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ON THE ASYMPTOTICS OF SOLUTIONS TO THE SECOND INITIAL BOUNDARY VALUE PROBLEM FOR SCHRÖDINGER SYSTEMS IN DOMAINS WITH CONICAL POINTS

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Abstract. In this paper, for the second initial boundary value problem for Schrödinger systems, we obtain a performance of generalized solutions in a neighborhood of conical points on the boundary of the base of infinite cylinders. The main result are asymptotic formulas for generalized solutions in case the associated spectrum problem has more than one eigenvalue in the strip considered.

Keywords: second initial boundary value problem, Schrödinger systems, generalized solution, regularity, asymptotic behavior

MSC 2010: 35B40, 35B65, 35G99

1. Introduction

The boundary value problems for Schrödinger equations whose coefficients are independent of the time variable have been previously proposed and analyzed by J.-L. Lions and E. Magenes, [15], [16]. In the finite cylinder $\Omega_T = \Omega \times (0,T)$, the first initial boundary value problem for this kind of equation with coefficients depending on both the time and spatial variables has been considered in [2]. In this paper, we study the second initial boundary value problem for general Schrödinger systems (coefficients depending on both the time and spatial variables) in the infinite cylinder $\Omega_\infty = \Omega \times (0,\infty)$ with conical points on the boundary of the base Ω. Existence, uniqueness of the generalized solution of this problem were considered in [5], the regularity of the generalized solution (with respect to both the time and

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spatial variables) were given in [6] and [7]. Our main purpose here is to study the asymptotic behavior of generalized solutions in a neighborhood of conical points.

In the previous papers [1], [2], [3], [4], to consider asymptotic behavior of solutions of boundary value problems for non-stationary systems, the authors just studied the case when the associated spectrum problem has simple eigenvalues. And now, by weakening restrictions on eigenvalues of the spectrum problem (extending them to semisimple eigenvalues having invariant multiplicity), we obtain generally asymptotic formulas of solutions as linear combinations of special singular vector functions and regular vector functions. Moreover, these vector functions and coefficients of the linear combinations are regular with respect to the time variable. Our results are extended to the case in which the considered strip has more than one eigenvalue. That causes more technical difficulties. The main method used in this paper can be shown as follows. At first, we study the asymptotic behavior of solutions of the second boundary value problem for elliptic systems depending on a parameter. After that, we take the term containing the derivative in time of the unknown vector function to the right-hand side of the system so that the problem can be viewed as an elliptic one (depending on the parameter t). Dividing m into 3 cases by comparing it with $\frac{1}{2}n$, where n is the dimension of Ω, we can manage to apply results for elliptic systems depending on a parameter to get the asymptotic behavior of solutions of our problem.

The paper is organized as follows. In Section 2, we introduce some notation and formulation of the problem. The main result is given in Section 3, where asymptotic formulas of solutions of the second initial boundary value problem for Schrödinger systems are shown. In Section 4, by giving some auxiliary lemmas, we prove our main result. Some examples are stated in Section 5. Finally, some conclusions will be given in the last section.

2. Notation and formulation of the problem

Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$. Moreover, we assume that $\Gamma = \partial\Omega \setminus \{0\}$ is a smooth manifold and Ω coincides with the cone $K = \beta\{x\colon\ x/|x| \in G\}$ in a neighborhood of the origin 0, where G is a smooth domain on the unit sphere S^{n-1} in $\mathbb{R}^n$.

Let A be a subset of $\mathbb{R}^n$. Denote $A_T = A \times (0,T)$ for some $T \in (0,\infty)$, $A_\infty = A \times (0,\infty)$ and $\overline{A}_\infty = \overline{A} \times [0,\infty]$. Let u be a complex valued vector function with components $u_1,\dots,u_s$ and let $\alpha = (\alpha_1,\dots,\alpha_n)$ ($\alpha_i \in \mathbb{N}$, $i = 1,\dots,n$) be a multi-index, $|\alpha| = \alpha_1 + \ldots + \alpha_n$. We use the notation $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}$, $|D^\alpha u|^2 = \sum\limits_{i=1}^{s}|D^\alpha u_i|^2$ and $u_{t^j} = (\partial^j u_1/\partial t^j,\dots,\partial^j u_s/\partial t^j)$. Denote by $\omega = (\omega_1,\dots,\omega_{n-1})$ a local coordinate system on S^{n-1}, $r = |x|$ and $\binom{k}{l} = k!/l!(k-l)!$ $(0 \leqslant l \leqslant k)$. Moreover, if β is a real number, we use the symbol $[\beta]$ for the maximum integer that is smaller than or equal to β.

Assume that l, h, k are nonnegative integer numbers, β is a real number and γ is a positive real number. In this paper we will use the usual function spaces: $\mathring{C}^\infty(\Omega)$, $L_2(\Omega)$, $H^l(\Omega)$, $H^{l-1/2}(\Gamma)$, $H^{l,k}(\Omega_T)$ when $T < \infty$ (see [2], [5] for the precise definitions). We define

- $H^l_\beta(\Omega)$—the space of all measurable complex functions $v(x)$ that satisfy

$$\|v\|_{H^l_\beta(\Omega)} = \Big(\sum_{|\alpha|=0}^{l} \int_\Omega r^{2(\beta+|\alpha|-l)} |D^\alpha v|^2 \,\mathrm{d}x\Big)^{1/2} < \infty; \tag{2.1}$$

- $H^{l-1/2}_\beta(\Gamma)$—the space of traces of functions from $H^l_\beta(\Omega)$ on Γ with the norm

$$\|v\|_{H^{l-1/2}_\beta(\Gamma)} = \inf\{\|w\|_{H^l_\beta(\Omega)}\colon\ w \in H^l_\beta(\Omega),\ w|_\Gamma = v\}; \tag{2.2}$$

- $W^l_\beta(\Omega)$—the space of all measurable complex functions $v(x)$ that have generalized derivatives up to order l with the norm

$$\|v\|_{W^l_\beta(\Omega)} := \Big(\sum_{|\alpha|=0}^{l} \int_\Omega r^{2\beta} |D^\alpha v|^2 \,\mathrm{d}x\Big)^{1/2} < \infty;$$

- $H^{l,k}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$—the space of all measurable complex functions $v(x,t)$ that have generalized derivatives up to order l with respect to x and up to order k with respect to t with the norm

$$\|v\|_{H^{l,k}(\mathrm{e}^{-\gamma t},\Omega_\infty)} = \Big(\int_{\Omega_\infty} \Big[\sum_{|\alpha|=0}^{l} |D^\alpha v|^2 + \sum_{j=0}^{k} |v_{t^j}|^2\Big] \mathrm{e}^{-2\gamma t} \,\mathrm{d}x \,\mathrm{d}t\Big)^{1/2} < \infty;$$

- $H^{l,k}_\beta(\mathrm{e}^{-\gamma t}, \Omega_\infty)$—the space of all measurable complex functions $v(x,t)$ with the norm

$$\begin{aligned} &\|v\|_{H^{l,k}_\beta(\mathrm{e}^{-\gamma t},\Omega_\infty)} \\ &\quad = \Big(\int_{\Omega_\infty} \Big(\sum_{|\alpha|=0}^{l} r^{2(\beta+|\alpha|-l)} |D^\alpha v|^2 + \sum_{j=1}^{k} |v_{t^j}|^2\Big) \mathrm{e}^{-2\gamma t} \,\mathrm{d}x \,\mathrm{d}t\Big)^{1/2} < \infty; \end{aligned} \tag{2.3}$$

- $V^l_\beta(\mathrm{e}^{-\gamma t},\Omega_\infty)$—the space of all measurable complex functions $v(x,t)$ that have generalized derivatives up to order l with respect to x and t, with the norm

$$\|v\|_{V^l_\beta(\mathrm{e}^{-\gamma t},\Omega_\infty)} = \Big(\int_{\Omega_\infty}\Big(\sum_{|\alpha|+j=1}^{l} r^{2(\beta+|\alpha|+j-l)}|D^\alpha l v_{t^j}|^2+|v|^2\Big)\mathrm{e}^{-2\gamma t}\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}<\infty.$$

The weighted spaces $H^l_\beta(K)$, $H^{l-1/2}_\beta(\partial K)$, $H^{l,k}_\beta(\mathrm{e}^{-\gamma t},K_\infty)$ are defined similarly to (2.1), (2.2), and (2.3) with Ω, Γ replaced by K and ∂K, respectively.

Let X be a Banach space and h a nonnegative integer. By $L^\infty(0,\infty;X)$ we denote the space of all X-valued functions defined on $(0,\infty)$ with the norm

$$\|v\|_{\infty,X}=\operatorname*{ess\,sup}_{t>0}\|v(t)\|_X<\infty.$$

Denote by $W^h(0,\infty;X)$ the Sobolev space of all X-valued functions defined on $(0,\infty)$ with the norm

$$\|f\|_{W^h(0,\infty;X)}=\Big(\sum_{k=0}^{h}\int_0^\infty\|f_{t^k}(t)\|^2_X\mathrm{e}^{-2\gamma t}\,\mathrm{d}t\Big)^{1/2}<\infty.$$

For short, we denote $L^h_2(\mathrm{e}^{-\gamma t},(0,\infty))=W^h(0,\infty;\mathbb{C})$,

$$\begin{aligned}
V^{l,0}_{\beta,h}(\mathrm{e}^{-\gamma t},\Omega_\infty)&=W^h(0,\infty;H^l_\beta(\Omega)),\\
V^{l-1/2,0}_{\beta,h}(\mathrm{e}^{-\gamma t},\Gamma_\infty)&=W^h(0,\infty;H^{l-1/2}_\beta(\Gamma)),\\
V^{l,0}_{\beta,h}(e^{-\gamma t},K_\infty)&=W^h(0,\infty;H^l_\beta(K)),\\
V^{l-1/2,0}_{\beta,h}(\mathrm{e}^{-\gamma t},\partial K_\infty)&=W^h(0,\infty;H^{l-1/2}_\beta(\partial K)),\\
C^{\infty,h}(\mathrm{e}^{-\gamma t},G_\infty)&=W^h(0,\infty;C^\infty(G)),\\
C^{\infty,h}(\mathrm{e}^{-\gamma t},\partial G_\infty)&=W^h(0,\infty;C^\infty(\partial G)).
\end{aligned}$$

Recall that an X-valued function $f(t)$ defined on $[0,\infty)$ is said to be continuous or analytic at ∞ if the function $g(t):=f(1/t)$ is continuous or analytic, respectively, at $t=0$ with a suitable value of $g(0)\in X$. In these cases we can regard $f(t)$ as a function defined on $[0,\infty]$ with $f(\infty):=g(0)$.

Denote by $C^a([0,\infty],X)$ the set of all X-valued functions defined and analytic on $[0,\infty]$. It is clear that if $f\in C^a([0,\infty],X)$ then f together with all its derivatives are bounded on $[0,\infty]$.

Let A be a subset of $\mathbb{R}^n$ and let $f(x,t)$ be a complex-valued function defined on $\overline{A}_\infty$. We will say that f belongs to the class $C^{\infty,a}(\overline{A}_\infty)$ if $f \in C^a([0,\infty], C^l(\overline{A}))$ for all nonnegative integers l.

For convenience, in the rest of this paper we say that the complex valued vector functions $u(x,t), f(x,t), g(x,t), \dots$ belong to some spaces if all of its components belong to them.

We now introduce a $2m$th-order differential operator

$$L = L(x,t,D) := \sum_{|p|,|q|=0}^{m} D^p(a_{pq}(x,t)D^q),$$

where a_{pq} are $s \times s$ matrices of functions belonging to $C^{\infty,a}(\overline{\Omega}_\infty)$ and $a_{pq} = (-1)^{|p|+|q|}a^*_{qp}$ (a^*_{qp} denotes the transposed conjugate matrix of a_{qp}). Suppose that a_{pq} are continuous in $x \in \overline{\Omega}$ uniformly with respect to $t \in [0,\infty]$ if $|p| = |q| = m$.

