof for the opposite case is similar) of equation (1.1). If $P = \infty$, then by Lemma 2.6, z_n does not belong to case (i) of Lemma 2.1. That is, z_n has to satisfy (ii), and from Lemma 2.2, we see that $\lim_{n\to\infty} x_n = 0$.

Next, we assume that $P < \infty$. We shall discuss two possibilities. If for z_n case (ii) holds, then exactly as above we are led, by Lemma 2.2, to $\lim_{n\to\infty} x_n = 0$.

Now, we assume that for z_n case (i) holds. Let w_n and r be defined by (2.7) and (2.8) respectively. Then from Lemma 2.6 we see that r satisfies the inequality

$$P \leqslant r - r^{1+1/\alpha}.$$

Using the inequality $Bu - Au^{1+1/\alpha} \leqslant (\alpha^\alpha/(\alpha+1)^{\alpha+1})B^{\alpha+1}/A^\alpha$ with $A = B = 1$ and $u = r$, we obtain that

$$P \leqslant \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}},$$

which contradicts (2.20). This completes the proof. □

Corollary 2.8. *Assume that condition* (2.3) *holds and* $\{a_n\}$ *is nondecreasing. Let* $\{x_n\}$ *be a solution of equation* (1.1). *If*

$$\liminf_{n\to\infty} \frac{n^\alpha}{a_n} \sum_{s=n}^{\infty} q_s \frac{(s-\tau)^{2\alpha}}{s^\alpha} > \frac{(2\alpha)^\alpha}{(\alpha+1)^{\alpha+1}(1-b)^\alpha}, \tag{2.21}$$

then $\{x_n\}$ *is oscillatory or* $x_n \to 0$ *as* $n \to \infty$.

P r o o f. We shall show that condition (2.21) implies condition (2.20). First note that for any $l \in (0,1)$ there exists an integer n_1 such that $n - \tau - N \geqslant l(n-\tau)$, $n \geqslant n_1$. Therefore,

$$P_l(n) \geqslant \frac{l^{2\alpha}(1-b)^\alpha}{2^\alpha} \frac{(n-\tau)^{2\alpha}}{n^\alpha} q_n, \ n \geqslant n_1. \tag{2.22}$$

On the other hand, (2.21) implies that for some $l \in (0,1)$

$$\liminf_{n\to\infty} \frac{n^\alpha}{a_n} \sum_{s=n}^{\infty} q_s \frac{(s-\tau)^{2\alpha}}{s^\alpha} > \frac{1}{l^{2\alpha}} \frac{(2\alpha)^\alpha}{(\alpha+1)^{\alpha+1}(1-b)^\alpha}. \tag{2.23}$$

Combining (2.22) with (2.23), we obtain (2.20). □

Theorem 2.9. *Assume that condition* (2.3) *holds and* $\{a_n\}$ *is non decreasing. Let* $\{x_n\}$ *be a solution of equation* (1.1). *If*

$$P + Q > 1, \tag{2.24}$$

then $\{x_n\}$ *is oscillatory or satisfies* $\lim_{n\to\infty} x_n = 0$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $\{x_n\}$ is a positive solution of equation (1.1). If $P = \infty$ or $Q = \infty$, then by Lemma 2.6, z_n does not belong to case (i) of Lemma 2.1. That is, z_n has to satisfy case (ii). From Lemma 2.2, we see that $\lim_{n\to\infty} x_n = 0$.

Next, we assume that $P < \infty$ and $Q < \infty$. We shall discuss two possibilities. If for z_n case (ii) holds, then exactly as above we are led, by Lemma 2.2, to $\lim_{n\to\infty} x_n = 0$. Now we assume that for z_n case (i) holds. Let w_n and r be defined by (2.7) and (2.8), respectively. Then from Lemma 2.6 we see that P and Q satisfy the inequality $P + Q \leqslant 1$, which contradicts (2.24). This completes the proof. □

As a consequence of Theorem 2.9, we have the following results.

Corollary 2.10. *Assume that condition* (2.3) *holds and* $\{a_n\}$ *is nondecreasing. Let* $\{x_n\}$ *be a solution of equation* (1.1). *If*

$$Q = \lim_{n\to\infty} \sup \frac{1}{n} \sum_{s=n_0}^{n-1} \frac{s^{\alpha+1}}{a_s} P_l(s) > 1, \tag{2.25}$$

then $\{x_n\}$ *is oscillatory or satisfies* $\lim_{n\to\infty} x_n = 0$.

