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# EFFICIENT ALGORITHM TO SOLVE OPTIMAL BOUNDARY CONTROL PROBLEM FOR BURGERS' EQUATION

ALAEDDIN MALEK, ROGHAYEH EBRAHIM NATAJ,  
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In this paper, we propose a novel algorithm for solving an optimal boundary control problem of the Burgers' equation. The solving method is based on the transformation of the original problem into a homogeneous boundary conditions problem. This transforms the original problem into an optimal distributed control problem. The modal expansion technique is applied to the distributed control problem of the Burgers' equation to generate a low-dimensional dynamical system. The control parametrization method is formulated for approximating the time-varying control by a finite term of the orthogonal functions with unknown coefficients determined through an optimization process. The minimization of the objective functional is performed by using a conjugate gradient method. The accuracy and convergent rate of this hybrid method are shown by some numerical examples .

*Keywords:* optimal boundary control, Burgers' equation, conjugate gradient method, modal expansion technique, control parametrization

*Classification:* 49M37, 35K55

## 1. INTRODUCTION

Active control of fluid flows has been considered in significant researches [1, 12, 13]. Active control involves continuous adjustment of a variable that affects the flow based on the measurements of the quantities of the flow field (feedback). Development of an efficient control system for a fluid flow should be based on the specific Navier–Stokes equations that describe the flow in order to exploit their ability to accurately predict the spatiotemporal behavior of the flow field. Burgers' equation is the simplest approximation that captures the nonlinear and non-planar aspects of the Navier–Stokes equation and was proposed as a model for turbulence and the viscous structure of the weak shock waves [3]. Thereby it is a useful tool for examining the robustness of numerical schemes for optimal control problem of the Navier–Stokes equation.

The aim of optimal control is to find the best parameters of the model to simulate the closest computed values to the observed ones. Some authors have already dealt with this kind of problem for Burgers' equation. Dean and Gubernatis [7] introduced a pointwise control. Lellouche et al. [15] focused their study on the control by boundary conditions. Since the control by boundary conditions is more influential than the control by initial

conditions and by the viscosity coefficient [16], different numerical techniques for the optimal control of Burgers' equation with Neumann and Dirichlet boundary controls are studied in [14, 19]. In [19] the problem is solved by employing the low dimensional model using the Karhunen–Loeve Galerkin procedure, where the minimization of the objective function is performed by using a conjugate gradient method (KLG-CG). In [14] the problem with two boundary controls and smooth target function is solved by transforming it into a distributed one and applying the modal space approach. The parametrization method is then implemented to approximate the control by finite terms of a Fourier series type. The optimization problem is thereby reduced to a finite-dimensional minimization problem which is solved by using simplex method.

This work proposes an efficient numerical approach for solving the optimal control problem of the Burgers' equation by using one boundary control. Objective functional is determined by the distance between the final state at the terminal time and the target function over the spatial domain, and the energy depended on the boundary control actuators over a given period of time. In [11] it was shown that one cannot reach an arbitrary target function in arbitrary time with the help of one control force for the Burgers' equation. So to ensure that the target function is a reachable state, we determine it by solving Burgers' equation numerically with a given control function as a boundary condition. In this procedure we compute an optimal control function that reduces the cost expenditure of control effort in the objective functional respect to the given known control function. For this purpose we transform the optimal boundary control problem to one with distributed control; then the method of modal expansion is employed to convert the distributed control problem to that of the optimal control of lumped parameter systems in a finite-dimensional space. Consequently, a direct approach, control parametrization method, is followed. Finally a conjugate gradient method is performed to minimize the objective functional.

Our work is organized in the following way. The optimal boundary control problem is proposed in the second section. In the next three sections the modal expansion technique, control parametrization approach and conjugate gradient method are discussed. In the final section, some numerical results are stated.

## 2. THE OPTIMAL BOUNDARY CONTROL PROBLEM

In this paper we propose the boundary control of viscous Burgers' equation with an external distributed force in the form

$$u_t(x, t) + \mathcal{L}[u(x, t)] = f(x, t) \quad \text{for } (x, t) \in \Omega_x \times \Omega_T, \quad (1)$$

where

$$\mathcal{L}[u(x, t)] = -\nu u_{xx}(x, t) + u(x, t)u_x(x, t). \quad (2)$$

The variable  $u(x, t)$  is interpreted as the velocity of fluid at a spatial point  $x$  and at time  $t$ ,  $f(x, t)$  is the specific external force acting on the fluid and  $\nu$  is the kinematic viscosity. The problem is subject to the Dirichlet boundary conditions

$$u(0, t) = 0, \quad u(1, t) = v(t), \quad (3)$$

and the initial condition

$$u(x, 0) = 0, \quad (4)$$

where  $v(t)$  is an arbitrary time-varying function. Let  $H(S)$  denotes the Hilbert space introduced as:

$$H(S) = \{f : \mathbb{R} \supseteq S \rightarrow \mathbb{R} \mid \|f\|_{H(S)}^2 < \infty\},$$

where the norm  $\|f\|_{H(S)}^2 = \langle f, f \rangle_{H(S)}$  and

$$\langle f, g \rangle_{H(S)} = \int_S f(r)g(r) \, dr,$$

for all  $f, g \in H(S)$ , in which  $S$  can be either  $\Omega_T = (0, T)$  or  $\Omega_x = (0, 1)$ . The Cartesian product of two copies of  $H(S)$  is denoted by  $H^2(S)$ .