Set

$$B[u,u](t) = \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_\Omega a_{pq}(x,t)D^q u(x,t)\overline{D^p u(x,t)}\,\mathrm{d}x.$$

We assume that $B[\cdot,\cdot](t)$ is $H^m(\Omega)$-coercive uniformly with respect to $t \in (0,\infty)$, i.e.,

$$B(t,u,u) \geqslant \mu_0 \|u\|^2_{H^m(\Omega)}$$

for all $u \in H^m(\Omega)$, $t \in (0,\infty)$, where μ_0 is a positive constant independent of u and t (see [5] for reference).

Assume that

$$B_j = B_j(x,t,D) := \sum_{|\alpha| \leqslant 2m-1-j} b_{j\alpha}(x,t)D^\alpha, \quad j = 0,1,\dots,m-1$$

is a system of boundary operators on Γ_∞, where coefficients $b_{j\alpha}$ are $s \times s$ matrices of functions belonging to $C^{\infty,a}(\partial\Omega \times [0,\infty])$. Suppose that $b_{j\alpha}$ are continuous in $x \in \partial\Omega$ uniformly with respect to $t \in [0,\infty]$ if $|\alpha| = 2m-1-j$ for all $j = 0,1,\dots,m-1$.

Moreover, we assume that the system $\{B_j,\ j = 0,\dots,m-1\}$ satisfies the Green formula

$$\int_\Omega Lu\overline{v}\,\mathrm{d}x = \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_\Omega a_{pq}(\cdot,t)D^q u\overline{D^p v}\,\mathrm{d}x + \sum_{j=0}^{m-1} \int_\Gamma B_j u \frac{\partial^j \overline{v}}{\partial \nu^j}\,\mathrm{d}s$$

for all $u,v \in C^\infty(\overline{\Omega})$ and almost all $t \in (0,\infty)$, where ν is the unit exterior normal to Γ.

In this paper, we consider asymptotic behavior near the conical point of solutions of *the second initial boundary value problem for the Schrödinger system*

$$\mathrm{i}(-1)^{m-1}Lu - u_t = f \quad \text{in } \Omega_\infty, \tag{2.4}$$

$$B_j u = 0 \quad \text{on } \Gamma_\infty,\ j = 0, 1, \ldots, m-1, \tag{2.5}$$

$$u|_{t=0} = 0 \quad \text{on } \Omega, \tag{2.6}$$

where u, f are vector functions.

The vector function $u(x,t)$ is then called *a generalized solution* in the space $H^{m,0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$ of the problem (2.4)–(2.6) if $u \in H^{m,0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$ and for each $T \in (0,\infty)$, the integral equality

$$\begin{aligned}\mathrm{i}(-1)^{m-1} \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_{\Omega_T} a_{pq}(x,t) D^q u(x,t) \overline{D^p \eta}(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{\Omega_T} u(x,t)\overline{\eta_t}(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f(x,t)\overline{\eta}(x,t)\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{2.7}$$

holds for all test vector functions $\eta(x,t) \in H^{m,1}(\Omega_T)$, $\eta(x,T) = 0$ for all $x \in \Omega$.

The solvability of this problem was considered in [5] (Theorems 3.1, 3.2). It can be formulated as follows.

Assume that

i) $|(\partial a_{pq}/\partial t)(x,t)| \leqslant \mu$ where $\mu = \text{const.} > 0$; $\forall\, 0 \leqslant |p|,|q| \leqslant m$; $\forall\,(x,t) \in \Omega_\infty$ and

ii) $f, f_t \in L^\infty(0,\infty, L_2(\Omega))$, $f(x,0) = 0$.

Then there exists a positive number γ_0 such that for every $\gamma > \gamma_0$ the second initial boundary value problem (2.4)–(2.6) has a unique generalized solution $u(x,t)$ in the space $H^{m,0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$.

Moreover, the regularity of the generalized solution (with respect to both the time and spatial variables) was given in [6] and [7]. The main point in this paper is to study the asymptotic behavior of generalized solutions in a neighborhood of conical points. The results are given in the next section.

3. Main result

Denote

$$\mathcal{L}(t,D) := \sum_{|p|=|q|=m} D^p(a_{pq}(0,t)\,D^q),$$
$$\mathcal{B}_j(t,D) := \sum_{|\alpha|=2m-1-j} b_{j\alpha}(0,t)D^\alpha, \quad j=0,1,\dots,m-1$$

to be the principal parts of operators $L(x,t,D)$, $B_j(x,t,D)$ at the origin 0.

Let $\omega = (\omega_1,\dots,\omega_{n-1})$ be a local coordinate system on S^{n-1}, $r=|x|$. We can check easily that $D^\alpha = r^{-|\alpha|}\sum\limits_{k=0}^{|\alpha|} P_{\alpha,k}(\omega,D\omega)(rD_r)^k$, where $P_{\alpha,k}(\omega,D\omega)$ is a linear operator with coefficients belonging to $C^{\infty,a}(\overline{G}_\infty)$, $D_r = \mathrm{i}\partial/\partial r$ and $D_\omega = \partial/\partial\omega_1\dots\partial\omega_{n-1}$. So the operators $\mathcal{L}(t,D)$, $\mathcal{B}_j(t,D)$ can be rewritten in the form

$$\mathcal{L}(t,D) = r^{-2m}\mathcal{L}(\omega,t,rD_r,D_\omega),$$
$$\mathcal{B}_j(t,D) = r^{-(2m-1-j)}\mathcal{B}_j(\omega,t,rD_r,D_\omega), \quad j=0,1,\dots,m-1,$$

where $\mathcal{L}(\omega,t,rD_r,D_\omega)$, $\mathcal{B}_j(\omega,t,rD_r,D_\omega)$, $j=0,1,\dots,m-1$, are linear operators with coefficients belonging to $C^{\infty,a}(\overline{G}_\infty)$.

We introduce the operator

$$\mathcal{U}(\lambda,t) = (\mathcal{L}(\omega,t,\lambda,D_\omega),\mathcal{B}_j(\omega,t,\lambda,D_\omega)), \quad \lambda\in\mathbb{C},\ t\in[0,\infty)$$

of the parameter-depending elliptic boundary-value problem

$$\mathcal{L}(\omega,t,\lambda,D_\omega)v = 0 \quad \text{in } G, \tag{3.1}$$

$$\mathcal{B}_j(\omega,t,\lambda,D_\omega)v = 0 \quad \text{on } \partial G,\ j=0,1,\dots,m-1. \tag{3.2}$$

(Here the parameters are λ and t.) For every fixed number $\lambda\in\mathbb{C}$, $t\in[0,\infty)$ this operator continuously maps

$$\mathcal{X}\equiv H^l(G) \text{ into } \mathcal{Y}\equiv H^{l-2m}(G)\times\prod_{j=1}^m H^{l-\mu_j-1/2}(\partial G) \quad (l\geqslant 2m).$$

We mention now some well-known definitions ([13]). Let $t_0\in[0,\infty)$ be a fixed number. If $\lambda_0\in\mathbb{C}$, $\varphi_0\in\mathcal{X}$ are such that $\varphi_0\neq 0$, $\mathcal{U}(\lambda_0,t_0)\varphi_0=0$, then λ_0 is called an eigenvalue of $\mathcal{U}(\lambda,t_0)$ and $\varphi_0\in\mathcal{X}$ is called an eigenvector corresponding to λ_0. $\Lambda = \dim\ker\mathcal{U}(\lambda_0,t_0)$ is called the geometric multiplicity of the eigenvalue λ_0.

If the elements $\varphi_1, \dots, \varphi_s$ of $\mathcal{X}$ satisfy the equations

$$\sum_{q=0}^{\sigma} \frac{1}{q!} \frac{\mathrm{d}^q}{\mathrm{d}\lambda^q} \mathcal{U}(\lambda, t_0)|_{\lambda=\lambda_0} \varphi_{\sigma-q} = 0 \quad \text{for } \sigma = 1, \dots, s,$$

then the ordered collection $\varphi_0, \varphi_1, \dots, \varphi_s$ is said to be a Jordan chain corresponding to the eigenvalue λ_0 of the length $s+1$. The rank of the eigenvector φ_0 ($\operatorname{rank} \varphi_0$) is the maximal length of the Jordan chains corresponding to the eigenvector φ_0.

A canonical system of eigenvectors of $\mathcal{U}(\lambda_0, t_0)$ corresponding to the eigenvalue λ_0 is a system of eigenvectors $\varphi_{1,0}, \dots, \varphi_{\Lambda,0}$ such that $\operatorname{rank} \varphi_{1,0}$ is maximal among the ranks of all eigenvectors corresponding to λ_0 and $\operatorname{rank} \varphi_{j,0}$ is maximal among the ranks of all eigenvectors in any direct complement in $\ker \mathcal{U}(\lambda_0, t_0)$ to the linear span of vectors $\varphi_{1,0}, \dots, \varphi_{j-1,0}$ ($j = 2, \dots, \Lambda$). The numbers $\kappa_j = \operatorname{rank} \varphi_{j,0}$ ($j = 1, \dots, \Lambda$) are called the partial multiplicities and the sum $\kappa = \kappa_1 + \ldots + \kappa_\Lambda$ is called the algebraic multiplicity of the eigenvalue λ_0.

The eigenvalue λ_0 is called simple if both its geometric multiplicity and the rank of the corresponding eigenvector equal one. The eigenvalue λ_0 is called semisimple if its algebraic multiplicity and its geometric multiplicity are equal. So all semisimple eigenvalue's partial multiplicities are equal to one (see [8, p. 70]).

It is well known [13] that for every $t \in [0, \infty)$, the spectrum of the problem (3.1)–(3.2) is an enumerable set of eigenvalues. In addition, it follows from [8, p. 70, p. 99] that if $\lambda(t)$ is a semisimple eigenvalue having invariant multiplicity for all $t \in [0, \infty]$ of the problem (3.1)–(3.2), then $\lambda(t)$ is analytic on $[0, \infty]$. Moreover, there exists a canonical system of eigenvectors $\{\varphi_k(\omega, t),\ k = 1, \dots, \Lambda\}$ of the problem (3.1)–(3.2) corresponding to the eigenvalue $\lambda(t)$ such that $\varphi_k(\omega, t)$ are analytic functions on $\overline{G}_\infty$ for all $k = 1, \dots, \Lambda$; Λ is the algebraic multiple of the eigenvalue $\lambda(t)$.

For $\gamma > 0$, denote $\gamma_k := (2k+1)\gamma$. The following theorem gives the asymptotic behavior of solutions of the second initial boundary value problem for Schrödinger system (2.4)–(2.6) in a neighborhood of a conical point.

Theorem 3.1. *Let l, h be nonnegative integers, let β, β' be real numbers that satisfy $\beta \geqslant \max\{m, 2m - \frac{1}{2}n\}$ and $0 \leqslant \beta' < \beta$. Assume that the vector function $u(x,t)$ is a generalized solution in $H^{m,0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$ of the problem (2.4)–(2.6); $f_{t^k} \in L^\infty(0, \infty, L_2(\Omega))$, $k \leqslant h + 2l + 2$; $f_{t^k}(x, 0) = 0$, $k \leqslant h + 2l$. Moreover, assume that the straight lines $\operatorname{Im} \lambda = -\beta + 2m - \frac{1}{2}n$, $\operatorname{Im} \lambda = -\beta' + 2m + l - \frac{1}{2}n$ do not contain any point of the spectrum of the problem (3.1)–(3.2) for every $t \in [0, \infty]$, and in the strip*

$$-\beta + 2m - \frac{1}{2}n < \operatorname{Im} \lambda < -\beta' + 2m + l - \frac{1}{2}n$$

there are semisimple eigenvalues $\lambda_1(t), \dots, \lambda_{N_0}(t)$ of the problem (3.1)–(3.2) that have invariant multiplicity for all $t \in [0, \infty]$ and satisfy for all $T \in (0, \infty)$

i) $\operatorname{Im}\lambda_1(t) < \dots < \operatorname{Im}\lambda_{N_0}(t)$, $t \in [0, T]$,
ii) $-\beta + 2m - \frac{1}{2}n < \operatorname{Im}\lambda_1(t) < -\beta + 2m + \mu_1^* - \frac{1}{2}n < \operatorname{Im}\lambda_2(t) < \dots < -\beta + 2m + \mu_{N_0-1}^* - \frac{1}{2}n < \operatorname{Im}\lambda_{N_0}(t) < -\beta' + 2m + l - \frac{1}{2}n$, $t \in (T, \infty]$,
iii) $\operatorname{Im}\lambda_j(t) \neq \operatorname{Im}\lambda_k(t) + z$, $z \in \mathbb{Z}$, $j \neq k \in \{1, \dots, N_0\}$, $t \in [0, \infty]$,

where μ_j^*, $j = 1, \dots, N_0$, are nonnegative numbers. Then the following representation holds:

$$u(x,t) = \sum_{j=0}^{N_0} \sum_{k=0}^{l+\kappa_j-1} r^{-\mathrm{i}\lambda_j(t)+k} P_{k,j}(\ln r) + w(x,t), \tag{3.3}$$

where $w(\cdot,\cdot) \in V^{2m+l,0}_{\beta',h+l}(\mathrm{e}^{-\gamma_{h+l}t}, \Omega_\infty)$, $P_{k,j}(\cdot)$ are vectors of polynomials of order less than $3l + \kappa_j$, whose coefficients are functions in the space $C^{\infty,h+l}(\mathrm{e}^{-\gamma_{h+l}t}, G_\infty)$; κ_j is the minimum integer greater than $-\beta' + 2m - \frac{1}{2}n - \operatorname{Im}\lambda_j(t)$ for all $t \in [0, \infty)$, $j = 1, \dots, N_0$.