Corollary 2.11. *Assume that condition* (2.3) *holds and* $\{a_n\}$ *is nondecreasing. Let* $\{x_n\}$ *be a solution of equation* (1.1). *If*

$$\lim_{n\to\infty} \sup \frac{1}{n} \sum_{s=n}^{\infty} \frac{s(s-\tau)^{2\alpha}}{a_s} q_s > \frac{2^\alpha}{(1-b)^\alpha},$$

then $\{x_n\}$ *is oscillatory or satisfies* $\lim_{n\to\infty} x_n = 0$.

The proof is similar to that of Corollary 2.8 and hence the details are omitted. We conclude this section with two examples.

Example 2.1. Consider the third order nonlinear difference equation

$$\Delta\Big(n\Big(\Delta^2\Big(x_n + \frac{1}{3}x_{n-1}\Big)\Big)^3\Big) + \frac{\lambda}{(n-2)^6}\, x_{n-1}^3 = 0,\ \lambda > 0. \tag{2.26}$$

It is easy to see that condition (2.3) holds. Hence, by Corollary 2.8, we see that every solution of equation (2.26) is either oscillatory or converges to zero as $n \to \infty$ provided that $\lambda > 3^6/2^7$.

E x a m p l e 2.2. Consider the third order difference equation
(2.27)

$$\Delta\Big(n\Big(\Delta^2\Big(x_n+\frac{(n-1)}{2n}x_{n-1}\Big)\Big)^3\Big)+\frac{27(8n^2+27n+27)(n-1)^3}{(n+3)^3(n+2)^3(n+1)^3n^2}\,x_{n-1}^3=0,\ n\geqslant 1.$$

By Corollary 2.8, every solution of equation (2.27) is either oscillatory or converges to zero as $n\to\infty$. In fact, $\{x_n\}=\{1/n\}$ is one such solution of equation (2.27).

3. Oscillation of equation (1.2)

In this section, we present oscillatory criteria for equation (1.2). We define

$$z_n=x_n-b_nx_{n-\delta}. \tag{3.1}$$

Lemma 3.1. *Let* $\{x_n\}$ *be a positive solution of equation* (1.2). *Then the corresponding function* z_n *defined in* (3.1) *satisfies*

(iii) $z_n>0$, $\Delta z_n>0$, $\Delta^2 z_n>0$;
(iv) $z_n>0$, $\Delta z_n<0$, $\Delta^2 z_n>0$;
(v) $z_n<0$, $\Delta z_n<0$, $\Delta^2 z_n>0$;
(vi) $z_n<0$, $\Delta z_n<0$, $\Delta^2 z_n<0$ *for* $n\geqslant n_1$, *where* n_1 *is sufficiently large.*

P r o o f. Let $\{x_n\}$ be a positive solution of equation (1.2). Then

$$\Delta(a_n(\Delta^2 z_n)^\alpha)=-q_nx^\alpha_{n+1-\tau}<0.$$

Thus $a_n(\Delta^2 z_n)^\alpha$ is decreasing and of one sign, which implies that $\Delta^2 z_n$ is of one sign. We have two possibilities for $\Delta^2 z_n$;

$$\Delta^2 z_n>0\quad\text{or}\quad\Delta^2 z_n<0\quad\text{for all } n\geqslant n_1.$$

The condition $\Delta^2 z_n<0$ implies that there exists a constant $M>0$ such that $a_n(\Delta^2 z_n)^\alpha\leqslant -M<0$ or $\Delta^2 z_n\leqslant -M^{1/\alpha}/a_n^{1/\alpha}$.

Summing the last inequality from n_1 to $n-1$, we obtain

$$\Delta z_n\leqslant\Delta z_{n_1}-M^{1/\alpha}\sum_{s=n_1}^{n-1}\frac{1}{a_s^{1/\alpha}}.$$

Letting $n\to\infty$ in the above inequality and using (1.3) we get $\Delta z_n<0$. But $\Delta z_n<0$ and $\Delta^2 z_n<0$ eventually, imply $z_n<0$ eventually. Thus for $\Delta^2 z_n<0$ case (vi) may occur.

