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ON HOLOMORPHICALLY PROJECTIVE MAPPINGS OF e -KÄHLER MANIFOLDS

IRENA HINTERLEITNER

ABSTRACT. In this paper we study fundamental equations of holomorphically projective mappings of e -Kähler spaces (i.e. classical, pseudo- and hyperbolic Kähler spaces) with respect to the smoothness class of metrics. We show that holomorphically projective mappings preserve the smoothness class of metrics.

1. INTRODUCTION

First we study the general dependence of holomorphically projective mappings of classical, pseudo- and hyperbolic Kähler manifolds (shortly e -Kähler) in dependence on the smoothness class of the metric. We present well known facts, which were proved by Domashev, Kurbatova, Mikeš, Prvanović, Otsuki, Tashiro etc., see [2, 3, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19]. In these results no details about the smoothness class of the metric were stressed. They were formulated “for sufficiently smooth” geometric objects.

2. KÄHLER MANIFOLDS

In the following definition we introduce generalizations of Kähler manifolds.

Definition 1. An n -dimensional (pseudo-)Riemannian manifold (M, g) is called an e -Kähler manifold K_n , if beside the metric tensor g , a tensor field F ($\neq \text{Id}$) of type $(1, 1)$ is given on the manifold M_n , called a *structure* F , such that the following conditions hold:

$$(1) \quad F^2 = e \text{Id}; \quad g(X, FX) = 0; \quad \nabla F = 0,$$

where $e = \pm 1$, X is an arbitrary vector of TM_n , and ∇ denotes the covariant derivative in K_n .

If $e = -1$, K_n is a (*pseudo*-)Kähler space (also *elliptic Kähler space*) and F is a *complex structure*. As A -spaces, these spaces were first considered by P. A. Shirokov, see [14]. Independently they were studied by E. Kähler [5].

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If $e = +1$, K_n is a *hyperbolic Kähler space* (also *para Kähler space*, see [1]) and F is a *product structure*. The spaces K_n^+ were considered by P. K. Rashevskij [13].

The e -Kähler spaces introduced here are called shortly “Kähler” in the literature [10, 16]. By our definition we want to give a unified notation for all classes.

3. HOLOMORPHICALLY PROJECTIVE MAPPING THEORY
FOR $K_n \rightarrow \bar{K}_n$ OF CLASS C^1

Assume the e -Kähler manifolds $K_n = (M, g, F)$ and $\bar{K}_n = (\bar{M}, \bar{g}, \bar{F})$ with metrics g and \bar{g} , structures F and \bar{F} , Levi-Civita connections ∇ and $\bar{\nabla}$, respectively. Here $K_n, \bar{K}_n \in C^1$, i.e. $g, \bar{g} \in C^1$ which means that their components $g_{ij}, \bar{g}_{ij} \in C^1$.

Likewise, as in [11] we introduce the following notations.

Definition 2. A curve ℓ in K_n which is given by the equation $\ell = \ell(t)$, $\lambda = d\ell/dt$, ($\neq 0$), $t \in I$, where t is a parameter is called *analytically planar*, if under the parallel translation along the curve, the tangent vector λ belongs to the two-dimensional distribution $D = \text{Span} \{ \lambda, F\lambda \}$ generated by λ and its conjugate $F\lambda$, that is, it satisfies

$$\nabla_t \lambda = a(t)\lambda + b(t)F\lambda,$$

where $a(t)$ and $b(t)$ are some functions of the parameter t .

Particularly, in the case $b(t) = 0$, an analytically planar curve is a geodesic.

Definition 3. A diffeomorphism $f: K_n \rightarrow \bar{K}_n$ is called a *holomorphically projective mapping* of K_n onto \bar{K}_n if f maps any analytically planar curve in K_n onto an analytically planar curve in \bar{K}_n .

Assume a holomorphically projective mapping $f: K_n \rightarrow \bar{K}_n$. Since f is a diffeomorphism, we can suppose local coordinate charts on M or \bar{M} , respectively, such that locally, $f: K_n \rightarrow \bar{K}_n$ maps points onto points with the same coordinates, and $\bar{M} = M$.