R e m a r k 3.1. The formula (3.3) is separated into two parts. The latter part $w(\cdot,\cdot) \in V^{2m+l,0}_{\beta',h+l}(\mathrm{e}^{-\gamma_{h+l}t}, \Omega_\infty)$ has good regularity near the conical points, and the former part is built on the eigenvalues λ_j ($j = 1, \dots, N_0$) and the distance r to the conical point. With the restriction on κ_j, all vector functions that have good regularity are combined to $w(\cdot,\cdot)$, the remaining vector functions are in the sum of $r^{-\mathrm{i}\lambda_j(t)+k}$, with coefficients being vectors of polynomials $P_{k,j}(\cdot)$. First, the vectors of polynomials $P_{k,j}(\cdot)$ are constructed on some quite simple sums (see Lemma 4.1, Lemma 4.2), based on canonical systems of eigenvectors of the problem (3.1)–(3.2) corresponding to its eigenvalues. However, when we combine formulas to build the asymptotic behavior of solutions of the parameter-depending elliptic boundary-value problem (3.1)–(3.2) and the problem (2.4)–(2.6) (Proposition 4.1 and the following results), because of the dependence of the coefficients on the time variable, all sums of coefficients must be changed into the form of vectors of polynomials $P_{k,j}(\cdot)$ whose coefficients are functions with respect to (ω, t). By choosing κ_j to be the minimum integer greater than $-\beta' + 2m - \frac{1}{2}n - \operatorname{Im}\lambda_j(t)$ for all $t \in [0, \infty)$, the formula has clear representation.

4. Proof of the main result

First, in the cone K we consider the second boundary value problem for elliptic system depending on a parameter t in the form

$$(-1)^m \mathcal{L}(t,D)u = f \quad \text{in } K_\infty, \tag{4.1}$$

$$\mathcal{B}_j(t,D)u = g_j \quad \text{on } \partial K_\infty,\ j=0,\dots,m-1. \tag{4.2}$$

The following lemma can be proved similarly to Lemma 4.1 in [1].

Lemma 4.1. *Let the vector function* $u \in V^{l_1,0}_{\beta_1,h}(\mathrm{e}^{-\gamma t}, K_\infty)$ *be a solution of the problem* (4.1)–(4.2), *where* $f \in V^{l_2-2m,0}_{\beta_2,h}(\mathrm{e}^{-\gamma t}, K_\infty)$, $g_j \in V^{l_2-2m+j+\frac{1}{2},0}_{\beta_2,h}(\mathrm{e}^{-\gamma t}, \partial K_\infty)$; $l_1, l_2 \geqslant 2m$, $\beta_2 - l_2 < \beta_1 - l_1$. *In addition, assume that the straight lines* $\operatorname{Im}\lambda = -\beta_i + l_i - \frac{1}{2}n$, $i=1,2$, *do not contain any point of the spectrum of the problem* (3.1)–(3.2) *for all* $t \in [0,\infty]$ *and the eigenvalues of the problem* (3.1)–(3.2) $\lambda_1(t)\dots,\lambda_N(t)$ *in the strip*

$$-\beta_1 + l_1 - \frac{1}{2}n < \operatorname{Im}\lambda < -\beta_2 + l_2 - \frac{1}{2}n$$

are semisimple and have invariant multiplicity for all $t \in [0,\infty]$. *Then the following representation holds:*

$$u(x,t) = \sum_{\mu=1}^{N} r^{-\mathrm{i}\lambda_\mu(t)} \sum_{k=1}^{\Lambda_\mu} c_{\mu k}(t)\varphi_{\mu k}(\omega,t) + w(x,t),$$

where $w \in V^{l_2,0}_{\beta_2,h}(\mathrm{e}^{-\gamma t}, K_\infty)$, $\quad c_{\mu k} \in L^h_2(\mathrm{e}^{-\gamma t}, (0,\infty))$ *and* $\{\varphi_{\mu k},\ k=1,\dots,\Lambda_\mu\}$ *is a canonical system of eigenvectors of the problem* (3.1)–(3.2) *corresponding to the eigenvalue* $\lambda_\mu(t)$, *in which* Λ_μ *is a multiple of* $\lambda_\mu(t)$, $\mu = 1,\dots,N$.

When the right-hand sides f, g_j of the problem (4.1)–(4.2) have special forms, by using an analogous method to that used in [17] we can prove the following lemma.

Lemma 4.2. *We consider the problem* (4.1)–(4.2), *where*

$$f = r^{-\mathrm{i}\lambda_0(t)-2m} \sum_{k=0}^{M} \ln^k r f_k(\omega,t),$$

$$g_j = r^{-\mathrm{i}\lambda_0(t)-2m+j+1} \sum_{k=0}^{M} \ln^k r_{j,k}(\omega,t), \quad j=0,\dots,m-1,$$

in which $f_k \in C^{\infty,h}(\mathrm{e}^{-\gamma t}, G_\infty)$ *and* $g_{j,k} \in C^{\infty,h}(\mathrm{e}^{-\gamma t}, \partial G_\infty)$ *for all* $k=0,\dots,M$. *Moreover, assume that if* $\lambda_0(t)$ *is an eigenvalue of the problem* (3.1)–(3.2) *for some*

$t \in [0, \infty]$ *then* $\lambda_0(t)$ *is a semisimple eigenvalue having invariant multiplicity of the problem* (3.1)–(3.2) *for all* $t \in [0, \infty]$. *Then there exists a solution of the problem* (4.1)–(4.2) *in the form*

$$u(x,t) = r^{-\mathrm{i}\lambda_0(t)} \sum_{k=0}^{M+\mu} \ln^k r \hat{u}_k(\omega, t),$$

where $\hat{u}_k \in C^{\infty,h}(\mathrm{e}^{-\gamma t}, G_\infty)$, $k = 0, \ldots, M + \mu$; $\mu = 1$ *or* $\mu = 0$ *according to whether* $\lambda_0(t)$ *is either an eigenvalue of the problem* (3.1)–(3.2) *or not.*

Now we consider the second boundary value problem for the elliptic system depending on a parameter t in the form

$$(-1)^m L(x,t,D)u = f \quad \text{in } K_\infty, \tag{4.3}$$

$$B_j(x,t,D)u = g_j \quad \text{on } \partial K_\infty,\ j = 0, \ldots, m-1. \tag{4.4}$$

The following proposition describes the asymptotic behavior of solutions of the problem (4.3)–(4.4) in a neighborhood of a conical point.

Proposition 4.1. *Let the vector function* $u \in V^{l_1,0}_{\beta_1,h}(\mathrm{e}^{-\gamma t}, K_\infty)$ *be a solution of the problem* (4.3)–(4.4) *and* $f \in V^{l_2-2m,0}_{\beta_2,h}(\mathrm{e}^{-\gamma t}, K_\infty)$, $g_j \in V^{l_2-2m+j+1/2,0}_{\beta_2,h}(\mathrm{e}^{-\gamma t}, \partial K_\infty)$, $j = 0, \ldots, m-1$, *where* l_1, l_2, h *are nonnegative integers,* $l_2 \geqslant l_1 \geqslant 2m$, β_1, β_2 *are real numbers that satisfy* $l_1 - \beta_1 < l_2 - \beta_2$. *Moreover, we assume that the straight lines* $\operatorname{Im}\lambda = -\beta_j + l_j - \frac{1}{2}n$, $j = 1, 2$ *do not contain any eigenvalue of the problem* (3.1)–(3.2) *for all* $t \in [0, \infty]$, *and in the strip*

$$-\beta_1 + l_1 - \frac{1}{2}n < \operatorname{Im}\lambda < -\beta_2 + l_2 - \frac{1}{2}n$$

there exists only one semisimple eigenvalue $\lambda_0(t)$ *having invariant multiplicity for all* $t \in [0, \infty]$ *of the problem* (3.1)–(3.2). *Then the solution* u *has the form*

$$u = \sum_{k=0}^{\kappa-1} r^{-\mathrm{i}\lambda_0(t)+k} P_k(\ln r) + w, \tag{4.5}$$

where $w \in V^{l_2,0}_{\beta_2,h}(\mathrm{e}^{-\gamma t}, K_\infty)$, $P_k(\cdot)$ *are vectors of polynomials of order less than* κ, *whose coefficients are functions in the space* $C^{\infty,h}(\mathrm{e}^{-\gamma t}, G_\infty)$, κ *is the minimum integer greater than* $-\beta_2 + l_2 - \frac{1}{2}n - \operatorname{Im}\lambda_0(t)$ *for all* $t \in [0, \infty]$.

P r o o f. Divide the interval $[\beta_2, \beta_1 + l_2 - l_1]$ into M subintervals by $\delta_0, \delta_1, \ldots, \delta_M$ such that $\delta_0 = \beta_1 + l_2 - l_1$, $\delta_M = \beta_2$, $0 < \delta_{d-1} - \delta_d \leqslant 1$, $d = 1, \ldots, M$. Denote Λ to be the multiple of $\lambda_0(t)$.

From Lemma 6.3.1 in [13] we have

$$u \in V^{l_2,0}_{\delta_0,h}(\mathrm{e}^{-\gamma t}, K_\infty). \tag{4.6}$$

Rewrite the problem (4.3)–(4.4) in the form

$$(-1)^m \mathcal{L}(t, D)u = \hat{f}(x, t), \tag{4.7}$$

$$\mathcal{B}_j(t, D)u = \hat{g}_j(x, t), \quad j = 0, \ldots, m-1, \tag{4.8}$$

where

$$\hat{f}(x, t) = f(x, t) + (-1)^m (\mathcal{L}(t, D) - L(x, t, D))u,$$
$$\hat{g}_j(x, t) = g_j(x, t) + (\mathcal{B}_j(t, D) - B_j(x, t, D))u, \quad j = 0, \ldots, m-1.$$

We have

$$\mathcal{L}(t, D) - L(x, t, D) = \sum_{|p|=|q|=m} D^p(a_{pq}(0, t) - a_{pq}(x, t))D^q u \sum_{\substack{|p|+|q|<2m \\ |p|,|q|\leqslant m}} D^p a_{pq}(x, t) D^q u.$$

Since $|a_{pq}(0, t) - a_{pq}(x, t)| \leqslant Cr$ for $|p| = |q| = m$ and $u \in V^{l_2,0}_{\delta_0,h}(\mathrm{e}^{-\gamma t}, K_\infty)$, we have

$$\sum_{|p|=|q|=m} D^p(a_{pq}(0, t) - a_{pq}(x, t))D^q u \in V^{l_2-2m,0}_{\delta_0-1,h}(\mathrm{e}^{-\gamma t}, K_\infty).$$

In another way, we have

$$\sum_{\substack{|p|+|q|<2m \\ |p|,|q|\leqslant m}} D^p a_{pq}(x, t) D^q u \in V^{l_2-2m+1,0}_{\delta_0,h}(\mathrm{e}^{-\gamma t}, K_\infty) \subset V^{l_2-2m,0}_{\delta_0-1,h}(\mathrm{e}^{-\gamma t}, K_\infty).$$

From $\delta_0 - 1 \leqslant \delta_1$ it can be seen that $V^{l_2-2m,0}_{\delta_0-1,h}(\mathrm{e}^{-\gamma t}, K_\infty) \subset V^{l_2-2m,0}_{\delta_1,h}(\mathrm{e}^{-\gamma t}, K_\infty)$. This implies

$$\hat{f} \in V^{l_2-2m,0}_{\delta_1,h}(\mathrm{e}^{-\gamma t}, K_\infty). \tag{4.9}$$

Using similar arguments, we obtain

$$\hat{g}_j \in V^{l_2-2m+j+1/2,0}_{\delta_1,h}(\mathrm{e}^{-\gamma t}, \partial K_\infty), \quad j = 0, \ldots, m-1. \tag{4.10}$$

We will prove that if $-\delta_0+l_2-\frac{1}{2}n<\operatorname{Im}\lambda_0(t)<-\delta_d+l_2-\frac{1}{2}n$, $1\leqslant d\leqslant M$ for all $t\in[0,\infty]$ then

$$u(x,t)=\sum_{k=0}^{\kappa_d-1} r^{-\mathrm{i}\lambda_0(t)+k}P_{k,d}(\ln r)+u_d(x,t), \tag{4.11}$$

where $u_d\in V^{l_2,0}_{\delta_d,h}(\mathrm{e}^{-\gamma t},K_\infty)$, κ_d is the minimum integer greater than $-\delta_d+l_2-\frac{1}{2}n-\operatorname{Im}\lambda_0(t)$ for all $t\in[0,\infty]$ and $P_{k,d}(\cdot)$ are vectors of polynomials of order less than κ_d whose coefficients are functions in $C^{\infty,h}(\mathrm{e}^{-\gamma t},G_\infty)$.

