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INITIAL BOUNDARY VALUE PROBLEM FOR GENERALIZED ZAKHAROV EQUATIONS*

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Abstract. This paper considers the existence and uniqueness of the solution to the initial boundary value problem for a class of generalized Zakharov equations in $(2+1)$ dimensions, and proves the global existence of the solution to the problem by a priori integral estimates and the Galerkin method.

Keywords: global solutions, modified Zakharov equations

MSC 2010: 35Q40

1. Introduction

The Zakharov equations, derived by Zakharov in 1972 [11], describe the propagation of Langmuir waves in an unmagnetized plasma. The usual Zakharov system defined in space-time $\mathbb{R}^{d+1}$ is given by

$$\mathrm{i}E_t + \Delta E = nE, \tag{1.1}$$

$$n_{tt} - \Delta n = \Delta |E|^2, \tag{1.2}$$

where $E\colon \mathbb{R}^{1+d} \to \mathbb{C}^d$ is the slowly varying amplitude of the high-frequency electric field, and $n\colon \mathbb{R}^{1+d} \to \mathbb{R}$ denotes the fluctuation of the ion-density from its equilibrium.

In the past decades, the Zakharov system was studied by many authors [2]–[5], [9]–[10]. In [2], B. Guo, J. Zhang, and X. Pu established global in time existence

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and uniqueness of smooth solution for a generalized Zakharov equation in the two dimensional case for small initial data, and proved global existence of a smooth solution in one spatial dimension without any smallness assumption for the initial data. F. Linares and C. Matheus [4] obtained global well-posedness results for the initial-value problem associated with the 1D Zakharov-Rubenchik system, and showed that the results are sharp in some situations by proving ill-posedness results otherwise. In [5], F. Linares and J.-C. Saut proved that the Cauchy problem for the three-dimensional Zakharov-Kuznetsov equation is locally well-posed for data in $H^s(\mathbb{R}^3)$, $s > \frac{9}{8}$. If $0 < p < 4$, the existence and uniqueness of the global classical solution for a generalized Zakharov equation were obtained in [10]. The nonlinear Schrödinger limit of the Zakharov equation was discussed in [7]–[8].

By using a quantum fluid approach, the following modified Zakharov equations are obtained [1]:

$$\mathrm{i}E_t + E_{xx} - H^2 E_{xxxx} = nE, \tag{1.3}$$

$$n_{tt} - n_{xx} + H^2 n_{xxxx} = |E|^2_{xx}, \tag{1.4}$$

where H is the dimensionless quantum parameter given by the ratio of the ion plasmon and electron thermal energies. The presence of a large value of H points to the possible experimental manifestation of quantum effects in the coupling between Langmuir and ion-acoustic modes in dense plasmas. Quantum effects may imply important consequences in the behavior of high density astrophysical plasmas. In this case, quantum effects cause an overall reduction in the wave-wave interaction level. This focusing effect may extend to quite long periods of time, indicating that the recurrence properties verified in the classical Zakharov equation are enhanced by the quantum effects.

In the present paper we are interested in studying the generalized Zakharov system

$$\mathrm{i}E_t + \Delta E - H^2\Delta^2 E - nE = 0, \tag{1.5}$$

$$n_{tt} - \Delta n + H^2\Delta^2 n - \Delta|E|^2 = 0, \tag{1.6}$$

with initial data

$$E|_{t=0} = E_0(x), \quad n|_{t=0} = n_0(x), \quad n_t|_{t=0} = n_1(x), \tag{1.7}$$

and boundary conditions

$$E|_{\partial\Omega} = \frac{\partial E}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial(\nabla E)}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad t \geqslant 0, \tag{1.8}$$

$$n|_{\partial\Omega} = 0, \quad \frac{\partial(\nabla n)}{\partial \nu}\Big|_{\partial\Omega} = 0, \tag{1.9}$$

where E is a complex valued unknown function vector, n is a real unknown function, H is a real constant and Ω is a bounded domain in $\mathbb{R}^2$ satisfying $\partial\Omega \in C^2$. We mainly consider the existence and uniqueness of the solution to the system.

To study the smooth solution of the modified Zakharov system, we transform it into the form

$$\mathrm{i}E_t + \Delta E - H^2\Delta^2 E - nE = 0, \tag{1.10}$$

$$n_t - \Delta\varphi = 0, \quad x \in \Omega, \tag{1.11}$$

$$\varphi_t - n + H^2\Delta n - |E|^2 = 0, \tag{1.12}$$

with initial data

$$E|_{t=0} = E_0(x), \quad n|_{t=0} = n_0(x), \quad \varphi|_{t=0} = \varphi_0(x), \tag{1.13}$$

and boundary conditions

$$E|_{\partial\Omega} = \frac{\partial E}{\partial\nu}\Big|_{\partial\Omega} = \frac{\partial(\nabla E)}{\partial\nu}\Big|_{\partial\Omega} = 0, \quad t \geqslant 0, \tag{1.14}$$

$$n|_{\partial\Omega} = 0, \quad \frac{\partial(\nabla n)}{\partial\nu}\Big|_{\partial\Omega} = 0, \quad \varphi|_{\partial\Omega} = 0, \tag{1.15}$$

where φ is a real unknown function.

For the sake of convenience of the reader, we introduce some notation. For $1 \leqslant q \leqslant \infty$, we denote by $L^q(\Omega)$ the space of all q-times integrable functions in Ω equipped with the norm $\|\cdot\|_{L^q(\Omega)}$ or simply $\|\cdot\|_{L^q}$, and by $H^{s,p}(\Omega)$ the Sobolev space with the norm $\|\cdot\|_{H^{s,p}(\Omega)}$. If $p = 2$, we write $H^s(\Omega)$ instead of $H^{s,2}(\Omega)$. Let $(f,g) = \int_\Omega f(x)\cdot\overline{g(x)}\,\mathrm{d}x$, where $\overline{g(x)}$ denotes the complex conjugate function of $g(x)$. Now we state the main results of the paper.

Theorem 1.1. *Suppose that $E_0(x) \in H^4(\Omega) \cap H_0^2(\Omega)$, $n_0(x) \in H_0^2(\Omega)$, $\varphi_0(x) \in H^2(\Omega)$. Then there exists a global generalized solution of the initial boundary value problem* (1.10)–(1.15),

$$\begin{aligned} \Delta^2 E(x,t) &\in L^\infty(0,T;L^2(\Omega)),\\ E_t(x,t) &\in L^\infty(0,T;L^2(\Omega)),\\ n(x,t) &\in L^\infty(0,T;H^2(\Omega)),\\ n_t(x,t) &\in L^\infty(0,T;L^2(\Omega)),\\ \Delta\varphi(x,t) &\in L^\infty(0,T;L^2(\Omega)),\\ \varphi_t(x,t) &\in L^\infty(0,T;L^2(\Omega)). \end{aligned}$$

Theorem 1.2. *Suppose that the conditions of Theorem* 1.1 *are satisfied. Then the global generalized solution of the initial boundary value problem* (1.10)–(1.12) *is unique.*

Theorem 1.3. *Suppose that* $E_0(x) \in H^{4+2l}(\Omega)\cap H_0^2(\Omega)$, $n_0(x) \in H^{2l}(\Omega)\cap H_0^2(\Omega)$, $\varphi_0(x) \in H^{2l}(\Omega)$, $l \geqslant 3$. *Then there exists a unique global smooth solution of the initial boundary value problem* (1.10)–(1.15),

$$\begin{aligned}
\Delta^4 E(x,t) &\in L^\infty(0,T;H^{2l-4}(\Omega)),\\
\Delta^2 E_t(x,t) &\in L^\infty(0,T;H^{2l-4}(\Omega)),\\
\Delta^2 n(x,t) &\in L^\infty(0,T;H^{2l-4}(\Omega)),\\
\Delta^2 n_t(x,t) &\in L^\infty(0,T;H^{2l-6}(\Omega)),\\
\Delta\varphi(x,t) &\in L^\infty(0,T;H^{2l-2}(\Omega)),\\
\Delta\varphi_t(x,t) &\in L^\infty(0,T;H^{2l-4}(\Omega)).
\end{aligned}$$