On the other hand, if $\Delta^2 z_n>0$, then Δz_n is of one sign. If $\Delta z_n>0$ for $n\geqslant n_1$, then $z_n>0$. So for $\Delta^2 z_n>0$ only the cases (iii), (iv) and (v) may occur. □

Lemma 3.2. *Let $\{x_n\}$ be a positive solution of equation* (1.2) *and let the corresponding z_n satisfy* (iv). *If* (2.3) *holds, then $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = 0$.*

Proof. Let $\{x_n\}$ be a positive solution of equation (1.2). It is clear that there exists a finite limit

$$\lim_{n\to\infty} z_n = l.$$

We claim that $l = 0$. Assume that $l > 0$. It follows from (3.1) that $z_n < x_n$. Combining this with equation (1.2), we are led to

$$\Delta(a_n(\Delta^2 z_n)^\alpha) \leqslant -q_n z_{n+1-\tau}^\alpha \leqslant -l^\alpha q_n.$$

Summing the last inequality from n to ∞ and then from n_1 to ∞, we obtain

$$z_{n_1} \geqslant l \sum_{n=n_1}^{\infty} \sum_{s=n}^{\infty} \Big(\frac{1}{a_s} \sum_{t=s}^{\infty} q_t\Big)^{1/\alpha}.$$

This contradicts (2.3). Therefore, $l = 0$. Moreover, the boundedness of x_n yields $\limsup\limits_{n\to\infty} x_n = d$, $0 \leqslant d < \infty$. Hence there exists a sequence $\{n_j\}$ such that $\lim\limits_{j\to\infty} n_j = \infty$, $\lim\limits_{j\to\infty} x_{n_j} = d$.

If $d > 0$, choosing $\varepsilon = \frac{1}{2}d(1-b)/b$ we see that $x_{n_j-\delta} < d + \varepsilon$ eventually.

Further,

$$0 = \lim_{j\to\infty} z_{n_j} \geqslant \lim_{j\to\infty} (x_{n_j} - b(d+\varepsilon)) = \frac{d}{2}(1-b) > 0.$$

Thus $d = 0$, and therefore $\lim\limits_{n\to\infty} x_n = 0$. The proof is now complete. □

For simplicity, we introduce the following notation: $\overline{P}_l(s) = l^\alpha q_s((s-\tau)/s)^\alpha \times ((s-\tau-N)/2)^\alpha$ with $l \in (0,1)$ arbitrarily chosen and N large enough,

$$\overline{P} = \liminf_{n\to\infty} \frac{n^\alpha}{a_n} \sum_{s=n}^{\infty} \overline{P}_l(s) \text{ and } \overline{Q} = \limsup_{n\to\infty} \frac{1}{n} \sum_{s=n_0}^{n-1} \frac{s^{\alpha+1}}{a_s} \overline{P}_l(s) \tag{3.2}$$

where w_n and r, R are defined in (2.6) and (2.7) respectively.

Lemma 3.3. *Assume that $\{a_n\}$ is nondecreasing. Let $\{x_n\}$ be a positive solution of* (1.2).

(I) *Let $\overline{P} < \infty$ and $\overline{Q} < \infty$. Suppose that the corresponding $\{z_n\}$ satisfies* (iii). *Then*

$$\overline{P} \leqslant r - r^{1+1/\alpha} \quad \text{and} \quad \overline{P} + \overline{Q} \leqslant 1. \tag{3.3}$$

(II) *If $\overline{P} = \infty$ or $\overline{Q} = \infty$, then z_n does not satisfy* (iii).

Proof. Part (I): Let $\{x_n\}$ be a positive solution of equation (1.2) and let z_n satisfy (iii). Since $0 < z_n < x_n$, equation (1.2) can be written in the form

$$\Delta(a_n(\Delta^2 z_n)^\alpha) + q_n z_{n+1-\tau}^\alpha < 0.$$

Thus

$$\Delta(a_n(\Delta^2 z_n)^\alpha) < 0.$$

Since $\Delta a_n \geqslant 0$, we have $\Delta^3 z_n \leqslant 0$. So, there exists an integer $N \geqslant n_0$ such that z_n satisfies $z_{n-\tau} > 0$, $\Delta z_n > 0$, $\Delta^2 z_n > 0$, $\Delta^3 z_n \leqslant 0$ for $n \geqslant N$.

From the definition of w_n and equation (1.2) we see that $w_n > 0$ and satisfies

$$\Delta w_n \leqslant -q_n\Big(\frac{z_{n+1-\tau}}{\Delta z_n}\Big)^\alpha - \frac{\alpha}{a_{n+1}^{1/\alpha}} w_{n+1}^{1+1/\alpha}. \tag{3.4}$$

From Lemma 2.3 with $u_n = \Delta z_n$, we have for all l, the same as in $\overline{P}_l(n)$

$$\frac{1}{\Delta z_n} \geqslant l\Big(\frac{n-\tau}{n}\Big)\frac{1}{\Delta z_{n-\tau}}, \quad n \geqslant N,$$

which with (3.4) gives

$$\Delta w_n \leqslant -l^\alpha q_n\Big(\frac{n-\tau}{n}\Big)^\alpha\Big(\frac{z_{n+1-\tau}}{\Delta z_{n-\tau}}\Big)^\alpha - \frac{\alpha}{a_{n+1}^{1/\alpha}} w_{n+1}^{1+1/\alpha}.$$