**Definition 2.1.** A time-varying function  $v(t)$  is said to be admissible control, if for a given  $T > 0$ ,  $v(t) \in H(\Omega_T)$ .

The problem of interest here is that of controlling the system above to produce a state function  $u(x, T)$  at the final time  $T$  that is as close as possible to a target function,  $u_T(x)$ . We aim at achieving this goal through adjusting the boundary condition  $v(t)$ , at  $x = 1$ . Our concern is to achieve the goal while minimizing the cost of the control function  $v(t)$ . This suggests the problem:

$$\min_{v \in H(\Omega_T)} J(v)$$

where

$$J(v) = \frac{1}{2} \int_0^1 (u(x, T) - u_T(x))^2 \, dx + \frac{\epsilon}{2} \int_0^T v(t)^2 \, dt, \quad (5)$$

and  $\epsilon$  is a small positive constant and it reflects the relative weight attached to a cost expenditure of control efforts.

### 3. MODAL EXPANSION TECHNIQUE

The idea of modal control is that one can control the motion of any point in the structure by controlling the mode of its vibration. Although a distributed structure has an infinite number of modes, in practice, only some of lower modes need to be controlled [2]. Using modal expansion method, the optimal control of distributed parameter systems is reduced to the optimal control of lumped parameter dynamical systems in finite-dimensional space in form of finite set of independent ordinary differential equations [20]. Since this method is applicable to the distributed control problems, an appropriate transformation is needed to convert the problem from one in which there are boundary controls to one in which there are distributed controls, the following transformation is considered [14]:

$$u(x, t) = xv(t) + w(x, t), \quad (6)$$

where  $w(x, t)$  is an auxiliary state function. Then, the Burgers' equation, Eq. (1) becomes:

$$w_t + \nu w_{xx} + ww_x + xv(t)w_x + v(t)w + xv^2(t) = f(x, t), \tag{7}$$

with homogeneous boundary conditions:

$$w(0, t) = 0, \quad w(1, t) = 0, \quad t \in \Omega_T, \tag{8}$$

and initial condition as below:

$$w(x, 0) = -xv(0), \quad x \in \Omega_x. \tag{9}$$

After this, we transform the distributed parameter control problem (7)–(9) into a modal control lumped-parameter problem by means of eigenfunction expansion technique. Before applying the eigenfunction procedure to reduce the degrees of freedom of the system, a set of eigenfunctions that satisfies the homogeneous boundary conditions in Eq. (8), needs to be specified. The set of orthogonal eigenfunctions

$$\{\psi_n(x)\}_{n=1}^\infty = \{\sqrt{2} \sin(n\pi x)\}_{n=1}^\infty \tag{10}$$

are the eigenfunctions of the operator  $\mathcal{L}[w] = \frac{\partial^2 w}{\partial x^2}$  and satisfy the boundary conditions. Since the set of eigenfunctions  $\{\psi_n\}_{n=1}^\infty$  is a complete orthogonal basis for  $H(\Omega_x)$ , each  $w(x, t) \in H(\Omega_x)$  has a unique representation in the following form [18]:

$$w(x, t) = \sum_{n=0}^\infty a_n(t)\psi_n(x). \tag{11}$$

Although the state variable expansion theoretically yields an infinite-dimensional system, it is necessary to limit the dimension since it is not feasible to control a large number of modes actively. This is achieved by selecting the eigenfunctions associated with the largest eigenvalues. Thus Eq. (11) can be written as a truncated Fourier series expansion:

$$w_N(x, t) = \sum_{n=0}^N a_n(t)\psi_n(x), \tag{12}$$

where

$$a_n(t) = \langle w_N(x, t), \psi_n(x) \rangle_{H(\Omega_x)}, \tag{13}$$

are Fourier coefficients of  $w_N(x, t)$  and  $\psi_n(x)$  is defined by Eq. (10). Substituting expansion (12) into Eq. (7), integrating over  $\Omega_x$ , and taking the inner product of both sides of Eq. (7) results in a finite system of ordinary differential equations as follow:

$$\dot{a}_n(t) + (\nu n^2 + v(t))a_n(t) + I_n(t) = \hat{f}(t), \quad n = 1, 2, \dots, N, \tag{14}$$