Theorem 1.4. *Suppose that* $E_0(x) \in H^{4+2l}(\Omega)\cap H_0^2(\Omega)$, $n_0(x) \in H^{2l}(\Omega)\cap H_0^2(\Omega)$, $n_1(x) \in H^{2l-2}(\Omega)$, $l \geqslant 3$. *Then there exists a unique global smooth solution of the initial boundary value problem* (1.5)–(1.9),

$$\begin{aligned}
\Delta^4 E(x,t) &\in L^\infty(0,T;H^{2l-4}(\Omega)),\\
\Delta^2 E_t(x,t) &\in L^\infty(0,T;H^{2l-4}(\Omega)),\\
\Delta^2 n(x,t) &\in L^\infty(0,T;H^{2l-4}(\Omega)),\\
\Delta^2 n_t(x,t) &\in L^\infty(0,T;H^{2l-6}(\Omega)),\\
n_{tt}(x,t) &\in L^\infty(0,T;H^{2l-4}(\Omega)).
\end{aligned}$$

The paper is organized as follows: In Section 2, we establish a priori estimates of the problem (1.10)–(1.15). In Section 3, first of all, we obtain the existence and uniqueness of the global smooth solution of the problem (1.10)–(1.15) by the Galerkin method [6]. Next, the existence and uniqueness of the global smooth solution of the problem (1.5)–(1.9) are obtained.

2. A priori estimations of problem (1.10)–(1.15)

Lemma 2.1. *Suppose that* $E_0(x) \in L^2(\Omega)$. *Then for the solution of problem* (1.10)–(1.15) *we have*

$$\|E(\cdot,t)\|^2_{L^2(\Omega)} = \|E_0(x)\|^2_{L^2(\Omega)}.$$

P r o o f. Taking the inner product of (1.10) and E, we infer that

$$(\mathrm{i}E_t + \Delta E - H^2\Delta^2 E - nE, E) = 0. \tag{2.1}$$

By virtue of

$$\operatorname{Im}(\mathrm{i}E_t, E) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E\|^2_{L^2}$$

and by Green's formula,

$$\begin{aligned}
&\operatorname{Im}(\Delta E - H^2\Delta E - nE, E) \\
&= \operatorname{Im}\int_\Omega (\Delta E - H^2\Delta^2 E - nE)\overline{E}\,\mathrm{d}x \\
&= \operatorname{Im}\Big[\int_{\partial\Omega}\frac{\partial E}{\partial\nu}\,\overline{E}\,\mathrm{d}s - \int_\Omega \nabla E\cdot\nabla\overline{E}\,\mathrm{d}x + H^2\int_\Omega \Delta E\Delta\overline{E}\,\mathrm{d}x \\
&\quad + H^2\int_{\partial\Omega}\Big(\frac{\partial(\Delta E)}{\partial\nu}\,\overline{E} - \Delta E\frac{\partial E}{\partial\nu}\Big)\,\mathrm{d}s - \int_\Omega nE\overline{E}\,\mathrm{d}x\Big] \\
&= \operatorname{Im}\Big[-\|\nabla E\|^2_{L^2} + H^2\|\Delta E\|^2_{L^2} - \int_\Omega n|E|^2\,\mathrm{d}x\Big] = 0,
\end{aligned} \tag{2.2}$$

hence from (2.1) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|E(\cdot,t)\|^2_{L^2} = 0,$$

i.e.

$$\|E(\cdot,t)\|^2_{L^2} = \|E_0(x)\|^2_{L^2}.$$

□

Lemma 2.2 (Sobolev's inequality)**.** *Assume that* $u \in L^q(\Omega)$, $D^m u \in L^r(\Omega)$, $1 \leqslant q, r \leqslant \infty$, $0 \leqslant j \leqslant m$. *Then we have the estimates*

$$\|D^j u\|_{L^p(\Omega)} \leqslant C\|u\|^\alpha_{W^{m,r}(\Omega)}\|u\|^{1-\alpha}_{L^q(\Omega)},$$

where C *is a positive constant,* $0 \leqslant j/m \leqslant \alpha \leqslant 1$,

$$\frac{1}{p} = \frac{j}{n} + \alpha\Big(\frac{1}{r} - \frac{m}{n}\Big) + (1-\alpha)\frac{1}{q}.$$

Lemma 2.3 (Gronwal's inequality). *Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies for a.e. t the differential inequality*

$$\eta'(t) \leqslant \phi(t)\eta(t) + \psi(t),$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0,T]$. Then

$$\eta(t) \leqslant \mathrm{e}^{\int_0^t \psi(s\tau)\,\mathrm{d}\tau}\Big[\eta(0) + \int_0^t \psi(\tau)\,\mathrm{d}\tau\Big]$$

for all $0 \leqslant t \leqslant T$.

Lemma 2.4. *Suppose that $E_0(x) \in H^2(\Omega)$, $n_0(x) \in H^1(\Omega)$, $\varphi_0(x) \in H^1(\Omega)$. Then we have*

$$\sup_{0\leqslant t\leqslant T}\big[\|E\|_{H^2}^2 + \|n\|_{H^1}^2 + \|\varphi\|_{H^1}^2\big] \leqslant C_1.$$

Proof. Taking the inner products of (1.10) and $-E_t$, we infer that

$$(\mathrm{i}E_t + \Delta E - H^2\Delta^2 E - nE, -E_t) = 0. \tag{2.3}$$

Since

$$\mathrm{Re}(\mathrm{i}E_t, -E_t) = 0,$$

$$\mathrm{Re}(\Delta E, -E_t) = \mathrm{Re}\Big(-\int_{\partial\Omega}\frac{\partial E}{\partial\nu}\overline{E}_t\,\mathrm{d}s + \int_\Omega \nabla E\cdot\nabla\overline{E}_t\,\mathrm{d}x\Big) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla E\|_{L^2}^2,$$

$$\begin{aligned}\mathrm{Re}(-H^2\Delta^2E, -E_t) &= \mathrm{Re}\Big(H^2\int_{\partial\Omega}\frac{\partial(\Delta E)}{\partial\nu}\,\overline{E}_t\,\mathrm{d}s - H^2\int_\Omega \nabla(\Delta E)\cdot\nabla\overline{E}_t\,\mathrm{d}x\Big)\\ &= \mathrm{Re}\Big(-H^2\int_{\partial\Omega}\frac{\partial(\nabla E)}{\partial\nu}\cdot\nabla\overline{E}_t\,\mathrm{d}s + H^2\int_\Omega \nabla(\nabla E)\cdot\nabla(\nabla\overline{E}_t)\,\mathrm{d}x\Big)\\ &= \frac{H^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[\|E_{x_1x_1}\|_{L^2}^2 + 2\|E_{x_1x_2}\|_{L^2}^2 + \|E_{x_2x_2}\|_{L^2}^2\big],\end{aligned}$$

$$\begin{aligned}\mathrm{Re}(-nE, -E_t) &= \frac{1}{2}\int_\Omega n|E|_t^2\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n|E|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega n_t|E|^2\,\mathrm{d}x\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n|E|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega n_t(\varphi_t - n + H^2\Delta n)\,\mathrm{d}x\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n|E|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega n_t\varphi_t\,\mathrm{d}x + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|n\|_{L^2}^2 + \frac{H^2}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n\|_{L^2}^2\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n|E|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega \Delta\varphi\varphi_t\,\mathrm{d}x + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|n\|_{L^2}^2 + \frac{H^2}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n\|_{L^2}^2\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n|E|^2\,\mathrm{d}x + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\varphi\|_{L^2}^2 + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|n\|_{L^2}^2 + \frac{H^2}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n\|_{L^2}^2,\end{aligned}$$

hence from (2.3) it follows that

$$\text{(2.4)}\quad \frac{\mathrm{d}}{\mathrm{d}t}\Big[\|\nabla E\|_{L^2}^2 + H^2(\|E_{x_1x_1}\|_{L^2}^2 + 2\|E_{x_1x_2}\|_{L^2}^2 + \|E_{x_2x_2}\|_{L^2}^2) + \int_\Omega n|E|^2\,\mathrm{d}x + \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\|n\|_{L^2}^2 + \frac{H^2}{2}\|\nabla n\|_{L^2}^2\Big] = 0.$$