A manifold K_n admits a holomorphically projective mapping onto \bar{K}_n if and only if the following equations [10, 16]:

$$(2) \quad \bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X + e\psi(FX)FY + e\psi(FY)FX$$

hold for any tangent fields X, Y and where ψ is a differential form. If $\psi \equiv 0$ than f is *affine* or *trivially holomorphically projective*. Beside these facts it was proved [10, 16] that $\bar{F} = \pm F$; for this reason we can suppose that $\bar{F} = F$. In local form:

$$\bar{\Gamma}_{ij}^h = \Gamma_{ij}^h + \psi_i \delta_j^h + \psi_j \delta_i^h + e\psi_i \delta_j^h + e\psi_j \delta_i^h,$$

where Γ_{ij}^h and $\bar{\Gamma}_{ij}^h$ are the Christoffel symbols of K_n and \bar{K}_n , ψ_i, F_i^h are components of ψ, F and δ_i^h is the Kronecker delta, $\psi_i = \psi_\alpha F_i^\alpha, \delta_i^h = F_i^h$.

Here and in the following we will use the conjugation operation of indices in the way

$$A_{\dots \bar{i} \dots} = A_{\dots k \dots} F_i^k.$$

Equations (2) are equivalent to the following equations

$$(3) \quad \begin{aligned} \nabla_Z \bar{g}(X, Y) &= 2\psi(Z)\bar{g}(X, Y) + \psi(X)\bar{g}(Y, Z) + \psi(Y)\bar{g}(X, Z) \\ &\quad - e\psi(FX)\bar{g}(FY, Z) - e\psi(FY)\bar{g}(FX, Z). \end{aligned}$$

In local form:

$$\bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik} - e\psi_{\bar{i}} \bar{g}_{\bar{j}\bar{k}} - e\psi_{\bar{j}} \bar{g}_{\bar{i}\bar{k}},$$

where “ $\bar{\cdot}$ ” denotes the covariant derivative on K_n . It is known that

$$\psi_i = \partial_i \Psi, \quad \Psi = \frac{1}{2(n+2)} \ln \left| \frac{\det \bar{g}}{\det g} \right|, \quad \partial_i = \partial / \partial x^i.$$

Domashev, Kurbatova and Mikeš [3, 6, 16] proved that equations (2) and (3) are equivalent to

$$(4) \quad \begin{aligned} \nabla_Z a(X, Y) &= \lambda(X)g(Y, Z) + \lambda(Y)g(X, Z) \\ &\quad - e\lambda(FX)g(FY, Z) - e\lambda(FY)g(FX, Z). \end{aligned}$$

In local form:

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} - e\lambda_{\bar{i}} g_{\bar{j}\bar{k}} - e\lambda_{\bar{j}} g_{\bar{i}\bar{k}},$$

where

$$(5) \quad (a) \quad a_{ij} = e^{2\Psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}; \quad (b) \quad \lambda_i = -e^{2\Psi} \bar{g}^{\alpha\beta} g_{\beta i} \psi_{\alpha}.$$

From (4) follows $\lambda_i = \partial_i \lambda = \partial_i (\frac{1}{4} a_{\alpha\beta} g^{\alpha\beta})$. On the other hand [10]:

$$(6) \quad \bar{g}_{ij} = e^{2\Psi} \tilde{g}_{ij}, \quad \Psi = \frac{1}{2} \ln \left| \frac{\det \tilde{g}}{\det g} \right|, \quad \|\tilde{g}_{ij}\| = \|g^{i\alpha} g^{j\beta} a_{\alpha\beta}\|^{-1}.$$

The above formulas are the criterion for holomorphically projective mappings $K_n \rightarrow \bar{K}_n$, globally as well as locally.

4. HOLOMORPHICALLY PROJECTIVE MAPPING THEORY FOR $K_n \rightarrow \bar{K}_n$ OF CLASS C^2

Let K_n and $\bar{K}_n \in C^2$ be e -Kähler manifolds, then for holomorphically projective mappings $K_n \rightarrow \bar{K}_n$ the Riemann and the Ricci tensors transform in this way

$$(7) \quad \begin{aligned} (a) \quad \bar{R}_{ijk}^h &= R_{ijk}^h + \delta_k^h \psi_{ij} - \delta_j^h \psi_{ik} - e\delta_k^h \psi_{i\bar{j}} + e\delta_j^h \psi_{i\bar{k}} + 2e\delta_i^h \psi_{j\bar{k}}; \\ (b) \quad \bar{R}_{ij} &= R_{ij} - (n+2)\psi_{ij}, \end{aligned}$$

where $\psi_{ij} = \psi_{i,j} - \psi_i \psi_j + \psi_{\bar{i}} \psi_{\bar{j}}$ ($\psi_{ij} = \psi_{ji} = -e\psi_{\bar{i}\bar{j}}$).