where the dot denotes the derivative with respect to time  $t$  in which

$$I_n(t) = \frac{2n}{\pi} \sum_{\substack{k=1 \\ k \neq n}}^N k \frac{v(t)(-1)^{1+k+n}}{n^2 - k^2} a_n(t) + \frac{\sqrt{2}}{n\pi} (-1)^{n+1} v^2(t), \tag{15}$$

and

$$\hat{f}(t) = \langle f(x, t), \psi_n(x) \rangle_{H(\Omega_x)}.$$

The modal equations in Eq. (14) form a system of  $N$  first-order nonlinear coupled differential equations subject to the modal initial conditions

$$a_n(0) = \frac{\sqrt{2}}{n\pi} (-1)^n v(0).$$

By using of the transformation (6) and the expansion (12), the objective functional (5) in the first  $N$  modes becomes

$$J_N(v) = \frac{1}{2} \int_0^1 (w_N(x, T) - u_T(x))^2 dx + \frac{\epsilon}{2} \int_0^T v(t)^2 dt, \tag{16}$$

and the optimal boundary control problem (5) is reduced to the following modal control problem:

$$\min_{v \in H(0, T)} J_N(v). \tag{17}$$

#### 4. CONTROL PARAMETRIZATION APPROACH

In general, this technique approximates the control functions by finite terms of orthogonal functions with unknown coefficients, thereby converting an optimal control problem into a mathematical programming problem. In this paper, the achieved modal control problem is approximated by using finite terms of Fourier series where the unknown coefficients giving a solution near the optimal (or sub-optimal) solutions are sought. Let the finite-dimensional subspace  $U_m \subset H(0, T)$  be the linear space spanned by  $P_i(t)$  for  $i = 1, 2, \dots, m$  that can be taken as standard families of polynomials or functions such as orthogonal polynomials [4], trigonometric functions [9] or polynomial splines [21]:

$$U_m = \{v_m(t) \in H(0, T) \mid v_m(t) = \sum_{i=0}^m \alpha_i P_i(t), \alpha_i \in \mathbb{R}\}.$$

Assume the set

$$U_\infty = \{v(t) \mid v(t) = \lim_{m \rightarrow \infty} v_m(t)\}. \tag{18}$$

It is dense in  $H(0, T)$ , in the sense that for each admissible control  $v(t) \in H(0, T)$  and  $\delta > 0$  there exists  $\tilde{v}(t) \in U_\infty$  such that

$$\|v(t) - \tilde{v}(t)\|_{H(0, T)} \leq \delta, \tag{19}$$

After applying the approximation theory [6] there exists a unique  $L_2$ -approximation  $v_m(t) \in U_m$  to  $v^0(t) \in H(0, 1)$ ,

$$\|v(t) - v^0(t)\|_{H(0, T)} = \inf\{\|v_m(t) - v^0(t)\|_{H(0, T)} \mid v_m(t) \in U_m\}. \tag{20}$$

In the control parametrization approach, the original problem of minimizing  $J_N$  over  $v(t) \in H(0, T)$  in Eq. (17) is replaced by the finite-dimensional problem of finding  $v_m(t) \in U_m$  that minimizes  $J_N$  over  $U_m$ . In this paper, the components of parametrization of  $v_m(t)$  is chosen to be of the form:

$$\begin{aligned}
 v_m(t) &= \sum_{i=1}^{2m} \beta_i \varphi_i(t) \\
 &= \beta_1 \sin(\pi t) + \beta_2 \cos(\pi t) + \dots + \beta_{2m-1} \sin(m\pi t) + \beta_{2m} \cos(m\pi t), \quad (21)
 \end{aligned}$$

where  $\beta = [\beta_1, \beta_2, \dots, \beta_{2m}]^T$  is to be determined optimally. Then our problem changes to the following programming problem:

Find the optimal value for vector  $\hat{\beta}$  such that

$$J_N(\hat{\beta}) = \inf_{\beta \in \mathbb{R}^{2m}} J_N(\beta). \quad (22)$$

The necessary condition of optimality is:

$$\frac{\partial J_N}{\partial \beta} = 0. \quad (23)$$

Solution of the latter system leads to a nonlinear system of equations for  $\beta$ .

### 5. CONJUGATE GRADIENT METHOD

Among various minimization techniques for solving system of equations (23), Fletcher-Reeves method, that is one of the conjugate gradient methods is used. The advantage of conjugate gradient methods is that they have simple formulae for updating the direction vector. These methods are slightly more complicated than steepest descent methods. However they converge faster. It has been shown that any minimization method that makes use of the conjugate directions like conjugate gradient methods is quadratically convergent. This property of quadratic convergence is very useful because it ensures that the method will minimize a quadratic function in which the number of required steps equals to the number of minimization variables or less. Since any general function can be approximated reasonably well by a quadratic near the optimum point, any quadratically convergent method is expected to find the optimum point in a finite number of iterations [10]. In this analysis, the search direction  $d$  at the first step is computed by