Letting

$$w(t) = \|\nabla E\|_{L^2}^2 + H^2(\|E_{x_1x_1}\|_{L^2}^2 + 2\|E_{x_1x_2}\|_{L^2}^2 + \|E_{x_2x_2}\|_{L^2}^2) + \int_\Omega n|E|^2\,\mathrm{d}x + \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\|n\|_{L^2}^2 + \frac{H^2}{2}\|\nabla n\|_{L^2}^2$$

and taking into account (2.4) it follows that

$$\text{(2.5)}\qquad w(t) = w(0).$$

By using Young's inequality with ε: $ab \leqslant \frac{1}{p}\varepsilon a^p + \frac{1}{q}b^q\varepsilon^{-q/p}(p^{-1}+q^{-1}-1)$, and the Sobolev inequality, it follows that

$$\text{(2.6)}\quad \left|\int_\Omega n|E|^2\,\mathrm{d}x\right| \leqslant \frac{1}{4}\|n\|_{L^2}^2 + \|E\|_{L^4}^4 \leqslant \frac{1}{4}\|n\|_{L^2}^2 + C\|E\|_{H^2}\|E\|_{L^2}^3 \leqslant \frac{1}{4}\|n\|_{L^2}^2 + \frac{H^2}{2}\big[\|E_{x_1x_1}\|_{L^2}^2 + \|E_{x_1x_2}\|_{L^2}^2 + \|E_{x_2x_2}\|_{L^2}^2\big] + C.$$

Hence, from (2.5) we get

$$\|\nabla E\|_{L^2}^2 + \|E_{x_1x_1}\|_{L^2}^2 + \|E_{x_1x_2}\|_{L^2}^2 + \|E_{x_2x_2}\|_{L^2}^2 + \|\nabla\varphi\|_{L^2}^2 + \|n\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 \leqslant \text{const.}$$

□

Lemma 2.5. *Suppose that $E_0(x) \in H^4(\Omega)$, $n_0(x) \in H^2(\Omega)$, $\varphi_0(x) \in H^2(\Omega)$. Then we have*

$$\sup_{0\leqslant t\leqslant T}\big[\|\Delta^2 E\|_{L^2}^2 + \|n\|_{H^2}^2 + \|\Delta\varphi\|_{L^2}^2 + \|E_t\|_{L^2}^2 + \|n_t\|_{L^2}^2 + \|\varphi_t\|_{L^2}^2\big] \leqslant C_2.$$

P r o o f. Differentiating (1.10) with respect to t, then taking the inner products of the resulting equation and E_t, we obtain that

$$\text{(2.7)}\qquad (\mathrm{i}E_{tt} + \Delta E_t - H^2\Delta^2 E_t - n_t E - nE_t, E_t) = 0.$$

Since

$$\operatorname{Im}(\mathrm{i}E_{tt}, E_t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E_t\|_{L^2}^2, \quad \operatorname{Im}(\Delta E_t - H^2\Delta^2 E_t - nE_t, E_t) = 0,$$
$$|\operatorname{Im}(-n_t E, E_t)| \leqslant C\|E\|_{L^\infty}\|n_t\|_{L^2}\|E_t\|_{L^2} \leqslant C(\|E_t\|_{L^2}^2 + \|n_t\|_{L^2}^2),$$

from (2.7) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|E_t\|_{L^2}^2 \leqslant C(\|E_t\|_{L^2}^2 + \|n_t\|_{L^2}^2).$$

Differentiating (1.11) with respect to t, then taking the inner products of the resulting equation and n_t, we conclude that

$$(n_{tt} - \Delta\varphi_t, n_t) = 0. \tag{2.9}$$

Since

$$(n_{tt}, n_t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n_t\|_{L^2}^2, \quad (-\Delta\varphi_t, n_t) = (-\Delta n + H^2\Delta^2 n - \Delta|E|^2, n_t),$$
$$(-\Delta n, n_t) = -\int_{\partial\Omega}\frac{\partial n}{\partial\nu}n_t\,\mathrm{d}s + \int_\Omega \nabla n\cdot\nabla n_t\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n\|_{L^2}^2,$$
$$\begin{aligned}(H^2\Delta^2 n, n_t) &= H^2\int_{\partial\Omega}\frac{\partial(\Delta n)}{\partial\nu}n_t\,\mathrm{d}s - H^2\int_\Omega\nabla(\Delta n)\cdot\nabla n_t\,\mathrm{d}x\\ &= -H^2\int_{\partial\Omega}\frac{\partial(\nabla n)}{\partial\nu}\cdot\nabla n_t\,\mathrm{d}s + H^2\int_\Omega\nabla(\nabla n)\cdot\nabla(\nabla n_t)\,\mathrm{d}x\\ &= \frac{H^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[\|n_{x_1x_1}\|_{L^2}^2 + 2\|n_{x_1x_2}\|_{L^2}^2 + \|n_{x_2x_2}\|_{L^2}^2\big],\end{aligned}$$
$$\begin{aligned}|(-\Delta|E|^2, n_t)| &\leqslant C(\|E\|_{L^\infty}\|E_{x_1x_1}\|_{L^2} + \|E_{x_1}\|_{L^\infty}\|E_{x_1}\|_{L^2}\\ &\quad + \|E\|_{L^\infty}\|E_{x_2x_2}\|_{L^2} + \|E_{x_2}\|_{L^\infty}\|E_{x_2}\|_{L^2})\|n_t\|_{L^2}\\ &\leqslant C(\|n_t\|_{L^2}^2 + 1),\end{aligned}$$

equality (2.9) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big[\|n_t\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + H^2(\|n_{x_1x_1}\|_{L^2}^2 + 2\|n_{x_1x_2}\|_{L^2}^2 + \|n_{x_2x_2}\|_{L^2}^2)\big]\\ \leqslant C(\|n_t\|_{L^2}^2 + 1).\end{aligned} \tag{2.10}$$

From (2.8) and (2.10) it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big[\|E_t\|_{L^2}^2 + \|n_t\|_{L^2}^2 + \|\nabla n\|_{L^2}^2\\ + H^2(\|n_{x_1x_1}\|_{L^2}^2 + 2\|n_{x_1x_2}\|_{L^2}^2 + \|n_{x_2x_2}\|_{L^2}^2)\big]\\ \leqslant C(\|n_t\|_{L^2}^2 + \|E_t\|_{L^2}^2 + 1).\end{aligned} \tag{2.11}$$

By using Gronwall's inequality, we obtain

$$\|E_t\|_{L^2}^2 + \|n_t\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + \|n_{x_1x_1}\|_{L^2}^2 + \|n_{x_1x_2}\|_{L^2}^2 + \|n_{x_2x_2}\|_{L^2}^2 \leqslant \text{const.}$$

From (1.10), (1.11), and (1.12) we get

$$\|\Delta^2 E\|_{L^2} \leqslant \text{const.}, \quad \|\Delta\varphi\|_{L^2} \leqslant \text{const.}, \quad \|\varphi_t\|_{L^2} \leqslant \text{const.}$$