Let $d=1$. The straight lines $\operatorname{Im}\lambda=-\delta_0+l_2-\frac{1}{2}n$, $\operatorname{Im}\lambda=-\delta_1+l_2-\frac{1}{2}n$ do not contain any point of the spectrum of the problem (3.1)–(3.2) and $-\delta_0+l_2-\frac{1}{2}n<\operatorname{Im}\lambda_0(t)<-\delta_1+l_2-\frac{1}{2}n$ for all $t\in[0,\infty]$. It follows from (4.6), (4.9), (4.10), and Lemma 4.1 that

$$u(x,t)=r^{-\mathrm{i}\lambda_0(t)}\sum_{k=1}^{\Lambda}c_k(t)\varphi_k(\omega,t)+u_1(x,t),$$

where $u_1\in V^{l_2,0}_{\delta_1,h}(\mathrm{e}^{-\gamma t},K_\infty)$, $c_k(\cdot)\in L^h_2(\mathrm{e}^{-\gamma t},(0,\infty))$, $\varphi_k(\omega,t)$ are infinitely differentiable vector functions of (ω,t) for all $k=1,\dots,\Lambda$. So (4.11) holds for $d=1$.

Assume that (4.11) holds for $d\leqslant M-1$. We have to prove that it is true for $d+1$. We distinguish the following cases.

Case 1: $-\delta_d+l_2-\frac{1}{2}n<\operatorname{Im}\lambda_0(t)<-\delta_{d+1}+l_2-\frac{1}{2}n$ for all $t\in[0,\infty]$. It follows that the strip

$$-\delta_0+l_2-\frac{1}{2}n\leqslant\operatorname{Im}\lambda\leqslant-\delta_1+l_2-\frac{1}{2}n$$

does not contain any eigenvalue of the problem (3.1)–(3.2) for all $t\in[0,\infty]$, hence from (4.6), (4.9), (4.10), and Lemma 4.1 we have $u\in V^{l_2,0}_{\delta_1,h}(\mathrm{e}^{-\gamma t},K_\infty)$. Using similar arguments in the proof of (4.9), (4.10) we have $\hat f\in V^{l_2-2m,0}_{\delta_2,h}(\mathrm{e}^{-\gamma t},K_\infty)$, $\hat g_j\in V^{l_2-2m+j+1/2,0}_{\delta_2,h}(\mathrm{e}^{-\gamma t},\partial K_\infty)$ for all $j=0,\dots,m-1$. By virtue of Lemma 4.1 we get $u\in V^{l_2,0}_{\delta_2,h}(\mathrm{e}^{-\gamma t},K_\infty)$. Repeating it d times we obtain $u\in V^{l_2,0}_{\delta_d,h}(\mathrm{e}^{-\gamma t},K_\infty)$ and then $\hat f\in V^{l_2-2m,0}_{\delta_{d+1},h}(\mathrm{e}^{-\gamma t},K_\infty)$, $\hat g_j\in V^{l_2-2m+j+1/2,0}_{\delta_{d+1},h}(\mathrm{e}^{-\gamma t},\partial K_\infty)$, $j=0,\dots,m-1$. Applying Lemma 4.1 again we get

$$u(x,t)=r^{-\mathrm{i}\lambda_0(t)}\sum_{k=1}^{\Lambda}c_k(t)\varphi_k(\omega,t)+u_1(x,t),$$

where $u_1\in V^{l_2,0}_{\delta_{d+1},h}(\mathrm{e}^{-\gamma t},K_\infty)$, $c_k(\cdot)\in L^h_2(\mathrm{e}^{-\gamma t},(0,\infty))$ and $\varphi_k(\omega,t)$ are infinitely differentiable vector functions of (ω,t). That shows (4.11) holds for $d+1$.

Case 2: $-\delta_0 + l_2 - \frac{1}{2}n < \operatorname{Im}\lambda_0(t) < -\delta_d + l_2 - \frac{1}{2}n$ for all $t \in [0,\infty]$. From the induction assumption we infer (4.11) with $u_d \in V^{l_2,0}_{\delta_d,h}(\mathrm{e}^{-\gamma t}, K_\infty)$. Putting

$$S_d(x,t) := \sum_{k=0}^{\kappa_d-1} r^{-\mathrm{i}\lambda_0(t)+k} P_{k,d}(\ln r),$$

we conclude

$$LS_d = A_d + R_d, \quad B_j S_d = E_{j,d} + F_{j,d}, \quad j = 0,\dots,m-1, \tag{4.12}$$

where

$$\begin{aligned}
A_d &= \sum_{k=0}^{\kappa_d-1} \sum_{k+k' \leqslant \kappa_{d+1}-1} r^{-\mathrm{i}\lambda_0(t)-2m+k+k'} P_{k,k',d}(\ln r)\\
&= \sum_{k=0}^{\kappa_{d+1}-1} r^{-\mathrm{i}\lambda_0(t)-2m+k} \hat{P}_{k,d}(\ln r),\\
R_d &= \sum_{k=0}^{\kappa_d-1} \sum_{k+k' \geqslant \kappa_{d+1}} r^{-\mathrm{i}\lambda_0(t)-2m+k+k'} P_{k,k',d}(\ln r),\\
E_{j,d} &= \sum_{k=0}^{\kappa_d-1} \sum_{k+k' \leqslant \kappa_{d+1}-1} r^{-\mathrm{i}\lambda_0(t)-2m+j+1+k+k'} Q_{j,k,k',d}(\ln r)\\
&= \sum_{k=0}^{\kappa_{d+1}-1} r^{-\mathrm{i}\lambda_0(t)-2m+j+1+k} \hat{Q}_{j,k,d}(\ln r),\\
F_{j,d} &= \sum_{k=0}^{\kappa_d-1} \sum_{k+k' \geqslant \kappa_{d+1}} r^{-\mathrm{i}\lambda_0(t)-2m+j+1+k+k'} Q_{j,k,k',d}(\ln r).
\end{aligned}$$

It is easy to see that $R_d \in V^{l_2-2m,0}_{\delta_{d+1},h}(\mathrm{e}^{-\gamma t}, K_\infty)$, $F_{j,d} \in V^{l_2-2m+j+1/2,0}_{\delta_{d+1},h}(\mathrm{e}^{-\gamma t}, \partial K_\infty)$ for all $j = 0,\dots,m-1$. Moreover, there exists

$$v_d = \sum_{k=0}^{\kappa_{d+1}-1} r^{-\mathrm{i}\lambda_0(t)+k} \hat{P}_{k,d+1}(\ln r) \tag{4.13}$$

such that

$$\mathcal{L}(t,D)v_d = -A_d, \tag{4.14}$$

$$\mathcal{B}_j(t,D)v_d = -E_{j,d}, \quad j = 0,\dots,m-1, \tag{4.15}$$

by virtue of Lemma 4.2, where $\hat{P}_{k,d+1}(\cdot)$ are vectors of polynomials of order less than κ_{d+1}, whose coefficients are functions in $C^{\infty,h}(\mathrm{e}^{-\gamma t}, G_\infty)$.

It follows from (4.11), (4.12), (4.14), and (4.15) that

$$(-1)^m \mathcal{L}(t,D)[u_d - v_d] = f + (-1)^m \mathcal{L}_1 u_d - (-1)^m R_d \in V^{l_2-2m,0}_{\delta_{d+1},h}(\mathrm{e}^{-\gamma t}, K_\infty),$$
$$\mathcal{B}_j(t,D)[u_d - v_d] = g_j + \mathcal{B}^1_j u_d - F_{j,d} \in V^{l_2-2m+j+1/2,0}_{\delta_{d+1},h}(\mathrm{e}^{-\gamma t}, \partial K_\infty)$$

for all $j = 0,\dots,m-1$, where $\mathcal{L}_1 := \mathcal{L}(t,D) - L(x,t,D)$ and $\mathcal{B}^1_j := \mathcal{B}_j(t,D) - B_j(x,t,D)$. By applying Lemma 4.1 and noting that $u_d - v_d \in V^{l_2,0}_{\delta_0,h}(\mathrm{e}^{-\gamma t}, K_\infty)$, we get

$$u_d - v_d = r^{-\mathrm{i}\lambda_0(t)} \sum_{j=1}^{\Lambda} c_j(t)\varphi_j(\omega,t) + u_{d+1}, \tag{4.16}$$

where $u_{d+1} \in V^{l_2,0}_{\delta_{d+1},h}(\mathrm{e}^{-\gamma t}, K_\infty)$, $c_k(\cdot) \in L^h_2(\mathrm{e}^{-\gamma t}, (0,\infty))$. From (4.11), (4.13), and (4.16) we have

$$u(x,t) = \sum_{k=0}^{\kappa_{d+1}-1} r^{-\mathrm{i}\lambda_0(t)+k} P_{k,d+1}(\ln r) + u_{d+1}, \tag{4.17}$$

where $P_{0,d+1} = P_{0,d} + \hat{P}_{0,d+1} + \sum\limits_{j=1}^{\Lambda} c_j(t)\varphi_j(\omega,t)$, $P_{k,d+1} = P_{k,d} + \hat{P}_{k,d+1}$ for all $k = 1,\dots,\kappa_d - 1$ and $P_{k,d+1} = \hat{P}_{k,d+1}$ for $k = \kappa_{d+1} - 1$. Clearly $P_{k,d+1}$ are vectors of polynomials of order less than κ_{d+1}, whose coefficients are functions in $C^{\infty,h}(\mathrm{e}^{-\gamma_h t}, G_\infty)$.

Case 3: There exists t_0 such that $\operatorname{Im}\lambda_0(t_0) = -\delta_d + l_2 - \frac{1}{2}n$. We may assume without loss of generality that $-\delta_d - \varepsilon + l_2 - \frac{1}{2}n < \operatorname{Im}\lambda_0(t) < -\delta_{d+1} - \varepsilon + l_2 - \frac{1}{2}n$ for $\varepsilon \in (0, \delta_d - \delta_{d+1})$. Since in the strip $-\delta_0 + l_2 - \frac{1}{2}n \leqslant \operatorname{Im}\lambda \leqslant -\delta_d - \varepsilon + l_2 - \frac{1}{2}n$, there is no point of the spectrum of the problem (3.1)–(3.2), by repeating arguments of the proof of Case 1, we obtain

$$u(x,t) = r^{-\mathrm{i}\lambda_0(t)} \sum_{k=1}^{\Lambda} c_k(t)\varphi_k(\omega,t) + \hat{u}_d(x,t),$$

where $\hat{u}_d \in V^{l_2,0}_{\delta_{d+1}+\varepsilon,h}(\mathrm{e}^{-\gamma t}, K_\infty)$. After that by using method similar to that used in the proof of Case 2 we obtain (4.17).