$$d^{(1)} = -\frac{\partial J_N}{\partial \beta^{(1)}} = -\int_0^1 (u(x, T) - u_T(x)) \frac{\partial u(x, T)}{\partial \beta^{(1)}} dx - \epsilon \int_0^T v(t) \Phi(t) dt, \quad (24)$$

where

$$\frac{\partial u(x, T)}{\partial \beta} = \left[ \frac{\partial u(x, T)}{\partial \beta_1}, \frac{\partial u(x, T)}{\partial \beta_2}, \dots, \frac{\partial u(x, T)}{\partial \beta_{2m}} \right]^T,$$

and

$$\Phi(t) = [\varphi_1, \varphi_2, \dots, \varphi_{2m}]^T.$$

The sensitivity matrix  $\left[\frac{\partial u}{\partial \beta}\right]$  is calculated by partially differentiating Eqs. (1)–(4) with respect to the parameter vector  $\beta$ :

$$\begin{aligned} \frac{\partial}{\partial t}\theta(k) - \nu \frac{\partial^2}{\partial x^2}\theta(k) + \theta(k)\frac{\partial u}{\partial x} + u\frac{\partial}{\partial x}\theta(k) &= 0 \quad (k = 1, 2, \dots, 2m) \quad (25) \\ \theta(x = 0, t, k) &= 0, \\ \theta(x = 1, t, k) &= \varphi_k(t), \\ \theta(x, t = 0, k) &= 0, \end{aligned}$$

where

$$\theta(k) = \theta(x, t, k) \equiv \frac{\partial u}{\partial \beta_k}. \quad (26)$$

We solve this set of equations by the explicit finite difference method [17]. The parameter vector  $\beta^{(i+1)}$ , where  $i$  denotes the iteration number, is calculated from  $\beta^{(i)}$  through moving in the conjugate direction  $\mathbf{d}^{(i)}$ :

$$\beta^{(i+1)} = \beta^{(i)} + \alpha \mathbf{d}^{(i)}. \quad (27)$$

The optimal step length  $\alpha$  is determined by minimizing  $J(\beta + \alpha \mathbf{d})$  with respect to  $\alpha$ . Since

$$u(\beta + \alpha \mathbf{d}) = u(\beta) + \sum_{k=1}^{2m} \frac{\partial u}{\partial \beta_k} \alpha d_k,$$

we have:

$$w_N(\beta + \alpha \mathbf{d}) \cong w_N(\beta) + \sum_{k=1}^{2m} \frac{\partial u}{\partial \beta_k} \alpha d_k. \quad (28)$$

Then

$$\begin{aligned} J_N(\beta + \alpha \mathbf{d}) &= \frac{1}{2} \int_0^1 (w_N(x, t) + \sum_{k=1}^{2m} \alpha \theta(k) d_k - u_T(x))^2 dx \\ &\quad + \frac{\epsilon}{2} \int_0^T \left( \sum_{k=1}^{2m} (\beta_k + \alpha d_k) \varphi_k(t) \right)^2 dt. \quad (29) \end{aligned}$$

Thus, the scalar  $\alpha$  is computed by partially differentiating Eq. (29) with respect to  $\alpha$  and setting the resulting equation equal to zero,

$$\alpha = - \frac{\int_0^1 (w_N(x, t) - u_T(x)) \left( \sum_{k=1}^{2m} \theta(x, T, k) d_k \right) dx + \epsilon \int_0^T \left( \sum_{k=1}^{2m} \beta_k \varphi_k(t) \right) \left( \sum_{k=1}^{2m} d_k \varphi_k(t) \right) dt}{\int_0^1 \left( \sum_{k=1}^{2m} \theta(x, T, k) d_k \right)^2 dx + \epsilon \int_0^T \left( \sum_{k=1}^{2m} d_k \varphi_k(t) \right)^2 dt}.$$



The conjugate direction or the search direction  $\mathbf{d}$  is calculated and renewed by Fletcher-Reeves method, which is expressed as follows:

$$\mathbf{d}^{(i+1)} = -\frac{\partial J_N^{(i+1)}}{\partial \boldsymbol{\beta}} + \rho \mathbf{d}^{(i)}, \tag{30}$$

where

$$\frac{\partial J_N}{\partial \boldsymbol{\beta}} = \left[ \frac{\partial J_N}{\partial \beta_1}, \frac{\partial J_N}{\partial \beta_2}, \dots, \frac{\partial J_N}{\partial \beta_{2m}} \right]^T, \tag{31}$$

with

$$\frac{\partial J_N}{\partial \beta_k} = -\int_0^1 (u(x, T) - u_T(x))\theta(t = T, k) dx - \epsilon \int_0^T v(t)\Phi(t) dt, \tag{32}$$

and

$$\rho = \frac{\sum_{k=1}^{2m} \left( \frac{\partial J_N^{(i+1)}}{\partial \beta_k} \right)^2}{\sum_{k=1}^{2m} \left( \frac{\partial J_N^{(i)}}{\partial \beta_k} \right)^2}. \tag{33}$$

This renewed  $\mathbf{d}^{(i+1)}$  is used for the search direction at the next iterative state  $i + 1$ .