□

Lemma 2.6. *Suppose that $E_0(x) \in H^8(\Omega)$, $n_0(x) \in H^4(\Omega)$, $\varphi_0(x) \in H^4(\Omega)$. Then we have*

$$\sup_{0\leqslant t\leqslant T} \left[\|E_t\|_{H^2}^2 + \|n_t\|_{H^2}^2 + \|E_{tt}\|_{L^2}^2 + \|n_{tt}\|_{L^2}^2\right] \leqslant C_3.$$

Proof. Differentiating (1.10) twice with respect to t, then taking the inner products of the resulting equation and E_{tt}, we obtain that

$$(\mathrm{i}E_{ttt} + \Delta E_{tt} - H^2\Delta^2 E_{tt} - n_{tt}E - 2n_tE_t - nE_{tt}, E_{tt}) = 0. \tag{2.12}$$

Since

$$\begin{aligned}
\operatorname{Im}(\mathrm{i}E_{ttt}, E_{tt}) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E_{tt}\|_{L^2}^2, \quad \operatorname{Im}(\Delta E_{tt} - H^2\Delta^2 E_{tt} - nE_{tt}, E_{tt}) = 0,\\
|\operatorname{Im}(-n_{tt}E, E_{tt})| &\leqslant C\|E\|_{L^\infty}\|n_{tt}\|_{L^2}\|E_{tt}\|_{L^2} \leqslant C(\|E_{tt}\|_{L^2}^2 + \|n_{tt}\|_{L^2}^2),\\
|\operatorname{Im}(-2n_tE_t, E_{tt})| &\leqslant C\|E_t\|_{L^\infty}\|n_t\|_{L^2}\|E_{tt}\|_{L^2}\\
&\leqslant C(\|E_{tt}\|_{L^2}^2 + \|E_t\|_{H^1}^2) \leqslant C(\|E_{tt}\|_{L^2}^2 + \|\nabla E_t\|_{L^2}^2),
\end{aligned}$$

from (2.12) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|E_{tt}\|_{L^2}^2 \leqslant C(\|E_{tt}\|_{L^2}^2 + \|n_{tt}\|_{L^2}^2 + \|\nabla E_t\|_{L^2}^2). \tag{2.13}$$

Differentiating (1.11) twice with respect to t, then taking the inner products of the resulting equation and n_{tt}, we obtain that

$$(n_{ttt} - \Delta\varphi_{tt}, n_{tt}) = 0. \tag{2.14}$$

Since

$$\begin{aligned}
(n_{ttt}, n_{tt}) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n_{tt}\|_{L^2}^2, \quad (-\Delta\varphi_{tt}, n_{tt}) = (-\Delta n_t + H^2\Delta^2 n_t - \Delta|E|_t^2, n_{tt}),\\
(-\Delta n_t, n_{tt}) &= -\int_{\partial\Omega}\frac{\partial(n_t)}{\partial\nu}n_{tt}\,\mathrm{d}s + \int_\Omega \nabla n_t\cdot\nabla n_{tt}\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n_t\|_{L^2}^2,
\end{aligned}$$

$$(H^2\Delta^2 n_t, n_{tt}) = H^2\int_{\partial\Omega}\frac{\partial(\Delta n_t)}{\partial\nu}\, n_{tt}\,\mathrm{d}s - H^2\int_\Omega \nabla(\Delta n_t)\cdot\nabla n_{tt}\,\mathrm{d}x$$

$$= -H^2\int_{\partial\Omega}\frac{\partial(\nabla n_t)}{\partial\nu}\cdot\nabla n_{tt}\,\mathrm{d}s + H^2\int_\Omega \nabla(\nabla n_t)\cdot\nabla(\nabla n_{tt})\,\mathrm{d}x$$

$$= \frac{H^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[\|n_{x_1x_1t}\|_{L^2}^2 + 2\|n_{x_1x_2t}\|_{L^2}^2 + \|n_{x_2x_2t}\|_{L^2}^2\big],$$

$$|(-\Delta|E|_t^2, n_{tt})|$$

$$\leqslant C\big(\|E\|_{L^\infty}\|E_{x_1x_1t}\|_{L^2} + \|E_t\|_{L^\infty}\|E_{x_1x_1}\|_{L^2} + \|E_{x_1}\|_{L^\infty}\|E_{x_1t}\|_{L^2}$$

$$+ \|E\|_{L^\infty}\|E_{x_2x_2t}\|_{L^2} + \|E_t\|_{L^\infty}\|E_{x_2x_2}\|_{L^2} + \|E_{x_2}\|_{L^\infty}\|E_{x_2t}\|_{L^2}\big)\|n_{tt}\|_{L^2}$$

$$\leqslant C\big(\|E_{x_1x_1t}\|_{L^2}^2 + \|E_t\|_{H^1}^2 + \|E_{x_1t}\|_{L^2}^2 + \|E_{x_2x_2t}\|_{L^2}^2 + \|E_{x_2t}\|_{L^2}^2 + \|n_{tt}\|_{L^2}^2\big)$$

$$\leqslant C(\|E_{x_1x_1t}\|_{L^2}^2 + \|E_{x_2x_2t}\|_{L^2}^2 + \|\nabla E_t\|_{L^2}^2 + \|n_{tt}\|_{L^2}^2),$$

equality (2.14) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\|n_{tt}\|_{L^2}^2 + \|\nabla n_t\|_{L^2}^2 + H^2(\|n_{x_1x_1t}\|_{L^2}^2 + 2\|n_{x_1x_2t}\|_{L^2}^2 + \|n_{x_2x_2t}\|_{L^2}^2)\big]$$
$$\leqslant C(\|E_{x_1x_1t}\|_{L^2}^2 + \|\nabla E_t\|_{L^2}^2 + \|E_{x_2x_2t}\|_{L^2}^2 + \|n_{tt}\|_{L^2}^2). \tag{2.15}$$

Differentiating (1.10) with respect to t, then taking the inner products of the resulting equation and $-E_{tt}$, we conclude that

$$(\mathrm{i}E_{tt} + \Delta E_t - H^2\Delta^2 E_t - n_tE - nE_t, -E_{tt}) = 0. \tag{2.16}$$

Since

$$\mathrm{Re}(\mathrm{i}E_{tt}, -E_{tt}) = 0,$$

$$\mathrm{Re}(\Delta E_t -, E_{tt}) = \mathrm{Re}\Big(-\int_{\partial\Omega}\frac{\partial E_t}{\partial\nu}\,\overline{E}_{tt}\,\mathrm{d}s + \int_\Omega \nabla E_t\cdot\nabla\overline{E}_{tt}\,\mathrm{d}x\Big) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla E_t\|_{L^2}^2,$$

$$\mathrm{Re}(-H^2\Delta^2 E_t, -E_{tt})$$

$$= \mathrm{Re}\Big(H^2\int_{\partial\Omega}\frac{\partial(\Delta E_t)}{\partial\nu}\,\overline{E}_{tt}\,\mathrm{d}s - H^2\int_\Omega\nabla(\Delta E_t)\cdot\nabla\overline{E}_{tt}\,\mathrm{d}x\Big)$$

$$= \mathrm{Re}\Big(-H^2\int_{\partial\Omega}\frac{\partial(\nabla E_t)}{\partial\nu}\cdot\nabla\overline{E}_{tt}\,\mathrm{d}s + H^2\int_\Omega\nabla(\nabla E_t)\cdot\nabla(\nabla\overline{E}_{tt})\,\mathrm{d}x\Big)$$

$$= \frac{H^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[\|E_{x_1x_1t}\|_{L^2}^2 + 2\|E_{x_1x_2t}\|_{L^2}^2 + \|E_{x_2x_2t}\|_{L^2}^2\big],$$

$$|\mathrm{Re}(-n_tE, -E_{tt})| \leqslant \|E\|_{L^\infty}\|n_t\|_{L^2}\|E_{tt}\|_{L^2} \leqslant C(\|E_{tt}\|_{L^2}^2 + 1),$$

$$|\mathrm{Re}(-nE_t, -E_{tt})| \leqslant C\|n\|_{L^\infty}\|E_t\|_{L^2}\|E_{tt}\|_{L^2} \leqslant C(\|E_{tt}\|_{L^2}^2 + 1),$$

from (2.16) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\|\nabla E_t\|_{L^2}^2+H^2(\|E_{x_1x_1t}\|_{L^2}^2+2\|E_{x_1x_2t}\|_{L^2}^2+\|E_{x_2x_2t}\|_{L^2}^2)\big]\leqslant C(\|E_{tt}\|_{L^2}^2+1). \tag{2.17}$$