Using the fact from Lemma 2.4 that $z_{n+1} \geqslant \frac{1}{2}(n-N)\Delta z_n$, we have

$$\Delta w_n + \overline{P}_l(n) + \frac{\alpha}{a_{n+1}^{1/\alpha}} w_{n+1}^{1+1/\alpha} \leqslant 0. \tag{3.5}$$

Now, we proceed similarly to the proof of Part (I) of Lemma 2.6 to verify that (3.3) holds.

Part (II): Let $\{x_n\}$ be a positive solution of equation (1.2) and let the corresponding z_n satisfy (iii). First assume that $\overline{P} = \infty$. Summing (3.5) from n to ∞, one obtains

$$w_n \geqslant \sum_{s=n}^{\infty} \overline{P}_l(s) + \alpha \sum_{s=n}^{\infty} \frac{w_{s+1}^{1+1/\alpha}}{a_{s+1}^{1/\alpha}} \quad \text{for } n \geqslant n_2. \tag{3.6}$$

Therefore

$$\frac{n^\alpha}{a_n} w_n \geqslant \frac{n^\alpha}{a_n} \sum_{s=n}^{\infty} \overline{P}_l(s).$$

Taking $\liminf$ on both sides as $n \to \infty$, we obtain in view of (2.14) that

$$1 \geqslant r \geqslant \infty,$$

which is a contradiction. Now assume that $\overline{Q} = \infty$. To obtain the desired contradiction one can proceed exactly as in the proof of Lemma 2.6 □

Theorem 3.4. *Assume that $\{a_n\}$ is nondecreasing and condition* (2.3) *holds. Let $\{x_n\}$ be a solution of equation* (1.2)*. If*

$$\overline{P} = \liminf_{n\to\infty} \frac{n^\alpha}{a_n} \sum_{s=n}^{\infty} \overline{P}_l(s) > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}, \tag{3.7}$$

then $\{x_n\}$ is oscillatory or $x_n \to 0$ as $n \to \infty$.

P r o o f. Let $\{x_n\}$ be a positive solution of equation (1.2). Then

$$\Delta(a_n(\Delta^2 z_n)^\alpha) + q_n x_{n+1-\tau}^\alpha = 0. \tag{3.8}$$

We claim that $\{x_n\}$ is bounded. If not, then there exists a sequence $\{n_j\}$ such that $\lim\limits_{j\to\infty} n_j = \infty$ and $\lim\limits_{j\to\infty} x_{n_j} = \infty$, and

$$x_{n_j} = \max\{x_s\colon\ n_0 \leqslant s \leqslant n_j\}.$$

Since $n-\delta \to \infty$ as $n \to \infty$, we can choose $n_j - \delta > n_0$. As $n - \delta \leqslant n$, we have

$$x_{n_j} - \delta \leqslant \max\{x_s\colon\ n_0 \leqslant s \leqslant n_j - \delta\}.$$

Therefore for all large j

$$z_{n_j} = x_{n_j} - b_{n_j} x_{n_j-\delta} \geqslant (1-b)x_{n_j}.$$

Thus, $z_{n_j} \to \infty$ as $j \to \infty$. So $\{z_n\}$ is positive and unbounded. It follows from Lemma 3.1 that case (iii) has to occur. Lemma 3.3 (I) yields

$$\overline{P} \leqslant r - r^{1+1/\alpha}.$$

Using the inequality

$$Bu - Au^{1+1/\alpha} \leqslant \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}$$

with $A = B = 1$ and $u = r$ we obtain

$$\overline{P}_\alpha \leqslant \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}},$$

which contradicts (3.7). So we can conclude that both $\{x_n\}$ and $\{z_n\}$ are bounded. Lemma 3.1 now implies that for z_n, either (iv) or (v) holds.