In the following, we propose ME-CG Algorithm, since the low-dimensional model is determined by the Modal expansion approach and the minimization of the objective function is computed by means of the conjugate gradient method suggested by Fletcher and Reeves. The iterative algorithm for solving optimal boundary control problem of the Burgers' equation is summarized as:

**ME-CG Algorithm**

1. Provide  $N$  the number of eigenfunctions and  $m$  the number of components of parametrization, choose the initial guess  $\boldsymbol{\beta}^{(1)}$ , Let  $v(t) \equiv v_m(t)$ .
2. Assume  $i = 1$ . Determine  $v(t)$  by Eq. (21) and  $\boldsymbol{\beta}^{(1)}$ . With this control  $v(t)$ , calculate  $a_n(t)$  by Eq. (14) employing the fourth order Runge-Kutta method with fifth order error [8], then apply Eqs. (11)–(12) to calculate  $w_N(x, t)$  and transformation (6) for computing  $u(x, t = T)$ . Calculate  $\theta(x, t = T, k)$  employing the explicit finite difference method [17].
3. Compute  $\mathbf{d}^{(1)} = -\frac{\partial J_N}{\partial \boldsymbol{\beta}}$  by Eq. (24).
4. Determine  $\alpha$  that minimizes  $J_N(\boldsymbol{\beta}^{(i)} + \alpha \mathbf{d}^{(i)})$  by Eq. (30).
5. Compute  $\boldsymbol{\beta}^{(i+1)}$  by Eq. (27).

6. Calculate  $u(\beta^{(i+1)})$  and  $\theta(\beta^{(i+1)})$ .
7. Compute  $\rho$  by Eq. (33).
8. Compute  $d^{(i+1)}$  by Eq. (30). If  $d^{(i+1)} \leq \epsilon_1$ , Check the stopping criterion:  $\|J_N - J_{N-1}\| \leq \epsilon_2$ , where  $\epsilon_1$  and  $\epsilon_2$  are prescribed small numbers, (In the first iteration set  $J_{N-1} = \infty$ ). If the stopping criterion is not satisfied set  $N = N + 1$ ,  $\beta^{(1)} = \beta^{(i+1)}$  and go to Step 2.
9. If  $d^{(i+1)} > \epsilon_1$ , set  $i = i + 1$  and go to Step 4.

6. NUMERICAL RESULTS AND DISCUSSION

Here, we assess the accuracy and efficiency of the ME-CG algorithm proposed in the previous section for solving the optimal boundary control problem for Burgers' equation. To ensure that the target function  $u_T(x)$  is a reachable state, we determine it by integrating Eqs. (1)–(4) with given  $v(t)$  as a the boundary condition [19], using the finite difference technique [17] and set  $u_T(x) = u(x, T)$ . In the following examples the sine waves are chosen as the boundary condition for the Burgers' equation due to the nature of problem, since some exact solutions of the Burgers' equation evolve from an original sine wave like Fay's solution [5]. In the procedure proposed in the ME-CG algorithm, we generate an optimal control function that reducing the cost expenditure of control effort in the objective functional respect to the given known control function. Numerical computations in the article are performed by a personal computer with a 2.4 GHz CPU and 2 GB of RAM. To stabilize the numerical simulation the specific parameter values adopted here are  $\nu = 0.1$  in Eq. (2),  $T = 1$ ,  $\epsilon = 0.001$  in Eq. (5),  $N = 2$  in Eq. (12) and  $m = 2$  in Eq. (21). Tolerances numbers in the algorithm are  $\epsilon_1 = 0.001$  and  $\epsilon_2 = 0.0001$ . We set the unknowns in Eq. (21) as a vector in form of  $\beta = [\beta_1, \beta_2, \beta_3, \beta_4]^T$ . The numerical procedure starts with the initial guess  $\beta = \vec{0}_{4 \times 1}$ . We consider two different target functions determined by any of the following controls. In the first case, it is considered:

**Example 1.**

$$v(t) = \sin(\pi t), \quad 0 \leq t \leq 1. \tag{34}$$

This control and corresponding target function  $u_T(x)$  are depicted in Figs. 1 (a,b) respectively. In Fig. 2 the target function is compared with the controlled one at final time by using ME-CG algorithm. The convergence of method is also could be examined with the following definition:

**Definition 6.1.** The optimal state function  $u(x, T)$  approximate the given target function  $u_T(x)$  with  $p$  correct significant digits if  $p$  is the smallest positive integer number that satisfy in

$$\sum_{i=1}^n \frac{|u(x_i, T) - u_T(x_i)|}{|u(x_i, T)|} \leq \frac{1}{2} \times 10^{-p+1}$$

where  $n$  is the number of nodes using for discretization in a given spatial interval.