From (2.13), (2.15), and (2.17) it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big[&\|E_{tt}\|_{L^2}^2+\|n_{tt}\|_{L^2}^2+\|\nabla n_t\|_{L^2}^2+\|\nabla E_t\|_{L^2}^2+\|n_{x_1x_1t}\|_{L^2}^2+\|n_{x_1x_2t}\|_{L^2}^2\\&+\|n_{x_2x_2t}\|_{L^2}^2+\|E_{x_1x_1t}\|_{L^2}^2+\|E_{x_1x_2t}\|_{L^2}^2+\|E_{x_2x_2t\|_{L^2}^2}\big]\\&\leqslant C(\|n_{tt}\|_{L^2}^2+\|E_{tt}\|_{L^2}^2+\|\nabla E_t\|_{L^2}^2+\|E_{x_1x_1t}\|_{L^2}^2\\&\quad+\|E_{x_2x_2t}\|^2L^2+1).\end{aligned} \tag{2.18}$$

By using Gronwall's inequality, we obtain

$$\begin{aligned}&\|E_{tt}\|_{L^2}^2+\|n_{tt}\|_{L^2}^2+\|\nabla n_t\|_{L^2}^2+\|\nabla E_t\|_{L^2}^2+\|n_{x_1x_1t}\|_{L^2}^2+\|n_{x_1x_2t}\|_{L^2}^2\\&\quad+\|n_{x_2x_2t}\|_{L^2}^2+\|E_{x_1x_1t}\|_{L^2}^2+\|E_{x_1x_2t}\|_{L^2}^2+\|E_{x_2x_2t\|_{L^2}^2}\leqslant \text{const.}\end{aligned}$$

From (1.10), (1.11), and (1.12) it follows that

$$\begin{aligned}&\|\Delta^4E\|_{L^2}\leqslant\text{const.},\quad \|\Delta^2n\|_{L^2}\leqslant\text{const.},\quad \|\Delta\varphi\|_{H^2}\leqslant\text{const.},\\&\|\Delta^2E_t\|_{L^2}\leqslant\text{const.},\quad \|\Delta\varphi_t\|_{L^2}\leqslant\text{const.},\quad \|\varphi_{tt}\|_{L^2}\leqslant\text{const.}\end{aligned}$$

□

Lemma 2.7. *Suppose that $E_0(x)\in H^{4+2l}(\Omega)$, $n_0(x)\in H^{2l}(\Omega)$, $\varphi_0(x)\in H^{2l}(\Omega)$, $l\geqslant 2$. Then we have*

$$\sup_{0\leqslant t\leqslant T}\big[\|\partial_t^{l-1}E\|_{H^2}^2+\|\partial_t^{l-1}n\|_{H^2}^2+\|\partial_t^{l}E\|_{L^2}^2+\|\partial_t^{l}n\|_{L^2}^2\big]\leqslant C.$$

P r o o f. Lemma 2.7 is true when $l=2$ (Lemma 2.6). Suppose Lemma 2.7 is true when $l=k$ ($k\geqslant 2$), i.e.

$$\sup_{0\leqslant t\leqslant T}\big[\|\partial_t^{k-1}E\|_{H^2}^2+\|\partial_t^{k-1}n\|_{H^2}^2+\|\partial_t^{k}E\|_{L^2}^2+\|\partial_t^{k}n\|_{L^2}^2\big]\leqslant \text{const.}$$

Differentiating (1.10) $(k+1)$-times with respect to t, then taking the inner products of the resulting equation and $\partial_t^{k+1}E$ yields that

$$(\mathrm{i}\partial_t^{k+2}E+\partial_t^{k+1}\Delta E-H^2\partial_t^{k+1}\Delta^2E-\partial_t^{k+1}(nE),\partial_t^{k+1}E)=0. \tag{2.19}$$

Since

$$\begin{aligned}
&\operatorname{Im}(\mathrm{i}\partial_t^{k+2}E,\partial_t^{k+1}E)=\frac12\,\frac{\mathrm d}{\mathrm dt}\,\|\partial_t^{k+1}E\|_{L^2}^2,\quad \operatorname{Im}(\partial_t^{k+1}\Delta E-H^2\partial_t^{k+1}\Delta^2E,\partial_t^{k+1}E)=0,\\
&\quad|\operatorname{Im}(\partial_t^{k+1}(nE),\partial_t^{k+1}E)|\\
&\qquad\leqslant C(\|E\|_{L^\infty}\|\partial_t^{k+1}n\|_{L^2}+\|E_t\|_{L^\infty}\|\partial_t^k n\|_{L^2}+\ldots+\|\partial_t^kE\|_{L^2}\|n_t\|_{L^\infty})\|\partial_t^{k+1}E\|_{L^2}\\
&\qquad\leqslant C(\|\partial_t^{k+1}E\|_{L^2}^2+\|\partial_t^{k+1}n\|_{L^2}^2+1),
\end{aligned}$$

from (2.19) we get

$$\frac{\mathrm d}{\mathrm dt}\,\|\partial_t^{k+1}E\|_{L^2}^2\leqslant C(\|\partial_t^{k+1}E\|_{L^2}^2+\|\partial_t^{k+1}n\|_{L^2}^2+1). \tag{2.20}$$

Differentiating (1.11) $(k+1)$-times with respect to t, then taking the inner products of the resulting equation and $\partial_t^{k+1}n$, we obtain that

$$(\partial_t^{k+2}n-\partial_t^{k+1}\Delta\varphi,\partial_t^{k+1}n)=0. \tag{2.21}$$