When $d = M$, we obtain (4.5) from (4.11). The proposition is proven completely. □

Now we go back to consider the second initial boundary value problem for the Schrödinger system (2.4)–(2.6). Denote by U_0 a neighborhood of 0 in which Ω coincides with the cone K. We have the following lemma.

Lemma 4.3. *Let the vector function $u(x,t)$ be a generalized solution in $H^{m,0}(\mathrm{e}^{-\gamma t},\Omega_\infty)$ of the problem (2.4)–(2.6) such that $u\equiv 0$ outside U_0, $f_{t^k}\in L^\infty(0,\infty,L_2(\Omega))$, $k\leqslant h+2$; $f_{t^k}(x,0)=0$, $k\leqslant h$. Moreover, assume that the straight lines $\operatorname{Im}\lambda=-\beta+2m-\frac{1}{2}n$, $\operatorname{Im}\lambda=-\beta'+2m-\frac{1}{2}n$ do not contain any eigenvalue of the problem (3.1)–(3.2) for all $t\in[0,\infty]$, and in the strip*

$$-\beta+2m-\frac{1}{2}n<\operatorname{Im}\lambda<-\beta'+2m-\frac{1}{2}n$$

there is only one semisimple eigenvalue $\lambda_0(t)$ having invariant multiplicity for all $t\in[0,\infty]$ of the problem (3.1)–(3.2), where β, β' are real numbers satisfying $\beta\geqslant\max\{m,2m-\frac{1}{2}n\}$, $0\leqslant\beta'<\beta$. Then the solution u has the representation

$$u(x,t)=\sum_{k=0}^{\kappa-1}r^{-\mathrm{i}\lambda_0(t)+k}P_k(\ln r)+w(x,t), \tag{4.18}$$

where $w\in V^{2m,0}_{\beta',h}(\mathrm{e}^{-\gamma_h t},\Omega_\infty)$, $P_k(\cdot)$ are vectors of polynomials of order less than κ, whose coefficients are functions in $C^{\infty,h}(\mathrm{e}^{-\gamma_h t},G_\infty)$; κ is the minimum integer greater than $-\beta'+2m-\frac{1}{2}n-\operatorname{Im}\lambda_0(t)$ for all $t\in[0,\infty]$.

Proof. Rewrite (2.4), (2.5) in the form

$$(-1)^m L(x,t,D)u=F,\quad \text{where } F=\mathrm{i}(u_t+f), \tag{4.19}$$

$$B_j(x,t,D)u=0. \tag{4.20}$$

We consider the following cases.

i) Case $m<\frac{1}{2}n$. From Theorem 4.1 in [5] we have $u_{t^j}\in H^{m,0}(\mathrm{e}^{-\gamma_j t},K_\infty)$ for all $j\leqslant h+1$. Then $u_{t^j}\in H^{2m,0}_m(\mathrm{e}^{-\gamma_j t},K_\infty)$ for all $j\leqslant h+1$ by virtue of Lemma 3.2 in [6]. It is easy to see that $u\in V^{l,0}_{\beta,h}(\mathrm{e}^{-\gamma t},K_\infty)$ if and only if $u_{t^j}\in H^{l,0}_\beta(\mathrm{e}^{-\gamma t},K_\infty)$ for all $j\leqslant h$. Therefore, $u\in V^{2m,0}_{m,h}(\mathrm{e}^{-\gamma_h t},K_\infty)\subset V^{2m,0}_{\beta,h}(\mathrm{e}^{-\gamma_h t},K_\infty)$. In another way we have $F=\mathrm{i}(u_t+f)\in V^{0,0}_{0,h}(\mathrm{e}^{-\gamma_h t},K_\infty)\subset V^{0,0}_{\beta',h}(\mathrm{e}^{-\gamma_h t},K_\infty)$. So by applying Proposition 4.1 to the problem (4.19)–(4.20) we obtain the representation (4.18).

ii) Case $m=\frac{1}{2}n$. Since on the straight line $\operatorname{Im}\lambda=-\beta+2m-\frac{1}{2}n$ there is no eigenvalue of the problem (3.1)–(3.2) and eigenvalues of this problem are either continuous in $[0,\infty]$ or have modulus tending to some point of $[0,\infty]$, there exists $\varepsilon>0$ such that in the strip $-\beta-\varepsilon+2m-\frac{1}{2}n\leqslant\operatorname{Im}\lambda\leqslant-\beta+2m-\frac{1}{2}n$ there is no eigenvalue of the problem (3.1)–(3.2) for all $t\in[0,\infty]$.

In another way, using similar arguments as those used in the proof of Lemma 3.2 in [6] we have $u_{t^k} \in H^{2m}_{m+\varepsilon}(K)$ for all $k \leqslant h+1$. Therefore $F \in V^{0,0}_{m,h}(\mathrm{e}^{-\gamma_h t}, K_\infty)$. Applying Proposition 4.1 we get $u \in V^{2m,0}_{m,h}(\mathrm{e}^{-\gamma_h t}, K_\infty)$. Then using analogous arguments as in the case $m < \frac{1}{2}n$ we can obtain the representation (4.18).

iii) Case $m > \frac{1}{2}n$. From Lemma 3.3 in [6] we find that

$$u(x,t) = \sum_{|\alpha| \leqslant n^*} c_\alpha(t) x^\alpha + u_0(x,t), \tag{4.21}$$

where $c_\alpha(\cdot) \in L^h_2(\mathrm{e}^{-\gamma_h t}, (0,\infty))$ for all $|\alpha| \leqslant n^*$ and $u_0 \in V^{2m,0}_{m,h}(\mathrm{e}^{-\gamma_h t}, K_\infty)$, with $n^* = \left[m - \frac{1}{2}n\right]$ if n is odd and $n^* = \left[m - \frac{1}{2}n - 1\right]$ if n is even.

Set $\hat{v}(x,t) := \sum\limits_{|\alpha| \leqslant n^*} c_\alpha(t) x^\alpha$. We have

$$\hat{v}(\cdot,t) \in W^{2m}_m(K), \quad u_0(\cdot,t) \in H^{2m}_m(K) \subset W^{2m}_m(K)$$

for almost all $t \in (0,\infty)$. Therefore, $u(\cdot,t) \in W^{2m}_m(K)$ for almost all $t \in (0,\infty)$. Since $\beta \geqslant \max\{m, 2m - \frac{1}{2}n\}$, then following Lemma 7.1.5 in [13] we get

$$u(\cdot,t) \in W^{2m}_m(K) \subset W^{2m}_\beta(K) \subset W^{2m}_{\beta+\varepsilon}(K) \equiv H^{2m}_{\beta+\varepsilon}(K)$$

for all $\varepsilon > 0$ and for almost all $t \in (0,\infty)$. Using analogous arguments for u_{t^k}, $k \leqslant h$ we obtain $u \in V^{2m,0}_{\beta+\varepsilon,h}(\mathrm{e}^{-\gamma_h t}, K_\infty)$.

On the other hand we have $F \in V^{0,0}_{\beta',h}(\mathrm{e}^{-\gamma_h t}, K_\infty)$. By using arguments similar to those in the case $m = \frac{1}{2}n$, we also get (4.18). The lemma is proven. □

Lemma 4.4. *Let l be a nonnegative integer; let β, β' be real numbers satisfying $\beta \geqslant \max\{m, 2m - \frac{1}{2}n\}$ and $0 \leqslant \beta' < \beta$. Assume that the vector function $u(x,t)$ is a generalized solution in $H^{m,0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$ of the problem (2.4)–(2.6) such that $u \equiv 0$ outside U_0, $f_{t^k} \in L^\infty(0,\infty, H^l_{\beta'}(\Omega))$, $k \leqslant h+2l+2$; $f_{t^k}(x,0) = 0$, $k \leqslant h+2l$. In addition, assume that the straight lines $\operatorname{Im}\lambda = -\beta + 2m - \frac{1}{2}n$, $\operatorname{Im}\lambda = -\beta' + 2m + l - \frac{1}{2}n$ do not contain any eigenvalue of the problem (3.1)–(3.2) for all $t \in [0,\infty]$ and in the strip*

$$-\beta + 2m - \frac{1}{2}n < \operatorname{Im}\lambda < -\beta' + 2m + l - \frac{1}{2}n$$

there is only one semisimple eigenvalue $\lambda_0(t)$ having invariant multiplicity for all $t \in [0,\infty]$ of the problem (3.1)–(3.2). Then we have

$$u(x,t) = \sum_{k=0}^{l+\kappa-1} r^{-\mathrm{i}\lambda_0(t)+k} P_k(\ln r) + w(x,t), \tag{4.22}$$

where $w \in V^{2m+l,0}_{\beta',h+l}(\mathrm{e}^{-\gamma_{h+l} t}, \Omega_\infty)$ and $P_k(\cdot)$ are vectors of polynomials of order less than $3l+\kappa$ whose coefficients are functions in $C^{\infty,h+l}(\mathrm{e}^{-\gamma_{h+l} t}, G_\infty)$; κ is the minimum integer greater than $-\beta' + 2m - \frac{1}{2}n - \operatorname{Im}\lambda_0(t)$ for all $t \in [0,\infty]$.

P r o o f. We use the induction by l. For $l = 0$ the assertion follows from Lemma 4.3. Assume that it is true for $l-1$ ($l \geqslant 1$). We have to prove that this lemma holds for l. Consider the following cases.

Case 1: $-\beta + 2m - \frac{1}{2}n < \operatorname{Im}\lambda_0(t) < -\beta' + 2m + l - 1 - \frac{1}{2}n$ for all $t \in [0,\infty]$. From induction hypothesis we have

$$u(x,t) = \sum_{k=0}^{l+\kappa-2} r^{-\mathrm{i}\lambda_0(t)+k} P_k(\ln r) + u_1(x,t), \tag{4.23}$$

where $P_k(\cdot)$ are vectors of polynomials of order less than $3l+\kappa-3$, whose coefficients are functions in $C^{\infty,h+l-1}(\mathrm{e}^{-\gamma_{h+l-1}t}, G_\infty)$; $u_1 \in V^{2m+l-1,0}_{\beta',h+l-1}(\mathrm{e}^{-\gamma_{h+l-1}t}, K_\infty)$.

Rewrite (2.4)–(2.5) in the form

$$\begin{aligned}(-1)^m \mathcal{L}(t,D)u_1 &= F_1 + (-1)^{m-1} L(x,t,D)S - \mathrm{i}S_t,\\ \mathcal{B}_j(t,D)u_1 &= g_{1,j} - B_j(x,t,D)S, \quad j = 0,\dots,m-1,\end{aligned}$$

where

$$\begin{aligned}F_1 &= \mathrm{i}((u_1)_t + f) + (-1)^m \mathcal{L}_1 u_1,\\ g_{1,j} &= \mathcal{B}^1_j u_1,\\ S &= \sum_{k=0}^{l+\kappa-2} r^{-\mathrm{i}\lambda_0(t)+k} P_k(\ln r).\end{aligned}$$

Since for almost all $t \in (0,\infty)$, $f_{t^k} \in H^l_{\beta'}(K)$, $k \leqslant 2l+h+2$ and $f_{t^k}(x,0) = 0$, $k \leqslant 2l+h$, so $f_{t^k} \in H^{l-1}_{\beta'}(K)$, $k \leqslant 2(l-1)+(h+2)+2$ and $f_{t^k}(x,0) = 0$, $k \leqslant 2(l-1)+(h+2)$. This implies $(u_1)_{t^k} \in H^{2m+l-1,0}_{\beta'}(\mathrm{e}^{-\gamma_k t}, K_\infty)$ for $k \leqslant h+l+1$. Therefore $F_1 \in V^{l,0}_{\beta',h+l}(\mathrm{e}^{-\gamma_{h+l}t}, K_\infty)$, $g_{1,j} \in V^{l+j+1/2,0}_{\beta',h+l}(\mathrm{e}^{-\gamma_{h+l}t}, \partial K_\infty)$ for all $j = 0,\dots,m-1$.