In this example according to the attained numerical results the optimal state function  $u(x, T)$  approximate the target function  $u_T(x)$  depicted in Fig. 1(b) with three correct significant digits. The optimal control  $v(t)$  calculated by the given ME-CG algorithm and the original control in Eq. (34) are depicted in Fig. 3. These figures show that the energy of the computed optimal control is less than the original control and this is what was desired in the objective functional Eq. (5).

Second example is the case where the target profile  $u_T(x)$  is determined by assuming the following control function as the boundary condition for the Burgers' equation.

### Example 2.

$$v(t) = \sin(4\pi t), \quad 0 \leq t \leq 1. \quad (35)$$

This control and the target profile associated with it are depicted in Figs. 4(a,b). The wavelength of sine wave is reduced compared to that of Example 1 to examine the robustness of the given algorithm in more challenging cases.

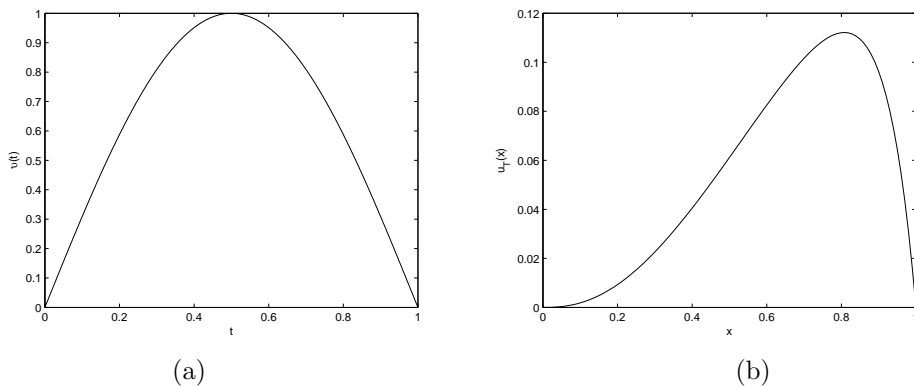
With the same assumptions, results are gathered in Figs. 5 and 6. According to Definition 6.1, the optimal state function  $u(x, T)$  approximate the target function  $u_T(x)$  depicted in Fig. 4(b) with two correct significant digits. In the Table 1, the difference between these two functions values are given in some nodes on the spatial interval for two Examples. The number of iterations needed in conjugate gradient method for getting results with given ME-CG algorithm are depicted in Figs. 7 and 8, associated to different cases respectively. It is shown that the value of the objective function decreases rapidly. Finally, it takes 1667s for the ME-CG algorithm to yield the convergent profile of the optimal control after  $N = 4$  iterations for the target profile shown in Fig. 1(b), while 1218s and  $N = 3$  iterations for the target profile determined in Fig. 4(b). This reduction in the CPU time was expected since we use low-dimensional model and conjugative gradient method, that provide quadratic convergence. In this case the performance functional is more closely quadratic and so convergent is more nearly assured.

## 7. CONCLUSION

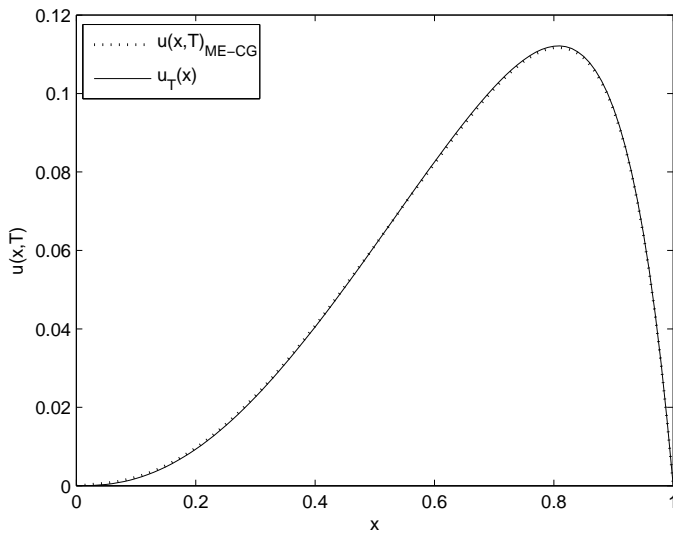
In this paper a class of the optimal control problems governed by Burgers' equation is considered. To develop a solution technique for such problems, the boundary control problem is transformed into a distributed one. By applying the modal space approach, the basic control problem is then reduced to a system of coupled nonlinear differential equations. The parametrization method is applied to approximate the control by a finite terms of Fourier series. The objective functional, which is determined by the distance between the final state  $u(x, T)$  and the target profile  $u_T(x)$  along with the energy of the control, is minimized by using the conjugate gradient method. Two examples are solved using the proposed approach. Numerical results indicate that the proposed algorithm is highly efficient and accurate. Applying this algorithm to similar problems with two boundary controls could be the subject of the future works.