Since

$$\begin{aligned}
(\partial_t^{k+2}n,\partial_t^{k+1}n)&=\frac12\,\frac{\mathrm d}{\mathrm dt}\,\|\partial_t^{k+1}n\|_{L^2}^2,\\
(-\partial_t^{k+1}\Delta\varphi,\partial_t^{k+1}n)&=(-\partial_t^k\Delta n+H^2\partial_t^k\Delta^2n-\partial_t^k\Delta|E|^2,\partial_t^{k+1}n),\\
(-\partial_t^k\Delta n,\partial_t^{k+1}n)&=\int_{\partial\Omega}\frac{\partial(\partial_t^kn)}{\partial\nu}\,\partial_t^{k+1}n\,\mathrm ds+\int_\Omega\nabla(\partial_t^kn)\cdot\nabla(\partial_t^{k+1}n)\,\mathrm dx\\
&=\frac12\,\frac{\mathrm d}{\mathrm dt}\,\|\nabla\partial_t^kn\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
&(H^2\partial_t^k\Delta^2n,\partial_t^{k+1}n)\\
&\quad=H^2\int_{\partial\Omega}\frac{\partial(\partial_t^k\Delta n)}{\partial\nu}\,\partial_t^{k+1}n\,\mathrm ds-H^2\int_\Omega\nabla(\partial_t^k\Delta n)\cdot\nabla(\partial_t^{k+1}n)\,\mathrm dx\\
&\quad=-H^2\int_{\partial\Omega}\frac{\partial(\nabla\partial_t^kn)}{\partial\nu}\cdot\nabla(\partial_t^{k+1}n)\,\mathrm ds+H^2\int_\Omega\nabla(\nabla\partial_t^kn)\cdot\nabla(\nabla\partial_t^{k+1}n)\,\mathrm dx\\
&\quad=\frac{H^2}{2}\,\frac{\mathrm d}{\mathrm dt}\,\big[\|\partial_t^kn_{x_1x_1}\|_{L^2}^2+2\|\partial_t^kn_{x_1x_2}\|_{L^2}^2+\|\partial_t^kn_{x_2x_2}\|_{L^2}^2\big],\\
&|(-\partial_t^k\Delta|E|^2,\partial_t^{k+1}n)|\\
&\quad\leqslant C(\|E\|_{L^\infty}\|\partial_t^kE_{x_1x_1}\|_{L^2}+\|E_{x_1}\|_{L^\infty}\|\partial_t^kE_{x_1}\|_{L^2}\\
&\qquad+\|E\|_{L^\infty}\|\partial_t^kE_{x_2x_2}\|_{L^2}+\|E_{x_2}\|_{L^\infty}\|\partial_t^kE_{x_2}\|_{L^2}+1)\|\partial_t^{k+1}n\|_{L^2}\\
&\quad\leqslant C(\|\partial_t^kE_{x_1x_1}\|_{L^2}^2+\|\partial_t^kE_{x_1}\|_{L^2}^2+\|\partial_t^kE_{x_2x_2}\|_{L^2}^2+\|\partial_t^kE_{x_2}\|_{L^2}^2+\|\partial_t^{k+1}n\|_{L^2}^2+1),
\end{aligned}$$

from (2.21) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\|\partial_t^{k+1}n\|_{L^2}^2+\|\nabla\partial_t^k n\|_{L^2}^2+\|\partial_t^k n_{x_1x_1}\|_{L^2}^2 + \|\partial_t^k n_{x_1x_2}\|_{L^2}^2+\|\partial_t^k n_{x_2x_2}\|_{L^2}^2\big] \leqslant C(\|\partial_t^k E_{x_1x_1}\|_{L^2}^2+\|\partial_t^k E_{x_1}\|_{L^2}^2+\|\partial_t^k E_{x_2x_2}\|_{L^2}^2 + \|\partial_t^k E_{x_2}\|_{L^2}^2+\|\partial_t^{k+1}n\|_{L^2}^2+1). \tag{2.22}$$

Differentiating (1.10) k-times with respect to t, then taking the inner products of the resulting equation and $-\partial_t^{k+1}E$, we conclude that

$$(\mathrm{i}\partial_t^{k+1}E+\partial_t^k\Delta E-H^2\partial_t^k\Delta^2E-\partial_t^k(nE),-\partial_t^{k+1}E)=0. \tag{2.23}$$

Since

$$\mathrm{Re}(\mathrm{i}\partial_t^{k+1}E,-\partial_t^{k+1}E)=0,$$

$$\mathrm{Re}(\partial_t^k\Delta E,-\partial_t^{k+1}E)=\mathrm{Re}\Big(-\int_{\partial\Omega}\frac{\partial(\partial_t^kE)}{\partial\nu}\,\partial_t^{k+1}\overline{E}\,\mathrm{d}s+\int_\Omega\nabla(\partial_t^kE)\cdot\nabla(\partial_t^{k+1}\overline{E})\,\mathrm{d}x\Big) =\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\partial_t^kE\|_{L^2}^2,$$

$$\begin{aligned}\mathrm{Re}(-H^2\partial_t^k\Delta^2E,-\partial_t^{k+1}E) &=\mathrm{Re}\Big(H^2\int_{\partial\Omega}\frac{\partial(\partial_t^kE)}{\partial\nu}\,\partial_t^{k+1}\overline{E}\,\mathrm{d}s - H^2\int_\Omega\nabla(\partial_t^k\Delta E)\cdot\nabla(\partial_t^{k+1}\overline{E})\,\mathrm{d}x\Big)\\ &=\mathrm{Re}\Big(-H^2\int_{\partial\Omega}\frac{\partial(\nabla\partial_t^kE)}{\partial\nu}\cdot\nabla(\partial_t^{k+1}\overline{E})\,\mathrm{d}s + H^2\int_\Omega\nabla(\nabla\partial_t^kE)\cdot\nabla(\nabla\partial_t^{k+1}\overline{E})\,\mathrm{d}x\Big)\\ &=\frac{H^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[\|\partial_t^kE_{x_1x_1}\|_{L^2}^2+2\|\partial_t^kE_{x_1x_2}\|_{L^2}^2+\|\partial_t^kE_{x_2x_2}\|_{L^2}^2\big],\end{aligned}$$

$$\begin{aligned}&|\mathrm{Re}(-\partial_t^k(nE),-\partial_t^{k+1}E)|\\ &\leqslant C(\|E\|_{L^\infty}\|\partial_t^kn\|_{L^2}+\|E_t\|_{L^\infty}\|\partial_t^{k-1}n\|_{L^2}+\ldots+\|n\|_{L^\infty}\|\partial_t^kE\|_{L^2})\|\partial_t^{k+1}E\|_{L^2}\\ &\leqslant C(\|\partial_t^{k+1}E\|_{L^2}^2+1),\end{aligned}$$

from (2.23) we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\|\nabla\partial_t^kE\|_{L^2}^2+\|\partial_t^kE_{x_1x_1}\|_{L^2}^2+\|\partial_t^kE_{x_1x_2}\|_{L^2}^2+\|\partial_t^kE_{x_2x_2}\|_{L^2}^2\big] \leqslant C(\|\partial_t^{k+1}E\|_{L^2}^2+1). \tag{2.24}$$

From (2.20), (2.22), and (2.24) it follows that

$$\begin{aligned}(2.25)\quad \frac{\mathrm{d}}{\mathrm{d}t}\big[&\|\partial_t^{k+1}E\|_{L^2}^2+\|\partial_t^{k+1}n\|_{L^2}^2+\|\nabla\partial_t^k n\|_{L^2}^2+\|\partial_t^k n_{x_1x_1}\|_{L^2}^2+\|\partial_t^k n_{x_1x_2}\|_{L^2}^2\\ &+\|\partial_t^k n_{x_2x_2}\|_{L^2}^2+\|\nabla\partial_t^k E\|_{L^2}^2+\|\partial_t^k E_{x_1x_1}\|_{L^2}^2+\|\partial_t^k E_{x_1x_2}\|_{L^2}^2\\ &+\|\partial_t^k E_{x_1x_1}\|_{L^2}^2\big]\\ \leqslant C\big(&\|\partial_t^{k+1}n\|_{L^2}^2+\|\partial_t^{k+1}E\|_{L^2}^2+\|\partial_t^k E_{x_1}\|_{L^2}^2\\ &+\|\partial_t^k E_{x_1x_1}\|_{L^2}^2+\|\partial_t^k E_{x_2x_2}\|_{L^2}^2+\|\partial_t^k E_{x_2}\|_{L^2}^2+1\big).\end{aligned}$$

By using Gronwall's inequality, we obtain

$$\begin{aligned}&\|\partial_t^{k+1}E\|_{L^2}^2+\|\partial_t^{k+1}n\|_{L^2}^2+\|\nabla\partial_t^k n\|_{L^2}^2+\|\partial_t^k n_{x_1x_1}\|_{L^2}^2+\|\partial_t^k n_{x_1x_2}\|_{L^2}^2+\|\partial_t^k n_{x_2x_2}\|_{L^2}^2\\ &\quad+\|\nabla\partial_t^k E\|_{L^2}^2+\|\partial_t^k E_{x_1x_1}\|_{L^2}^2+\|\partial_t^k E_{x_1x_2}\|_{L^2}^2+\|\partial_t^k E_{x_1x_1}\|_{L^2}^2\leqslant \text{const.}\end{aligned}$$