The tensor of holomorphically projective curvature, which is defined in the following form

$$(8) \quad P_{ijk}^h = R_{ijk}^h + \frac{1}{n+2} (\delta_k^h R_{ij} - \delta_j^h R_{ik} - e\delta_k^h R_{i\bar{j}} + e\delta_j^h R_{i\bar{k}} + 2e\delta_i^h R_{j\bar{k}}),$$

is invariant with respect to holomorphically projective mappings, i.e. $\bar{P}_{ijk}^h = P_{ijk}^h$.

The integrability conditions of equations (4) have the following form

$$(9) \quad \begin{aligned} a_{i\alpha}R_{jkl}^\alpha + a_{j\alpha}R_{ikl}^\alpha &= g_{ik}\lambda_{j,l} + g_{jk}\lambda_{i,l} - g_{il}\lambda_{j,k} - g_{jl}\lambda_{i,k} \\ &\quad - eg_{ik}\lambda_{\bar{j},l} - eg_{jk}\lambda_{\bar{i},l} + eg_{il}\lambda_{\bar{j},k} + eg_{jl}\lambda_{\bar{i},k}. \end{aligned}$$

We make the remark that the formulas introduced above, (7), (8) and (9), are not valid when $K_n \notin C^2$ or $\bar{K}_n \notin C^2$.

After contraction with g^{jk} we get:

$$a_{i\alpha}R_k^\alpha + a_{\alpha\beta}R_{ik}^{\alpha\beta} = e\lambda_{\bar{i},\bar{k}} - (n - 1)\lambda_{i,k},$$

where $R^\alpha_{il}{}^\beta = g^{\beta k}R^\alpha_{ilk}$; $R_l^\alpha = g^{\alpha j}R_{jl}$ and $\mu = \lambda_{\alpha,\beta}g^{\alpha\beta}$.

We contract this formula with $F_i^j F_k^r$ and from the properties of the Riemann and the Ricci tensors of K_n we obtain

$$(10) \quad \lambda_{\bar{i},\bar{k}} = -e\lambda_{i,k},$$

and ([3, 9, 10, 15])

$$(11) \quad n\lambda_{i,k} = \mu g_{ik} + a_{i\alpha}R_k^\alpha + a_{\alpha\beta}R_{ik}^{\alpha\beta}.$$

Because λ_i is a gradient-like covector, from equation (11) follows $a_{i\alpha}R_j^\alpha = a_{j\alpha}R_i^\alpha$.

From (10) follows that the vector field $\lambda_{\bar{i}} (\equiv \lambda_\alpha F_i^\alpha)$ is a Killing vector field, i.e. $\lambda_{\bar{i},j} + \lambda_{\bar{j},i} = 0$.

5. HOLOMORPHICALLY PROJECTIVE MAPPINGS BETWEEN $K_n \in C^r$ ($r > 2$) AND $\bar{K}_n \in C^2$

We proof the following theorem

Theorem 1. *If $K_n \in C^r$ ($r > 2$) admits holomorphically projective mappings onto $\bar{K}_n \in C^2$, then $\bar{K}_n \in C^r$.*

The proof of this theorem follows from the following lemmas.

Lemma 1 (see [4]). *Let $\lambda^h \in C^1$ be a vector field and ϱ a function. If*

$$(12) \quad \partial_i \lambda^h - \varrho \delta_i^h \in C^1$$

then $\lambda^h \in C^2$ and $\varrho \in C^1$.

In a similar way we can prove the following: *if $\lambda^h \in C^r$ ($r \geq 1$) and $\partial_i \lambda^h - \varrho \delta_i^h \in C^r$ then $\lambda^h \in C^{r+1}$ and $\varrho \in C^r$.*

Lemma 2. *If $K_n \in C^3$ admits a holomorphically projective mapping onto $\bar{K}_n \in C^2$, then $\bar{K}_n \in C^3$.*

Proof. In this case equations (4) and (11) hold. According to the assumptions $g_{ij} \in C^3$ and $\bar{g}_{ij} \in C^2$. By a simple check-up we find $\Psi \in C^2$, $\psi_i \in C^1$, $a_{ij} \in C^2$, $\lambda_i \in C^1$ and $R_{ijk}^h, R^h_{ij}{}^k, R_{ij}, R_i^h \in C^1$.