In another way, by using arguments similar to those in the case 2 of Proposition 4.1 we get

$$(-1)^{m-1} L(x,t,D)S - \mathrm{i}S_t = F_2 + \sum_{k=0}^{l+\kappa-1} r^{-\mathrm{i}\lambda_0(t)-2m+k}\hat{P}_k(\ln r),$$

$$-B_j(x,t,D)S = g_{2,j} + \sum_{k=0}^{l+\kappa-1} r^{-\mathrm{i}\lambda_0(t)-2m+j+k+1}\hat{Q}_k(\ln r), \quad j = 0,\dots,m-1,$$

where $\hat{P}_k$ are vectors of polynomials of order less than $3l+\kappa-1$ whose coefficients are functions in $C^{\infty,h+l}(\mathrm{e}^{-\gamma_{h+l}t}, G_\infty)$, $\hat{Q}_k$ are vectors of polynomials of order less than $3l+\kappa-1$ whose coefficients are functions in $C^{\infty,h+l}(\mathrm{e}^{-\gamma_{h+l}t}, \partial G_\infty)$ and $F_2 \in V^{l,0}_{\beta',l+h}(\mathrm{e}^{-\gamma_{l+h}t}, K_\infty)$, $g_{2,j} \in V^{l+j+1/2,0}_{\beta',l+h}(\mathrm{e}^{-\gamma_{l+h}t}, \partial K_\infty)$ for all $j = 0,\dots,m-1$.

According to Lemma 4.2, there exists

$$v=\sum_{k=0}^{l+\kappa-1} r^{-\mathrm{i}\lambda_0(t)+k}P_k(\ln r) \tag{4.24}$$

such that

$$(-1)^m\mathcal{L}(t,D)v=\sum_{k=0}^{l+\kappa-1} r^{-\mathrm{i}\lambda_0(t)-2m+k}\hat{P}_k(\ln r),$$

$$\mathcal{B}_j(t,D)v=\sum_{k=0}^{l+\kappa-1} r^{-\mathrm{i}\lambda_0(t)-2m+j+k+1}\hat{Q}_k(\ln r),\quad j=0,\dots,m-1,$$

where P_k are vectors of polynomials of order less than $3l+\kappa$ whose coefficients are functions in $C^{\infty,h+l}(\mathrm{e}^{-\gamma_{h+l}t},G_\infty)$. So we get

$$(-1)^m\mathcal{L}(t,D)(u_1-v)=F_3, \tag{4.25}$$

$$\mathcal{B}_j(t,D)(u_1-v)=g_{3,j}, \tag{4.26}$$

where

$$F_3=F_1+F_2\in V^{l,0}_{\beta',h+l}(\mathrm{e}^{-\gamma_{l+h}t},K_\infty),$$

$$g_{3,j}=g_{1,j}+g_{2,j}\in V^{l+j+1/2,0}_{\beta',h+l}(\mathrm{e}^{-\gamma_{l+h}t},\partial K_\infty),\quad j=0,\dots,m-1.$$

Applying Lemma 4.1 to the problem (4.25)–(4.26) and noting that $u_1-v\in V^{2m+l-1,0}_{\beta'+m+l-1,h+l}(\mathrm{e}^{-\gamma_{h+l}}t,K_\infty)$ we infer that

$$u_1(x,t)-v(x,t)=\sum_{j=0}^{\Lambda} r^{-\mathrm{i}\lambda_0(t)}c_j(t)\varphi_j(\omega,t)+u_2(x,t), \tag{4.27}$$

where $c_j(\cdot)\in L_2^{h+l}(\mathrm{e}^{-\gamma_{h+l}t},(0,\infty))$, $u_2\in V^{2m+l,0}_{\beta',h+l}(\mathrm{e}^{-\gamma_{h+l}t},K_\infty)$. It follows from (4.23), (4.24), and (4.27)

$$u(x,t)=\sum_{k=0}^{l+\kappa-1} r^{-\mathrm{i}\lambda_0(t)+k}P_k(\ln r)+u_2(x,t),$$

where $u_2\in V^{2m+l,0}_{\beta',h+l}(\mathrm{e}^{-\gamma_{h+l}t},K_\infty)$ and P_k are vectors of polynomials of order less than $3l+\kappa$ whose coefficients are functions in $C^{\infty,h+l}(\mathrm{e}^{-\gamma_{h+l}t},G_\infty)$.

Case 2: $-\beta'+2m+l-1-\frac{1}{2}n<\operatorname{Im}\lambda_0(t)<-\beta'+2m+l-\frac{1}{2}n$. By using the induction on j we prove that if for all $t\in(0,\infty)$, $f_{t^k}\in H^j_{\beta'}(K)$, $k\leqslant 2j+h+2$ and $f_{t^k}(x,0)=0$, $k\leqslant 2j+h$, then $u\in V^{2m+j,0}_{\beta',h+2l-j}(\mathrm{e}^{-\gamma_{h+2l-j}t},K_\infty)$ for all $j\leqslant l-1$.

Indeed, from Lemma 4.3 for $\varepsilon > 0$ arbitrary we have $u \in V^{2m,0}_{\beta+\varepsilon,h+2l}(\mathrm{e}^{-\gamma_{h+2l}t}, K_\infty)$ and $F \in V^{0,0}_{\beta',h+2l}(\mathrm{e}^{-\gamma_{h+2l}t}, K_\infty)$. By applying Proposition 4.1 one obtains $u \in V^{2m,0}_{\beta',h+2l}(\mathrm{e}^{-\gamma_{h+2l}t}, K_\infty)$. That shows the assertion is true for $j = 0$.

Assume that the assertion holds up to $j-1$. Since $f_{t^k} \in H^{j-1,0}_{\beta'}(\mathrm{e}^{-\gamma_k t}, K_\infty)$, $k \leqslant 2(j-1) + (h+2) + 2$ and $f_{t^k}(x,0) = 0$, $k \leqslant 2(j-1) + h + 2$, hence $u_{t^k} \in H^{2m+j-1,0}_{\beta'}(\mathrm{e}^{-\gamma_k t}, K_\infty)$, $k \leqslant h + 2l - j + 3$. So $u_{t^{k+1}} \in H^{2m+j-1,0}_{\beta'}(\mathrm{e}^{-\gamma_k t}, K_\infty)$, $k \leqslant h + 2l - j + 2$. On the other hand $F \in V^{j-1,0}_{\beta'-1,h+2l-j}(\mathrm{e}^{-\gamma_{h+2l-j}t}, K_\infty)$. Therefore by applying Proposition 4.1 we have $u \in V^{2m+j-1,0}_{\beta'-1,h+2l-j}(\mathrm{e}^{-\gamma_{h+2l-j}t}, K_\infty)$. From Lemma 6.3.1 in [13] one gets $u \in V^{2m+j,0}_{\beta',h+2l-j}(\mathrm{e}^{-\gamma_{h+2l-j}t}, K_\infty)$. That shows the assertion holds for all $j \leqslant l-1$.

It follows that for $j = l-1$ we have $u \in V^{2m+l-1,0}_{\beta',h+l+1}(\mathrm{e}^{-\gamma_{h+l+1}t}, K_\infty)$. Since $F \in V^{l,0}_{\beta',h+l}(\mathrm{e}^{-\gamma_{h+l}t}, K_\infty)$, by applying Proposition 4.1 we obtain

$$u(x,t) = \sum_{k=1}^{\Lambda} c_k(t) r^{-\mathrm{i}\lambda_0(t)} \varphi_k(\omega, t) + u_1(x,t),$$

where $u_1 \in V^{2m+l,0}_{\beta',h+l}(\mathrm{e}^{-\gamma_{h+l}t}, K_\infty)$, φ_k are infinitely differentiable with restpect to (ω, t) and $c_k(\cdot) \in L^{h+l}_2(\mathrm{e}^{-\gamma_{h+l}t}, (0,\infty))$ for all $k = 1, \ldots, \Lambda$, Λ is the multiple of $\lambda_0(t)$.

Case 3: There exists t_0 such that $\operatorname{Im} \lambda_0(t_0) = -\beta' + 2m + l - 1 - \frac{1}{2}n$. Using analogous arguments as in the proof of case 3 of Proposition 4.1 we obtain the representation (4.22). This completes the proof. □

Now we can prove Theorem 3.1.

Proof. 1. First we prove it for the case $u \equiv 0$ outside U_0.

For any $t_0 \in [0, \infty]$ there exists $\varepsilon > 0$ such that $-\beta + 2m + \mu_{j-1} - \frac{1}{2}n < \operatorname{Im} \lambda_j(t) < -\beta + 2m + \mu_j - \frac{1}{2}n$, $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, where μ_j, $j = 1, \ldots, N_0$ are nonnegative constants. Since $[0,T]$ is compact, we can divide $[0,T]$ into subintervals by $0 = \tau_0, \tau_1, \ldots, \tau_M = T$ such that $-\beta + 2m + \mu_{j-1,k} - \frac{1}{2}n < \operatorname{Im} \lambda_j(t) < -\beta + 2m + \mu_{j,k} - \frac{1}{2}n$, $t \in [\tau_{k-1}, \tau_k]$, where $\mu_{j,k}$ and $\mu_{j-1,k}$ are nonnegative constants, $j = 1, \ldots, N_0$, $k = 1, \ldots, M$. Therefore without loss of generality we can assume that

$$\begin{gathered} -\beta + 2m - \frac{1}{2}n < \operatorname{Im} \lambda_1(t) < -\beta + 2m + \mu_1 - \frac{1}{2}n < \operatorname{Im} \lambda_2(t) < \ldots \\ < -\beta + 2m + \mu_{N_0-1} - \frac{1}{2}n < \operatorname{Im} \lambda_{N_0}(t) < -\beta' + 2m + l - \frac{1}{2}n, \quad t \in [0,T]. \end{gathered}$$

We use the induction on N_0. For $N_0 = 1$ the assertion of the theorem follows from Lemma 4.4. Assume that the theorem holds for $N_0 - 1$.

Set $\hat{\mu}_0 := \max\{\mu_{N_0-1}, \mu^*_{N_0-1}\} \geqslant 0$, $l_0 := [\hat{\mu}_0 - \beta + \beta'] \leqslant \hat{\mu}_0 - \beta + \beta'$. Without loss of generality we can assume that $0 \leqslant l_0 < l$. Then $-\beta + \hat{\mu}_0 = -\beta' + l_0 + \delta$, $\delta \in [0, 1)$. Set $\beta_1 := \beta' - \delta \leqslant \beta'$.

Since in the strip $-\beta + 2m - \frac{1}{2}n < \operatorname{Im}\lambda < -\beta_1 + 2m + l_0 - \frac{1}{2}n$ there are $N_0 - 1$ eigenvalues we have

$$u(x,t) = \sum_{j=1}^{N_0-1} \sum_{k=0}^{l_0+\kappa_j-1} r^{-\mathrm{i}\lambda_j(t)+k} P_{k,j}(\ln r) + u_1(x,t), \tag{4.28}$$

where $P_{k,j}$ are vectors of polynomials of order less than $3l_0 + \kappa_j$ with coefficients in the space $C^{\infty,h+l_0}(\mathrm{e}^{-\gamma_{h+l_0}}, K_\infty)$, $u_1 \in V^{2m+l_0,0}_{\beta_1,h+l_0}(\mathrm{e}^{-\gamma_{h+l_0}}, K_\infty)$.

Using arguments similar to case 1 in the proof of Lemma 4.4, one gets if $-\beta_1 + 2m + l_1 - \frac{1}{2}n < \operatorname{Im}\lambda_{N_0}(t) < -\beta_1 + 2m + l_1 + 1 - \frac{1}{2}n$ for all $t \in [0, \infty]$ with $l_0 \leqslant l_1 < l$, hence we obtain the representation (3.3), where $P_{k,j}$ are vectors of polynomials of order less than $3l_1 + \kappa_j + 3$ with coefficients in the space $C^{\infty,h+l_1+1}(\mathrm{e}^{-\gamma_{h+l_1+1}}, K_\infty)$, $w \in V^{2m+l_1+1,0}_{\beta_1,h+l_1+1}(\mathrm{e}^{-\gamma_{h+l_1+1}}, K_\infty)$.