Hence,

$$\sup_{0\leqslant t\leqslant T}\big[\|\partial_t^k E\|_{H^2}^2+\|\partial_t^k n\|_{H^2}^2+\|\partial_t^{k+1}E\|_{L^2}^2+\|\partial_t^{k+1}n\|_{L^2}^2\big]\leqslant \text{const.}$$

The Lemma 2.7 is proved completely. □

3. The existence of global smooth solution for problem (1.5)–(1.9)

Definition 3.1. The functions $E(x,t)\in L^\infty(0,T;H^4(\Omega))\cap W^{1,\infty}(0,T;L^2(\Omega))$, $n(x,t)\in L^\infty(0,T;H^2(\Omega))\cap W^{1,\infty}(0,T;L^2(\Omega))$, $\varphi(x,t)\in L^\infty(0,T;H^2(\Omega))\cap W^{1,\infty}(0,T;L^2(\Omega))$ are called the *generalized solution of problem* (1.10)–(1.15) if they satisfy the integral equality

$$(3.1)\qquad (\mathrm{i}E_{kt},v)+(\Delta E_k,v)-(H^2\Delta^2E_k,v)-(nE_k,v)=0,$$

$$(3.2)\qquad (n_t,v)-(\Delta\varphi,v)=0,$$

$$(3.3)\qquad (\varphi_t,v)-(n,v)+(H^2\Delta n,v)-(|E|^2,v)=0,$$

with

$$(3.4)\quad (E_k(x,0),v)=(E_{k0}(x),v),\ \ (n(x,0),v)=(n_0(x),v),\ \ (\varphi(x,0),v)=(\varphi_0(x),v),$$

and

$$(3.5)\quad E_k|_{\partial\Omega}=\frac{\partial(\nabla E_k)}{\partial\nu}\Big|_{\partial\Omega}=0,\quad n(x,t)|_{\partial\Omega}=\frac{\partial(\nabla n)}{\partial\nu}\Big|_{\partial\Omega}=0,\quad \varphi(x,t)|_{\partial\Omega}=0,$$
$$k=1,2,\dots,d,\quad \forall v\in L^2(\Omega).$$

Theorem 3.1. *Suppose that $E_0(x) \in H^4(\Omega) \cap H_0^2(\Omega)$, $n_0(x) \in H_0^2(\Omega)$, $\varphi_0(x) \in H^2(\Omega)$. Then there exists a global generalized solution of the initial boundary value problem* (1.10)–(1.15),

$$
\begin{aligned}
\Delta^2 E(x,t) &\in L^\infty(0,T;L^2(\Omega)),\\
E_t(x,t) &\in L^\infty(0,T;L^2(\Omega)),\\
n(x,t) &\in L^\infty(0,T;H^2(\Omega)),\\
n_t(x,t) &\in L^\infty(0,T;L^2(\Omega)),\\
\Delta\varphi(x,t) &\in L^\infty(0,T;L^2(\Omega)),\\
\varphi_t(x,t) &\in L^\infty(0,T;L^2(\Omega)).
\end{aligned}
$$

Proof. By using the Galerkin method, choose basic functions $\{\omega_j(x)\} \subset H_0^2(\Omega)$ as follows:

$$
\begin{gathered}
-\Delta\omega_j(x) = \lambda_j\omega_j(x), \quad j = 1,2,\dots,l,\\
\omega_j(x)|_{\partial\Omega} = \frac{\partial(\nabla\omega_j)}{\partial\nu}\Big|_{\partial\Omega} = \frac{\partial(\nabla\omega_j)}{\partial\nu}\Big|_{\partial\Omega} = 0.
\end{gathered}
$$

The approximate solution of problem (1.10)–(1.15) can be written as

$$
E_l(x,t) = \sum_{j=1}^{l}\alpha_{jl}(t)\omega_j(x), \quad n_l(x,t) = \sum_{j=1}^{l}\beta_{jl}(t)\omega_j(x), \quad \varphi_l(x,t) = \sum_{j=1}^{l}\gamma_{jl}(t)\omega_j(x),
$$

where

$$
\alpha_{jl}(t) = \big(\alpha_{jl}^{(1)}(t), \alpha_{jl}^{(2)}(t), \dots, \alpha_{jl}^{(d)}(t)\big).
$$

According to the Galerkin method, these undetermined coefficients $\alpha_{jl}(t)$, $\beta_{jl}(t)$, and $\gamma_{jl}(t)$ need to satisfy the following initial value problem of the system of ordinary differential equations:

$$
\text{i}(E_l^{(k)}, \omega_s) + (\Delta E_l^{(k)}, \omega_s) - (H^2\Delta^2 E_l^{(k)}, \omega_s) - (n_l E_l^{(k)}, \omega_s) = 0, \tag{3.6}
$$

$$
(n_{lt}, \omega_s) - (\Delta\varphi_l, \omega_s) = 0, \quad k = 1,\dots,d, \quad s = 1,\dots,l, \tag{3.7}
$$

$$
(\varphi_{lt}, \omega_s) - (n_l, \omega_s) + (H^2\Delta n_l, \omega_s) - (|E_l|^2, \omega_s) = 0, \tag{3.8}
$$

$$
E_l^{(k)}(x,0) = E_{0l}^{(k)}(x), \quad n_l(x,0) = n_{0l}(x), \quad \varphi_l(x,0) = \varphi_{0l}(x), \quad x \in \overline{\Omega}, \tag{3.9}
$$

where

$$
\begin{gathered}
E_{0l}^{(k)}(x) \xrightarrow{H^4} E_0^{(k)}(x), \quad n_{0l}(x) \xrightarrow{H^2} n_0(x), \quad \varphi_{0l}(x) \xrightarrow{H^2} \varphi_0(x),\\
\text{as} \quad l \to \infty, \ k = 1,2,\dots,d.
\end{gathered}
$$

Similarly to the proofs of Lemma 2.1 and Lemma 2.5, for the solution $E_l(x,t)$, $n_l(x,t)$, $\varphi_l(x,t)$ of problem (3.6)–(3.9) we can establish the estimates

$$\sup_{0\leqslant t\leqslant T}\big[\|\Delta^2 E_l(x,t)\|_{L^2}+\|n_l(x,t)\|_{H^2}+\|\Delta\varphi_l(x,t)\|_{L^2}\big]\leqslant C_4, \tag{3.10}$$

$$\sup_{0\leqslant t\leqslant T}\big[\|E_{lt}(x,t)\|_{L^2}+\|n_{lt}\|_{L^2}+\|\varphi_{lt}\|_{L^2}\big]\leqslant C_5, \tag{3.11}$$

where the constants C_4 and C_5 are independent of l. By compactness argument, we can choose subsequences $E_{l_m}(x,t)$, $n_{l_m}(x,t)$, $\varphi_{l_m}(x,t)$, as $m\to\infty$ such that

$$\begin{gathered}
E_{l_m}(x,t)\to E(x,t) \text{ in } L^\infty(0,T;H^2) \text{ weakly star},\\
E_{l_mt}(x,t)\to E_t(x,t) \text{ in } L^\infty(0,T;L^2) \text{ weakly star},\\
E_{l_m}(x,t)\to E(x,t) \text{ in } L^2(Q_T) \text{ strongly and a.e.},\\
n_{l_m}(x,t)\to n(x,t) \text{ in } L^\infty(0,T;H^2) \text{ weakly star},\\
n_{l_mt}(x,t)\to n_t(x,t) \text{ in } L^\infty(0,T;L^2) \text{ weakly star},\\
n_{l_m}(x,t)\to n(x,t) \text{ in } L^2(Q_T) \text{ strongly and a.e.},\\
\varphi_{l_m}(x,t)\to \varphi(x,t) \text{ in } L^\infty(0,T;H^2) \text{ weakly star},\\
\varphi_{l_mt}(x,t)\to \varphi_t(x,t) \text{ in } L^\infty(0,t;L^2) \text{ weakly star},\\
n_{l_m}E_{l_m}\to nE \text{ in } L^\infty(0,T;L^2) \text{ weaky star},\\
|E_{l_m}(x,t)|^2\to |E(x,t)|^2 \text{ in } L^\infty(0,T;L^2) \text{ weakly star},
\end{gathered}$$

where $Q_T=\Omega\times[0,T]$. Hence, taking $l=l_m\to\infty$ from (3.6)–(3.9), by using the density of ω_j in $L^2(\Omega)$ we get the existence of a local generalized solution for the problem (1.10)–(1.15). By the continuous extension principle, from the conditions of the theorem and a priori estimates in Section 2 we can get the existence of the global solution for problem (1.10)–(1.15). □