From the above-mentioned conditions we easily convince ourselves that we can write equation (11) in the form (12), where

$$\lambda^h = g^{h\alpha}\lambda_\alpha \in C^1, \varrho = \mu/n \text{ and } f_i^h = \frac{1}{n}(-\lambda^\alpha \Gamma_{\alpha i}^h - g^{h\gamma} a_{\alpha\gamma} R_i^\alpha + g^{h\gamma} a_{\alpha\beta} R^\alpha_{i\gamma}{}^\beta) \in C^1.$$

From Lemma 1 follows that $\lambda^h \in C^2$, $\varrho \in C^1$, and evidently $\lambda_i \in C^2$. Differentiating (4) twice we convince ourselves that $a_{ij} \in C^3$. From this and formula (6) follows that also $\Psi \in C^3$ and $\bar{g}_{ij} \in C^3$. \square

Further we notice that for holomorphically projective mappings between e -Kähler manifolds K_n and \bar{K}_n of class C^3 holds the following third set of equations [6, 8, 9, 15, 10, 16]:

$$(13) \quad \mu_{,k} = 2\lambda_\alpha R_k^\alpha.$$

If $K_n \in C^r$ and $\bar{K}_n \in C^2$, then by Lemma 2, $\bar{K}_n \in C^3$ and (13) holds. Because the system (4), (11) and (13) is closed, we can differentiate equations (4) $(r-1)$ times. So we convince ourselves that $a_{ij} \in C^r$, and also $\bar{g}_{ij} \in C^r$ ($\equiv \bar{K}_n \in C^r$).

Remark. Moreover, in this case from equation (13) follows that the function $\mu \in C^{r-1}$.

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REFERENCES

- [1] Alekseevsky, D. V., *Pseudo-Kähler and para-Kähler symmetric spaces*. Handbook of pseudo-Riemannian geometry and supersymmetry, IRMA Lect. Math. Theor. Phys. **16** (2010), 703–729.
- [2] Beklemishev, D. V., *Differential geometry of spaces with almost complex structure*, Akad. Nauk SSSR Inst. Naučn. Informacii, Moscow, 1965 Geometry (1963), 165–212.
- [3] Domashev, V. V., Mikeš, J., *Theory of holomorphically projective mappings of Kählerian spaces*, Math. Notes **23** (1978), 160–163.
- [4] Hinterleitner, I., Mikeš, J., *Geodesic mappings and Einstein spaces*, Geometric Methods in Physics, XXX Workshop in Białowieża, Poland 2011, arXiv:1201.2827v1 [math.DG], 2012.
- [5] Kähler, E., *Über eine bemerkenswerte Hermitesche Metrik*, Sem. Hamburg. Univ. **9** (1933), 173–186.
- [6] Kurbatova, I. N., *HP-mappings of H-spaces*, Ukrain. Geom. Sb. **27** (1984), 75–83.
- [7] Mikeš, J., *Geodesic and holomorphically projective mappings of special Riemannian spaces*, Ph.D. thesis, Odessa, 1979.
- [8] Mikeš, J., *On holomorphically projective mappings of Kählerian spaces*, Ukrain. Geom. Sb. **23** (1980), 90–98.
- [9] Mikeš, J., *Holomorphically projective mappings and their generalizations*, J. Math. Sci. **89** (1998), 1334–1353.
- [10] Mikeš, J., Vanžurová, A., Hinterleitner, I., *Geodesic mappings and some generalizations*, Palacky University Press, Olomouc, 2009.
- [11] Otsuki, T., Tashiro, Y., *On curves in Kaehlerian spaces*, Math. J. Okayama Univ. **4** (1954), 57–78.
- [12] Prvanović, M., *Holomorphically projective transformations in a locally product space*, Math. Balkanica **1** (1971), 195–213.
- [13] Rashevsky, P. A., *Scalar fields in fibered space*, Trudy Sem. Vektor. Tenzor. Anal. **6** (1948), 225–248.
- [14] Shirokov, P. A., *Selected investigations on geometry*, Kazan' University Press, 1966.

- [15] Sinyukov, N. S., *Geodesic mappings of Riemannian spaces*, Moscow, Nauka, 1979.
- [16] Sinyukov, N. S., Kurbatova, I. N., Mikeš, J., *Holomorphically projective mappings of Kähler spaces*, Odessa, Odessk. Univ., 1985.
- [17] Škodová, M., Mikeš, J., Pokorná, O., *On holomorphically projective mappings from equiaffine symmetric and recurrent spaces onto Kählerian spaces*, Rend. Circ. Mat. Palermo (2) Suppl. **75** (2005), 309–316.
- [18] Tachibana, S.-I., Ishihara, S., *On infinitesimal holomorphically projective transformations in Kählerian manifolds*, Tôhoku Math. J. (2) **12** (1960), 77–101.
- [19] Yano, K., *Differential geometry of complex and almost complex spaces*, Pergamon Press, 1965.

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