Because in the strip $-\beta_1 + 2m + l_1 + 1 - \frac{1}{2}n \leqslant \operatorname{Im}\lambda \leqslant -\beta' + 2m + l - \frac{1}{2}n$ there is no eigenvalue of the problem (3.1)–(3.2) we receive (3.3) with $w \in V^{2m+l,0}_{\beta',h+l}(\mathrm{e}^{-\gamma_{h+l}t}, K_\infty)$.

If there is $t_0 \in [0, \infty]$ such that $\operatorname{Im}\lambda_{N_0}(t_0) = -\beta_1 + 2m + l_1 - \frac{1}{2}n$, then using arguments similar to case 3 of the proof of Lemma 4.4 we also obtain (3.3).

2. We now prove the general case. Denote $u^0 = \varphi_0 u$, where $\varphi_0 \in \mathring{C}^\infty(U_0)$ and $\varphi_0 \equiv 1$ in a neighborhood of 0. The vector function u^0 satisfies $(-1)^m \mathrm{i} L u^0 - (u^0)_t = \varphi_0 f + L_1 u$, where $L_1 u$ is a linear differential operator having order less than $2m$. Coefficients of this operator depend on the choice of the vector function φ_0 and equal 0 outside U_0. Denote $u^1 := \varphi_1 u = (1 - \varphi_0)u$. It is easy to see that u^1 is equal to 0 in a neighborhood of a conical point. Therefore, we can apply results of the smoothness of a solution of the elliptic problem in a smooth domain to this vector function to conclude that $(u^1)_{t^k} \in H^{2m+l}_{\beta'}(\Omega)$ for all $k \leqslant h + l$, for almost all $t \in (0, \infty)$. That shows $u^1 \in V^{2m+l,0}_{\beta',h+l}(\mathrm{e}^{-\gamma_{h+l}t}, \Omega_\infty)$.

In another way, the vector functions u^0 and $\hat{f} = \varphi_0 f + L_1 u$ satisfy the hypotheses of part 1, so u^0 has the representation (3.3). It follows that $u = u^0 + u^1$ has the representation (3.3) too. The theorem is proven completely. □

R e m a r k 4.1. From the proof of Theorem 3.1 it follows that the hypotheses of the semisimple property and the invariant multiplicity property of eigenvalues of the problem (3.1)–(3.2) are sufficient conditions to ensure that these eigenvalues and hence the eigenvectors are smooth enough with respect to t. If we can choose eigenvalues, eigenvectors and generalized eigenvectors of the problem (3.1)–(3.2) such that they are smooth with respect to t up to some order (for example $h + l$), then the results of Theorem 3.1 are also true.

5. Examples

In this example we consider the Cauchy-Neumann problem for the Schrödinger equation in quantum mechanics

$$\mathrm{i}\Delta u - u_t = f \quad \text{in } \Omega_\infty, \tag{5.1}$$

$$u|_{t=0} = 0 \quad \text{on } \Omega, \tag{5.2}$$

$$\frac{\partial u}{\partial \nu} := \sum_{k=1}^{n} \frac{\partial u}{\partial x_k} \cos(x_k, \nu) = 0 \quad \text{on } \Gamma_\infty, \tag{5.3}$$

where Δ is the Laplace operator and ν is a unit exterior normal to Γ_∞.

We can rewrite Laplace operator in local coordinates (r, ω) in the form

$$\Delta u(r,\omega) = \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\Big(r^{n-1}\frac{\partial}{\partial r}\Big) u(r,\omega) + \frac{1}{r^2}\Delta_\omega u(r,\omega),$$

where Δ_ω is the Laplace-Beltrami operator in the unit sphere S^{n-1}. So the problem (3.1)–(3.2) is the Neumann problem for the equation

$$\Delta_\omega v + [(\mathrm{i}\lambda)^2 + \mathrm{i}(2-n)\lambda]v = 0, \quad \omega \in G. \tag{5.4}$$

Denote by k_j, $j = 0, 1, \ldots$, the nonnegative eigenvalues of the Neumann problem for the equation

$$\Delta_\omega v + kv = 0, \quad \omega \in G. \tag{5.5}$$

Note that the values k_j, $j = 0, 1, \ldots$ are countably many nonegative real numbers (see [14, p. 46, p. 397]). Then $\lambda_j = \mathrm{i}\Big(1 - \frac{1}{2}n \pm \sqrt{(1-\frac{1}{2}n)^2 + k_j}\Big)$, $j = 0, 1, \ldots$, are eigenvalues of the Neumann problem for equation (5.4). We consider the following cases.

Case $n \geqslant 4$. The following corollary deals with the regularity of solutions of the problem (5.1)–(5.3).

Corollary 5.1. *Let $u(x,t)$ be a generalized solution in $H^{1,0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$ of the problem (5.1)–(5.3) and let $f, f_t, f_{tt}, f_{ttt} \in L^\infty(0, \infty, L_2(\Omega))$, $f(x,0) = f_t(x,0) = 0$ for $x \in \Omega$. Then we have*

i) *if $n > 4$ then $u \in H^2_0(\mathrm{e}^{-\gamma t}, \Omega_\infty)$,*

ii) *if $n = 4$ then $u \in H^2(\mathrm{e}^{-\gamma t}, \Omega_\infty)$.*

P r o o f. i) It is clear that if $n > 4$, the strip

$$1 - \frac{1}{2}n \leqslant \operatorname{Im}\lambda \leqslant 2 - \frac{1}{2}n$$

contains no eigenvalue of the Neumann problem for (5.4). Therefore we have $u \in H_0^2(\mathrm{e}^{-\gamma t}, \Omega_\infty)$ by virtue of Lemma 4.3.

ii) In case $n = 4$, since in the strip $-1 < \operatorname{Im}\lambda < 0$ there is no eigenvalue of the Neumann problem for (5.4) and $\lambda = 0$ is an eigenvalue of the Neumann problem for (5.4) on the straight line $\operatorname{Im}\lambda = 0$, so from formula (1.26) in [10] we have $Du \in H^{1,0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$. Hence

$$\sum_{|\alpha|=2} \int_{\Omega_\infty} |D^\alpha u|^2 \mathrm{e}^{-2\gamma t} \,\mathrm{d}x \,\mathrm{d}t < \infty. \tag{5.6}$$

In another way, from Theorem 4.1 in [5] we have $u_t, u_{tt} \in H^{1,0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$. That shows

$$\int_{\Omega_\infty} \Big[|u_t|^2 + |u_{tt}|^2 + \sum_{|\alpha|=1} |D^\alpha u_t|^2 \Big] \mathrm{e}^{-2\gamma t} \,\mathrm{d}x \,\mathrm{d}t < \infty. \tag{5.7}$$

From (5.6), (5.7), and $u \in H^{1,0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$ one gets $u \in H^2(\mathrm{e}^{-\gamma t}, \Omega_\infty)$. □

Case $n = 3$. From Theorem 2 in [12] it follows that the strip $-\frac{1}{2} \leqslant \operatorname{Im}\lambda \leqslant 0$ contains only one simple eigenvalue $\lambda_0 = 0$ of the Neumann problem for (5.4). In another way,

$$\lambda_j = \mathrm{i}\Big(-\frac{1}{2} + \sqrt{\frac{1}{4} + k_j}\Big), \quad \lambda_j^* = \mathrm{i}\Big(-\frac{1}{2} - \sqrt{\frac{1}{4} + k_j}\Big), \quad j = 0, 1, \ldots$$

are eigenvalues of the Neumann problem for (5.4), where $0 = k_0 < k_1 \leqslant k_2 \leqslant \ldots$ are eigenvalues of the Neumann problem for (5.5). It is easy to see that λ_j, $j = 0, 1, 2 \ldots$ are simple eigenvalues. Then we have the following result.

Corollary 5.2. *Let $u(x,t)$ be a generalized solution in $H^{1,0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$ of the problem* (5.1)–(5.3) *and let $f_{t^k} \in L^\infty(0, \infty, L_2(\Omega))$, $k \leqslant h + 2$, $f_{t^k}(x, 0) = 0$, $k \leqslant h$ for all $x \in \Omega$. Then*

$$u(x,t) = c(t) + u_1(x,t),$$

where $c(\cdot) \in L_2^h(\mathrm{e}^{-\gamma_h t}, (0, \infty))$ and $|u_1| \leqslant C|x|^{\min\{1/2, \operatorname{Im}\lambda_1\}}$, $C > 0$.

P r o o f. We distinguish the following cases.

Case 1: $\operatorname{Im}\lambda_1 > \frac{1}{2}$. Because on the straight lines $\operatorname{Im}\lambda = -\frac{1}{2}$ and $\operatorname{Im}\lambda = \frac{1}{2}$ there is no eigenvalue of the Neumann problem for (5.4) and the strip $-\frac{1}{2} < \operatorname{Im}\lambda < \frac{1}{2}$ contains a simple eigenvalue $\lambda_0 = 0$, from Lemma 4.3 we have

$$u(x,t) = c(t) + u_1(x,t),$$

where $c(\cdot) \in L_2^h(\mathrm{e}^{-\gamma_h t}, (0,\infty))$ and $u_1 \in V_{0,h}^{2,0}(\mathrm{e}^{-\gamma_h t}, \Omega_\infty)$.

Denote $\Omega^\varrho := \{x \in \Omega\colon \frac{1}{2}\varrho < |x| < 2\varrho,\ \varrho > 0\}$. Assume that ϱ is small enough so that Ω^ϱ coincides with the cone K. Set $v(x',t) = u_1(\varrho x', t)$. Since $u_1 \in H_0^{2,0}(\mathrm{e}^{-\gamma_h t}, \Omega_\infty)$, by applying with embedding theorem to the domain $K' = \{x' \in K\colon \frac{1}{2} < |x'| < 2\}$, we have

$$|v(x',t)|^2 \leqslant C_1 \int_{K'} \Big[v^2 + |\operatorname{grad} v|^2 + \sum_{|\alpha|=2} |D^\alpha v|^2\Big]\,\mathrm{d}x',$$

where $C_1 > 0$. Let $x = \varrho x'$. Then one gets

$$|u_1(x',t)|^2 \leqslant C_1 \int_{\Omega^\varrho} \Big[\varrho^{-3} u_1^2 + \varrho^{-1}|\operatorname{grad} u_1|^2 + \varrho \sum_{|\alpha|=2} |D^\alpha u_1|^2\Big]\,\mathrm{d}x.$$

That shows

$$\begin{aligned}\varrho^{-1}|u_1(x',t)|^2 &\leqslant C_1 \int_{\Omega^\varrho} \Big[\varrho^{-4} u_1^2 + \varrho^{-2}|\operatorname{grad} u_1|^2 + \sum_{|\alpha|=2} |D^\alpha u_1|^2\Big]\,\mathrm{d}x \\ &\leqslant C_2 \int_{\Omega^\varrho} \Big[r^{-4} u_1^2 + r^{-2}|\operatorname{grad} u_1|^2 + \sum_{|\alpha|=2} |D^\alpha u_1|^2\Big]\,\mathrm{d}x \\ &\leqslant C_3 \|u_1(x,t)\|^2_{H_0^2(\Omega)} \leqslant C_4 \|f(x,t)\|^2_{L_2(\Omega)}\end{aligned}$$

for almost all $t \in (0,\infty)$, where $C_i > 0$, $i = 2,3,4$. When $|x| = \varrho$ we have $|u_1(x,t)| \leqslant C|x|^{1/2}$, $C > 0$.

Case 2: $\operatorname{Im}\lambda_1 \leqslant \frac{1}{2}$. Assume that $0 = \lambda_0, \lambda_1, \ldots, \lambda_{N_0}$ are the eigenvalues of the Neumann problem for (5.4) such that $-\frac{1}{2} < \operatorname{Im}\lambda_1 < \ldots < \operatorname{Im}\lambda_{N_0} \leqslant \frac{1}{2}$.