Theorem 3.2. *Suppose that the conditions of Theorem* 3.1 *are satisfied. Then the global generalized solution of the problem* (1.10)–(1.15) *is unique.*

Proof. Suppose that there are two solutions E_1,n_1,φ_1 and E_2,n_2,φ_2. Let

$$E=E_1-E_2,\quad n=n_1-n_2,\quad \varphi=\varphi_1-\varphi_2.$$

From (1.10)–(1.15) we get

$$\mathrm{i}E_t+\Delta E-H^2\Delta^2E-n_1E_1+n_2E_2=0, \tag{3.12}$$

$$n_t-\Delta\varphi=0, \tag{3.13}$$

$$\varphi_t-n+H^2\Delta n-|E_1|^2+|E_2|^2=0, \tag{3.14}$$

with the initial data

$$E|_{t=0} = 0, \quad n|_{t=0} = 0, \quad \varphi|_{t=0} = 0, \tag{3.15}$$

and boundarty conditions

$$E|_{\partial\Omega} = \frac{\partial(\nabla E)}{\partial \nu}\Big|_{\partial\Omega} = 0, \tag{3.16}$$

$$n(x,t)|_{\partial\Omega} = \frac{\partial(\nabla n)}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad \varphi(x,t)|_{\partial\Omega} = 0. \tag{3.17}$$

Taking the inner product of (3.12) and E yields

$$(\mathrm{i}E_t + \Delta E - H^2\Delta^2 E - n_1 E_1 + n_2 E_2, E) = 0, \tag{3.18}$$

$$\begin{aligned}
\operatorname{Im}(\mathrm{i}E_t, E) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E\|_{L^2}^2, \quad \operatorname{Im}(\Delta E - G^2\Delta^2 E, E) = 0,\\
|\operatorname{Im}(n_1E_1 - n_2E_2, E)| &\leqslant |(nE_1 + n_2E, E)|\\
&\leqslant C(\|E_1\|_{L^\infty}\|n\|_{L^2} + \|n_2\|_{L^\infty}\|E\|_{L^2})\|E\|_{L^2}\\
&\leqslant C(\|n\|_{L^2}^2 + \|E\|_{L^2}^2),
\end{aligned}$$

from (3.18) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|E\|_{L^2}^2 \leqslant C(\|n\|_{L^2}^2 + \|E\|_{L^2}^2). \tag{3.19}$$

Taking the inner product of (3.14) and φ, we obtain

$$(\varphi_t - n + H^2\Delta n - |E_1|^2 + |E_2|^2, \varphi) = 0. \tag{3.20}$$

Since

$$\begin{aligned}
(\varphi_t, \varphi) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_{L^2}^2,\\
|(-n, \varphi)| &\leqslant C(\|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2),\\
(H^2\Delta n, \varphi) &= H^2\int_{\partial\Omega}\Big(\frac{\partial n}{\partial \nu}\varphi - n\frac{\partial \varphi}{\partial \nu}\Big)\,\mathrm{d}s + H^2\int_{\partial\Omega} n\Delta\varphi\,\mathrm{d}x\\
&= H^2\int_{\Omega} n n_t\,\mathrm{d}x = \frac{H^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n\|_{L^2}^2,\\
|(-|E_1|^2 + |E_2|^2, \varphi)| &= |((E_1 - E_2)\overline{E}_1 + E_2(\overline{E}_1 - \overline{E}_2), \varphi)|\\
&\leqslant C(\|E_1\|_{L^\infty}\|E\|_{L^2} + \|E_2\|_{L^\infty}\|E\|_{L^2})\|\varphi\|_{L^2}\\
&\leqslant C(\|E\|_{L^2}^2 + \|\varphi\|_{L^2}^2),
\end{aligned}$$

from (3.20) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\varphi\|_{L^2}^2+\|n\|_{L^2}^2\right]\leqslant C(\|E\|_{L^2}^2+\|n\|_{L^2}^2+\|\varphi\|_{L^2}^2). \tag{3.21}$$

Hence, from (3.19) and (3.21) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|E\|_{L^2}^2+\|n\|_{L^2}^2+\|\varphi\|_{L^2}^2\right]\leqslant C\left[\|E\|_{L^2}^2+\|n\|_{L^2}^2+\|\varphi\|_{L^2}^2\right]. \tag{3.22}$$

By using Gronwall's inequality and due to (3.15), it follows that

$$E\equiv 0,\quad n\equiv 0,\quad \varphi\equiv 0.$$

The theorem is proved. □

Theorem 3.3. *Suppose that* $E_0(x)\in H^{4+2l}(\Omega)\cap H_0^2(\Omega)$, $n_0(x)\in H^{2l}(\Omega)\cap H_0^2(\Omega)$, $\varphi_0(x)\in H^{2l}(\Omega)$, $l\geqslant 3$. *Then there exists a unique global smooth solution of the initial boundary value problem* (1.10)–(1.15),

$$\begin{aligned}
\Delta^4E(x,t)&\in L^\infty(0,T;H^{2l-4}(\Omega)),\\
\Delta^2E_t(x,t)&\in L^\infty(0,T;H^{2l-4}(\Omega)),\\
\Delta^2n(x,t)&\in L^\infty(0,T;H^{2l-4}(\Omega)),\\
\Delta^2n_t(x,t)&\in L^\infty(0,T;H^{2l-6}(\Omega)),\\
\Delta\varphi(x,t)&\in L^\infty(0,T;H^{2l-2}(\Omega)),\\
\Delta\varphi_t(x,t)&\in L^\infty(0,T;H^{2l-4}(\Omega)).
\end{aligned}$$

P r o o f. Using Lemma 2.7 and the embedding theorems of Sobolev spaces, the results of Theorem 3.3 are obvious. □

Theorem 3.4. *Suppose that* $E_0(x)\in H^{4+2l}(\Omega)\cap H_0^2(\Omega)$, $n_0(x)\in H^{2l}(\Omega)\cap H_0^2(\Omega)$, $n_1(x)\in H^{2l-2}(\Omega)$, $l\geqslant 3$. *Then there exists a unique global smooth solution of the initial boundary value problem* (1.5)–(1.9),

$$\begin{aligned}
\Delta^4E(x,t)&\in L^\infty(0,T;H^{2l-4}(\Omega)),\\
\Delta^2E_t(x,t)&\in L^\infty(0,T;H^{2l-4}(\Omega)),\\
\Delta^2n(x,t)&\in L^\infty(0,T;H^{2l-4}(\Omega)),\\
\Delta^2n_t(x,t)&\in L^\infty(0,T;H^{2l-6}(\Omega)),\\
n_{tt}(x,t)&\in L^\infty(0,T;H^{2l-4}(\Omega)).
\end{aligned}$$

P r o o f. Using Lemma 2.7 and the embedding theorems of Sobolev spaces, the proof of Theorem 3.4 is easy. □

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