If case (iv) holds, then Lemma 3.2 ensures that $\lim_{n\to\infty} x_n = 0$. On the other hand, if the case (v) holds, then there exists a finite limit $\lim_{n\to\infty} z_n = -d < 0$. We know that $0 < x_n$ is bounded, so

$$\limsup_{n\to\infty} x_n = c,\ \ 0 \leqslant c < \infty.$$

We claim that $c = 0$. If not, then there exists a sequence $\{n_j\}$ such that $\lim_{j\to\infty} n_j = \infty$ and $\lim_{j\to\infty} x_{n_j} = c$. It is easy to see that for $\varepsilon = \frac{1}{2}c(1-b)/b > 0$, we have $x_{n_j-\delta} < c+\varepsilon$. Moreover,

$$0 > -d = \lim_{j\to\infty} z_{n_j} \geqslant \lim_{j\to\infty} (x_{n_j} - b(c+\varepsilon)) = \frac{c}{2}(1-b) > 0,$$

which is a contradiction. Thus $c = 0$ and $\lim_{n\to\infty} x_n = 0$. This completes the proof. □

Corollary 3.5. *Let $\{a_n\}$ be nondecreasing and let condition* (2.3) *hold. Let $\{x_n\}$ be a solution of equation* (1.2)*. If*

$$\liminf_{n\to\infty} \frac{n^\alpha}{a_n} \sum_{s=n}^{\infty} q_s \frac{(s-\tau)^{2\alpha}}{s^\alpha} > \frac{(2\alpha)^\alpha}{(\alpha+1)^{\alpha+1}}, \tag{3.9}$$

then $\{x_n\}$ is oscillatory or satisfies $\lim_{n\to\infty} x_n = 0$.

Proof. It is easy to verify that (3.9) implies (3.7). □

The proof of the next result is similar to that of Theorem 2.9, so it is omitted.

Theorem 3.6. *Assume that $\{a_n\}$ is nondecreasing and condition* (2.3) *holds. Let $\{x_n\}$ be a solution of equation* (1.2)*. If*

$$\overline{P} + \overline{Q} > 1,$$

then $\{x_n\}$ is oscillatory or satisfies $\lim_{n\to\infty} x_n = 0$.

Corollary 3.7. *Assume $\{a_n\}$ is nondecreasing and condition* (2.3) *holds. Let $\{x_n\}$ be a solution of equation* (1.2)*. If*

$$\overline{Q} = \limsup_{n\to\infty} \frac{1}{n} \sum_{s=n_0}^{n-1} \frac{s^{\alpha+1}}{a_s} \overline{P}_l(s) > 1,$$

then $\{x_n\}$ is oscillatory or satisfies $\lim_{n\to\infty} x_n = 0$.

Simplifying the last result, we have

Corollary 3.8. *Assume* $\{x_n\}$ *is nondecreasing and condition* (2.3) *holds. Let* $\{x_n\}$ *be a solution of equation* (1.2). *If*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{s=n_0}^{n-1} \frac{s(s-\tau)^{2\alpha}}{a_s} q_s > 2^\alpha,$$

then $\{x_n\}$ *is oscillatory or satisfies* $\lim_{n\to\infty} x_n = 0$.

Example 3.1. Consider the third order half-linear difference equation

$$\Delta\Big(n\Big(\Delta^2\Big(x_n - \frac{1}{3}x_{n-1}\Big)\Big)^3\Big) + \frac{\lambda}{(n-2)^6}\, x_{n-1}^3 = 0, \quad \lambda > 0. \tag{3.10}$$

Corollary 3.5 implies that every solution of equation (3.10) is either oscillatory or converges to zero as $n \to \infty$ provided that $\lambda > \frac{27}{16}$.

Example 3.2. Consider a third order difference equation

$$\Delta\Big(n\Big(\Delta^2\Big(x_n - \frac{(n-1)}{2n}x_{n-1}\Big)\Big)^3\Big) + \frac{(8n^2+27n+27)(n-1)^3}{(n+3)^3(n+2)^3(n+1)^3n^2}\, x_{n-1}^3 = 0, \quad n \geqslant 1. \tag{3.11}$$

By Corollary 3.5, every solution of equation (3.11) is either oscillatory or converges to zero as $n \to \infty$. In fact, $\{x_n\} = \{1/n\}$ is one such solution of equation (3.11).

We conclude this section with the following remarks.

Remark 3.3. If we relax condition (2.3) in Theorems 2.7, 2.9, 3.4, 3.6, and Corollaries 2.8, 2.10, 3.5, 3.7, then the assertion of these results may be reformulated as: every nonoscillatory solution of equations (1.1) and (1.2) is bounded.

Remark 3.4. Theorems 2.7, 2.9, 3.4, 3.6 complement the results presented in [1], [2], [3] for nonlinear difference equations of the form

$$\Delta(b_n(\Delta(a_n\Delta x_n))^\alpha) + q_n x_n^\alpha = 0.$$

Acknowledgement. The authors thank the referee for his/her valuable suggestions which improve the consents of the paper.

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