Spatial nodes $x$	Example(1) $ u(x, T) - u_T(x) $	Example(2) $ u(x, T) - u_T(x) $
0	$0.0 \times 10^{-4}$	$0.0 \times 10^{-4}$
0.1	$0.336 \times 10^{-3}$	$0.3184 \times 10^{-3}$
0.2	$0.372 \times 10^{-3}$	$0.156 \times 10^{-3}$
0.3	$0.63 \times 10^{-4}$	$0.43 \times 10^{-4}$
0.4	$0.75 \times 10^{-4}$	$0.796 \times 10^{-4}$
0.5	$0.84 \times 10^{-4}$	$0.4378 \times 10^{-3}$
0.6	$0.378 \times 10^{-3}$	$0.398 \times 10^{-3}$
0.7	$0.462 \times 10^{-3}$	$0.452 \times 10^{-3}$
0.8	$0.42 \times 10^{-3}$	$0.1194 \times 10^{-3}$
0.9	$0.48 \times 10^{-3}$	$0.246 \times 10^{-3}$
1	$0.126 \times 10^{-3}$	$0.1194 \times 10^{-3}$

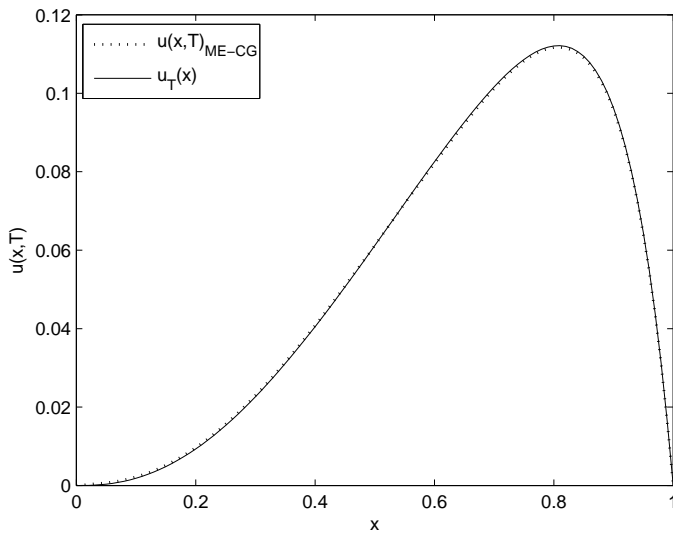
**Tab. 1.** Difference between the calculated optimal state function  $u(x, T)_{ME-CG}$  and the target function  $u_T(x)$  in some nodes on the spatial interval  $[0, 1]$  for the Examples 1 and 2.



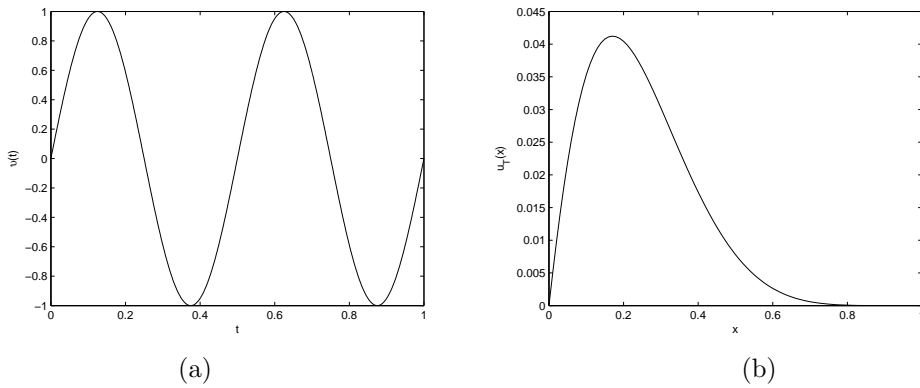
**Fig. 1.** Example 1, (a) The control given by Eq. (34), (b) The target profile  $u_T(x)$  calculated by numerically solving Burgers' equation using the boundary condition  $u(1, t) = \sin(\pi t)$ .



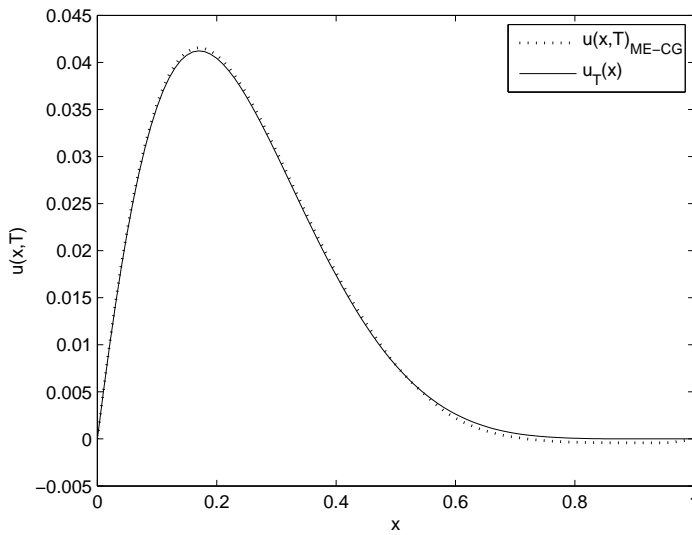
**Fig. 2.** Example 1, Comparison between the target function  $u_T(x)$  and the final state  $u(x, T)$  calculated by using the ME-CG algorithm.