(i) If on the straight line $\operatorname{Im}\lambda = \frac{1}{2}$ there is no eigenvalue of the Neumann problem for (5.4), then repeating the arguments from the proof of Theorem 3.1 we obtain

$$u(x,t) = c(t) + \sum_{j=1}^{N_0} c_j(t) r^{\operatorname{Im}\lambda_j} \phi_j(\omega,t) + u_0(x,t), \tag{5.8}$$

where ϕ_j are infinitely differentiable with respect to (ω,t), and

$$c_j(\cdot) \in L_2^h(\mathrm{e}^{-\gamma_h t}, (0,\infty)), \quad j = 0,1,\ldots,N_0, \quad u_0 \in V_{0,h}^{2,0}(\mathrm{e}^{-\gamma_h t}, \Omega_\infty).$$

Using arguments similar to those in the proof of case 1, we obtain $|u_0(x,t)| \leqslant C|x|^{1/2}$, $C > 0$. From this and (5.8) one has

$$u(x,t) = c(t) + u_1(x,t),$$

where $c(\cdot) \in L_2^h(\mathrm{e}^{-\gamma_h t}, (0,\infty))$ and $|u_1| \leqslant C|x|^{\operatorname{Im}\lambda_1}$, $C > 0$.

(ii) If $\operatorname{Im}\lambda_{N_0} = \frac{1}{2}$, then there exists $\varepsilon > 0$ such that on the straight line $\operatorname{Im}\lambda = \frac{1}{2}+\varepsilon$ there is no eigenvalue of the Neumann problem for (5.4) and $0 < \operatorname{Im}\lambda_1 < \ldots < \operatorname{Im}\lambda_{N_0} < \frac{1}{2} + \varepsilon$. Using arguments similar to those in the proof of Theorem 3.1 we obtain

$$u(x,t) = \sum_{j=0}^{N_0} c_j(t) r^{\operatorname{Im}\lambda_j} \phi_j(\omega,t) + u_0(x,t),$$

where ϕ_j are infinitely differentiable with respect to ω, $c_j(\cdot) \in L_2^h(\mathrm{e}^{-\gamma_h t}, (0,\infty))$, $j = 0,1,\ldots,N_0$ and $u_0 \in V_{-\varepsilon,h}^{2,0}(\mathrm{e}^{-\gamma_h t}, \Omega_\infty) \subset V_{0,h}^{2,0}(\mathrm{e}^{-\gamma_h t}, \Omega_\infty)$.

Repeating the arguments from the proof of part (i), we have $u(x,t) = c(t)+u_1(x,t)$, where $c(\cdot) \in L_2^h(\mathrm{e}^{-\gamma_h t}, (0,\infty))$ and $|u_1| \leqslant C|x|^{\operatorname{Im}\lambda_1}$, $C > 0$. The proof is completed. □

Case $n = 2$. We can assume that $K = \{x = (x_1,x_2) \in \mathbb{R}^2\colon\ r > 0,\ 0 < \omega < \omega_0\}$, where (r,ω) are the polar coordinates of $x = (x_1,x_2)$ and $0 < \omega_0 < 2\pi$. Then the Neumann problem for (5.4) has simple form

$$\frac{\partial^2 v}{\partial\omega^2} - \lambda^2 v = 0, \quad 0 < \omega < \omega_0, \tag{5.9}$$

$$\frac{\partial v}{\partial\omega}\Big|_{\omega=0} = \frac{\partial v}{\partial\omega}\Big|_{\omega=\omega_0} = 0. \tag{5.10}$$

We have that

$$\lambda_k = \frac{\mathrm{i}k\pi}{\omega_0}, \quad k = \pm1, \pm2, \ldots \tag{5.11}$$

are eigenvalues of the problem (5.9)–(5.10) with eigenvectors

$$\varphi_k(\omega) = \cos\frac{k\pi\omega}{\omega_0}. \tag{5.12}$$

Moreover, $\lambda_0 \equiv 0$ is an eigenvalue of the problem (5.9)–(5.10) with the multiple equal to 2, the eigenvector $u_0 = 1$, and the generalized eigenvector $u_1 = 1$.

Corollary 5.3. *Let $u(x,t)$ be a generalized solution in $H^{1,0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$ of the problem* (5.1)–(5.3) *and let $f, f_t, f_{tt}, f_{ttt} \in L^\infty(0,T,L_2(\Omega))$, $f(x,0) = f_t(x,0) = 0$ for $x \in \Omega$. We have*

- *if* $\omega_0 < \pi$ *then* $u \in H^2(\mathrm{e}^{-\gamma t}, \Omega_\infty)$ *and*
- *if* $\omega_0 > \pi$ *then*

$$u(x,t) = c(x,t) r^{\pi/\omega_0} \cos(\pi\omega/\omega_0) + u_1(x,t),$$

where $c(\cdot,\cdot) \in V^2_{\pi/\omega_0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$, $u_1 \in H^2(\mathrm{e}^{-\gamma t}, \Omega_\infty)$.

Proof. i) If $\omega_0 < \pi$ then in the strip $-\varepsilon < \operatorname{Im}\lambda < 1$, $\varepsilon \in (0,1)$, there is only one eigenvalue $\lambda_0 \equiv 0$ with multiple 2, eigenvector $u_0 = 1$ and generalized eigenvector $u_1 = 1$. We can see that the multiple of λ_0 is constant and the eigenvectors and the generalized eigenvector are infinitely differentiable to (ω, t). So by using Remark 4.1 and arguments similar to those in the proof of Theorem 3.1, we have

$$u(x,t) = c_1(t) + c_2(t)\ln r + u_0(x,t),$$

where $c_i(\cdot) \in L^2_2(\mathrm{e}^{-\gamma t}, (0,\infty))$, $i = 1,2$, $u_0 \in H^{2,0}_0(\mathrm{e}^{-\gamma t}, \Omega_\infty)$.

Rewrite u in the form

$$u(x,t) = c_2(t)\ln r + u_1(x,t),$$

where $c_2(\cdot) \in L^2_2(\mathrm{e}^{-\gamma t}, (0,\infty))$, $u_1 \in H^{2,0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$. It is easy to check that

$$\frac{\partial}{\partial x_j} c_2(t)\ln r = c_2(t)\frac{x_j}{r^2}.$$

Since $u \in H^{1,0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$ and $u_1 \in H^{2,0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$, we have

$$|c_2(t)|^2 \int_\Omega r^{-2}\,\mathrm{d}x \leqslant C\int_\Omega [|Du|^2 + |Du_1|^2]\,\mathrm{d}x < \infty, \quad C > 0.$$

It follows that $c_2(t) \equiv 0$ and $u = u_1 \in H^{2,0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$. From Theorem 4.1 in [5] we have $u_t, u_{tt} \in H^{1,0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$. Therefore, $u \in H^2(\mathrm{e}^{-\gamma t}, \Omega_\infty)$.

ii) If $\omega_0 > \pi$ then in the strip $-\varepsilon < \operatorname{Im}\lambda < 1$ $(0 < \varepsilon < \omega_0/\pi)$ there are $\lambda_0 = 0$ (multiple 2) and $\lambda_1 = \mathrm{i}\pi/\omega_0$ (simple). From Theorem 3.1 we have

$$u(x,t) = c(t) r^{\pi/\omega_0}\cos(\pi\omega/\omega_0) + u_1(x,t),$$

where $c(\cdot) \in L^2_2(\mathrm{e}^{-\gamma t}, (0,\infty)$, $u_1 \in H^{2,0}_0(\mathrm{e}^{-\gamma t}, \Omega_\infty)$.

Denote

$$D_1 = \cos\frac{\pi\omega}{\omega_0}\frac{\partial}{\partial r} - \frac{1}{r}\sin\frac{\pi\omega}{\omega_0}\frac{\partial}{\partial\omega}.$$

We have $(\omega_0/\pi)r^{1-\pi/\omega_0}D_1u = c(t) + (\omega_0/\pi)r^{1-\pi/\omega_0}D_1u_1$. Set $c_1(x,t) = (\pi/\omega_0)\times r^{1-\pi/\omega_0}D_1u$. Since $u \in H^{1,0}(\mathrm{e}^{-\gamma t},\Omega_\infty)$, we have

$$\begin{aligned}\int_{\Omega_\infty}|c_1|^2\mathrm{e}^{-2\gamma t}\,\mathrm{d}x\,\mathrm{d}t &\leqslant C\int_{\Omega_\infty}r^{2(1-\pi/\omega_0)}|D_1u|^2\mathrm{e}^{-2\gamma t}\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant C\int_{\Omega_\infty}|Du|^2\mathrm{e}^{-2\gamma t}\,\mathrm{d}x\,\mathrm{d}t<\infty,\end{aligned}\tag{5.13}$$

where $C>0$. From Theorem 4.1 in [5] one has $u_t \in H^{1,0}(\mathrm{e}^{-\gamma t},\Omega_\infty)$. This yields

$$\begin{aligned}&\int_{\Omega_\infty}r^{2(-1+\pi/\omega_0)}|(c_1)_t|^2\mathrm{e}^{-2\gamma t}\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\leqslant C\int_{\Omega_\infty}|D_1u_t|^2\mathrm{e}^{-2\gamma t}\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\leqslant C\int_{\Omega_\infty}|Du_t|^2\mathrm{e}^{-2\gamma t}\,\mathrm{d}x\,\mathrm{d}t<\infty,\end{aligned}\tag{5.14}$$

where $C>0$. On the other hand, since $u_1 \in H_0^{2,0}(\mathrm{e}^{-\gamma t},\Omega_\infty)$, we have

$$\begin{aligned}&\int_{\Omega_\infty}r^{2(-1+\pi/\omega_0)}|Dc_1|^2\mathrm{e}^{-2\gamma t}\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\leqslant C\int_{\Omega_\infty}[r^{-1}D_1u_1+DD_1u_1]^2\mathrm{e}^{-2\gamma t}\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\leqslant C\sum_{|\alpha|=1}^{2}\int_{\Omega_\infty}r^{2(|\alpha|-2)}|D^\alpha u_1|^2\mathrm{e}^{-2\gamma t}\,\mathrm{d}x\,\mathrm{d}t<\infty.\end{aligned}\tag{5.15}$$

From (5.13), (5.14), and (5.15) one gets

$$\begin{aligned}&\sum_{|\alpha|+k=1}\int_{\Omega_\infty}r^{2(-1+\pi/\omega_0+k+|\alpha|-1)}|D^\alpha(c_1)_{t^k}|^2\mathrm{e}^{-2\gamma t}\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad+\int_{\Omega_\infty}|c_1|^2\mathrm{e}^{-2\gamma t}\,\mathrm{d}x\,\mathrm{d}t<\infty,\end{aligned}\tag{5.16}$$

or $c_1 \in V^1_{-1+\pi/\omega_0}(\mathrm{e}^{-\gamma t},\Omega_\infty)$.

It follows from Lemma 2 in [11] that $\hat{c}_1(x,t)$ satisfies

$$\begin{aligned}\|\hat{c}_1\|_{V^2_{\pi/\omega_0}(\mathrm{e}^{-\gamma t},\Omega_\infty)} &\leqslant C\|c_1\|_{V^1_{-1+\pi/\omega_0}(\mathrm{e}^{-\gamma t},\Omega_\infty)}\\ &\qquad+\int_{\Omega_\infty}r^{2(-2+\pi/\omega_0)}|c_1-\hat{c}_1|^2\mathrm{e}^{-2\gamma t}\,\mathrm{d}x\,\mathrm{d}t<\infty,\end{aligned}$$

where $C > 0$. So from (5.13) and (5.16) we have

$$u(x,t) = \hat{c}_1(x,t) r^{\pi/\omega_0} \cos(\pi\omega/\omega_0) + u_2(x,t),$$

where $u_2 = (c - \hat{c}_1) r^{\pi/\omega_0} \cos(\pi\omega/\omega_0) + u_1 \in H^2(\mathrm{e}^{-\gamma t}, \Omega_\infty)$ and $\hat{c}_1 \in V^2_{\pi/\omega_0}(\mathrm{e}^{-\gamma t}, \Omega_\infty)$. The proof is completed. □

6. Conclusions

In this paper, by reducing some conditions on eigenvalues of the spectrum problem, we obtained generally asymptotic expansions of solutions of the second boundary value problem for Schrödinger systems in a neighborhood of conical points (to compare see [3], [4] for example). Results are obtained for the second initial boundary value problem in infinite cylinders with the coefficients depending on both the time and spatial variables, while previous results were given for the first initial boundary value problem [2], [4] or in a finite cylinder [2], [15], [16] or for coefficients independent of the time variable [9], [15], [16], [18] or for other kinds of systems [3], [4], [9], [18].

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