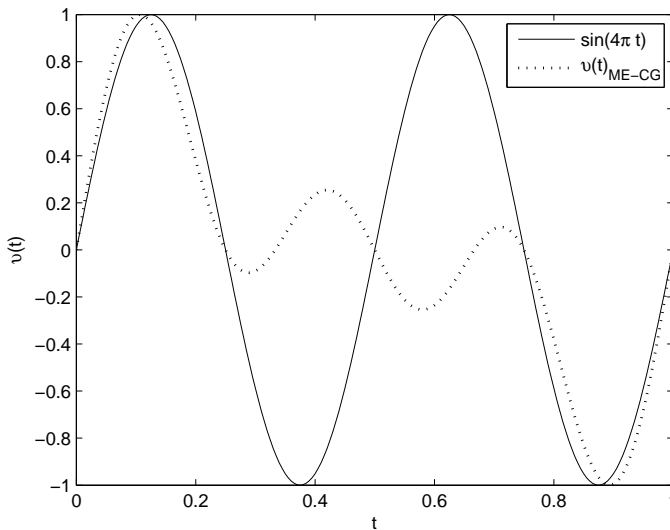
**Fig. 3.** Example 1, The optimal control computed by the ME-CG algorithm and the original control given by Eq. (34).



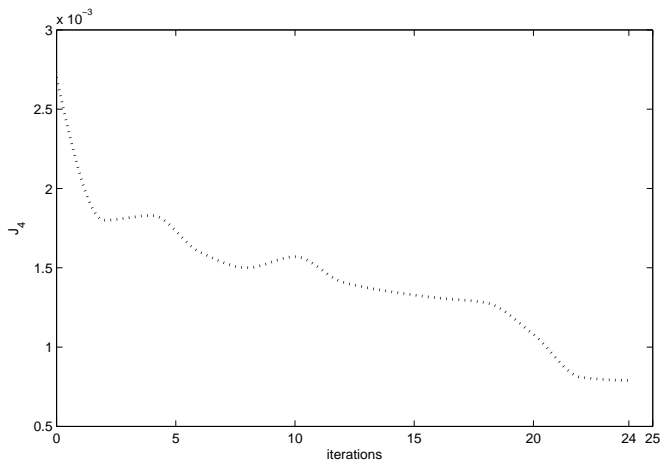
**Fig. 4.** Example 2, (a) The control given by Eq. (35), (b) The target profile  $u_T(x)$  calculated by solving the Burgers' equation using the boundary condition  $u(1, t) = \sin(4\pi t)$ .



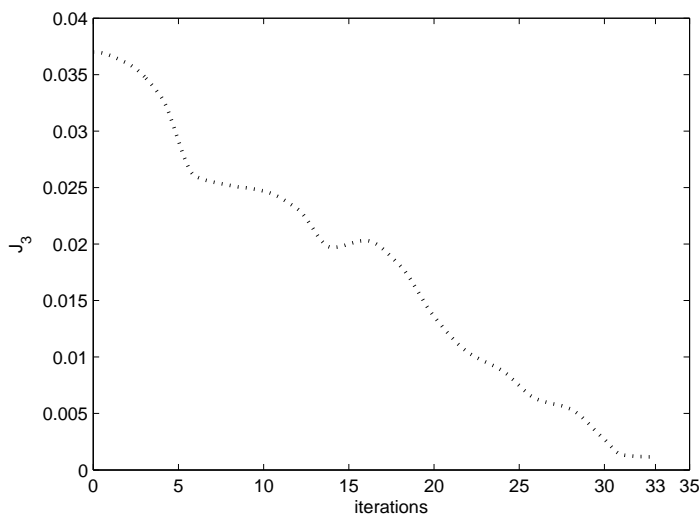
**Fig. 5.** Example 2, Comparison between the final state  $u(x, T)_{ME-CG}$  and the target function.



**Fig. 6.** Example 2, The profiles of the optimal control computed by the ME-CG algorithm and the original control in Eq. (34).



**Fig. 7.** Example 1, Convergent rate of the conjugate gradient method. Here  $J_4$  is the value of the objective functional in the final iteration  $N = 4$ , using the ME-CG algorithm.



**Fig. 8.** Example 2, Convergent rate of the conjugate gradient method. Here  $J_3$  is the value of the objective functional in the final iteration  $N = 3$ , using the ME-CG algorithm.

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