Reiner Vanselow

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NEW RESULTS CONCERNING THE DWR METHOD FOR SOME NONCONFORMING FEM

Reiner Vanselow, Dresden

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Cordially dedicated to Professor Christian Großmann on his 65th birthday anniversary.

Abstract. This paper presents a unified framework for the dual-weighted residual (DWR) method for a class of nonconforming FEM. Our approach is based on a modification of the dual problem and uses various ideas from literature which are combined in a new manner. The results are new error identities for some nonconforming FEM. Additionally, a posteriori error estimates with respect to the discrete H^1-seminorm are derived.

Keywords: nonconforming finite elements, dual-weighted residual method, a posteriori error estimate

MSC 2010: 65N15, 65N30

1. Introduction

We study the model problem

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^2, \qquad u = 0 \quad \text{on } \Gamma = \partial\Omega,$$

in a bounded, polygonal and simply connected domain Ω, where $f \in L^2(\Omega)$ is a given function.

A weak formulation is: Find $u \in V = H_0^1(\Omega)$ such that

$$a(u, v) = F(v) \qquad \forall\, v \in V \tag{1.1}$$

with

$$a(v, w) = (\mathbf{grad}\, v, \mathbf{grad}\, w)_\Omega = \iint_\Omega [\mathbf{grad}\, v]^{\mathrm{T}}\, \mathbf{grad}\, w \quad \text{and} \quad F(v) = (f, v)_\Omega.$$

We study the Galerkin-FEM on regular partitions $\mathcal{T}_h$ with $\overline{\Omega} = \overline{\Omega}_h$ $(= \bigcup_{K \in \mathcal{T}_h} K)$. Furthermore, for the seminorm and norm in the Sobolev space $H^m(G)$ we use the notation $|\cdot|_{m,G}$ and $\|\cdot\|_{m,G}$, respectively. The number C denotes a generic positive constant, independent of the step size, and P_k denotes the space of polynomials of degree k.

As a starting point of our study, we give a brief summary of some aspects and results of a posteriori error indicators and the DWR method.

For $K \in \mathcal{T}_h$ a local a posteriori error indicator η_K provides an information about the error $e_h = u - u_h$ of the FEM solution u_h. This is important for measuring the global accuracy within a prescribed tolerance with a minimal amount of work. Such indicators only use the computed numerical solution and the data of the PDE.

However, if we want to determine the accuracy of FEM-solutions, the question which occurs is: Which measure of accuracy can we use? Traditionally, norms like the H^1-seminorm or the L^2-norm are used. A norm induced accuracy leads to a posteriori error indicators. The DWR method is a generalization of this approach.

First, let us discuss a posteriori error indicators with respect to a norm $\|\cdot\|$.

For given local a posteriori error indicators η_K, $K \in \mathcal{T}_h$, the associated global a posteriori error indicator η is defined by

$$\eta = \Big(\sum_{K \in \mathcal{T}_h} |\eta_K|^2\Big)^{1/2}. \tag{1.2}$$

Usually, η_K and η satisfy

$$\|u - u_h\|_\Omega \leqslant C\eta \ \ \text{(reliability)} \quad \text{and} \quad \eta_K \leqslant C\|u - u_h\|_{\omega_K} \ \ \text{(efficiency)} \tag{1.3}$$

or comparable inequalities. Here, ω_K is a suitable and possibly small neighbourhood of K.

Relations (1.3) state the upper (global) and lower (local) boundedness of the norm $\|\cdot\|$ of the error $e_h = u - u_h$ in terms of the global and of the local error indicators η and η_K, respectively. If η is small enough the first inequality in (1.3) provides the wanted global accuracy. If in addition the second inequality in (1.3) is satisfied the local a posteriori error indicator η_K is of the same scale and does not overestimate the error on ω_K. If the first inequality in (1.3) holds with $C = 1$ the error indicators are called error estimators (see [4]).

Now, let a nonconforming FEM be defined by: Find $u_h \in V_h^{\rm nc}$ such that

$$a_h(u_h, v_h) = F(v_h) \quad \forall\, v_h \in V_h^{\rm nc} \tag{1.4}$$

with

$$a_h(v,w) = \sum_{K\in\mathcal{T}_h} a_K(v,w) \quad \text{and} \quad a_K(v,w) = (\mathbf{grad}\, v, \mathbf{grad}\, w)_K.$$

For a nonconforming FEM we have $V_h^{\mathrm{nc}} \not\subset V$ in contrast to $V_h^{\mathrm{c}} \subset V$ for a conforming FEM.

As a measure of accuracy, the discrete H^1-seminorm on Ω can be used:

$$|v|_{1,\Omega,h} = \Bigl(\sum_{K\in\mathcal{T}_h} |v|_{1,K}^2\Bigr)^{1/2}. \tag{1.5}$$

In [8], for our model problem, a triangular partition and the linear nonconforming FEM, a local a posteriori error indicator is given comparable to

$$\eta_K = \Bigl\{h_K^2\|f+\Delta u_h\|_{0,K}^2 + \frac{1}{2}\, h_K\Bigl[\|J_{h,\partial K\setminus\Gamma,\mathrm{n}}(u_h)\|_{0,\partial K\setminus\Gamma}^2 + \|J_{h,\partial K\setminus\Gamma,\mathrm{t}}(u_h)\|_{0,\partial K\setminus\Gamma}^2\Bigr] + h_K\|\partial_{\mathrm{t}_K} u_h\|_{\partial K\cap\Gamma}^2\Bigr\}^{1/2}. \tag{1.6}$$

Here, h_K is the diameter of K, t_K is a tangent vector to ∂K defined as

$$\mathrm{t}_K = \mathrm{t}_K(\mathrm{n}_K) = [-n_2, n_1]^{\mathrm{T}} \tag{1.7}$$

with $\mathrm{n}_K = [n_1, n_2]^{\mathrm{T}}$ being the outer unit normal vector of K, and the terms

$$J_{h,\partial K\setminus\Gamma,\nu}(u_h) = [\nu_K]^{\mathrm{T}} \bigl[\!\bigl|\, \mathbf{grad}_h\, u_h \bigr|\!\bigr] \quad \text{for } \nu\in\{\mathrm{n},\mathrm{t}\} \tag{1.8}$$

are defined on $\partial K\setminus\Gamma$ and denote the jump of $\partial_{\nu_K} u_h = [\nu_K]^{\mathrm{T}}\,\mathbf{grad}_h\, u_h$ with respect to K (for the definition of the discrete gradient $\mathbf{grad}_h$ see (2.4)). More precisely: Let $\mathcal{E}_h$ be the set of all edges of the triangulation $\mathcal{T}_h$,

$$\mathcal{E}_{h,\mathrm{in}} = \{E\in\mathcal{E}_h \colon \exists\, K_1, K_2 \in \mathcal{T}_h \text{ with } K_1\neq K_2 \text{ and } E = K_1\cap K_2\} \tag{1.9}$$

and $w\in\{v_h, \mathbf{grad}_h\, v_h\}$ with $v_h\in V_h^{\mathrm{nc}}$. Then for $K\in\mathcal{T}_h$ and $E\in\mathcal{E}_{h,\mathrm{in}}$ with $E\subset\partial K$, on $\mathrm{int}(E)$ the jump $\bigl[\!\bigl| w \bigr|\!\bigr]$ with respect to K is given by

$$\bigl[\!\bigl| w \bigr|\!\bigr](x) = w\big|_{E\cap\partial K_{\mathrm{out}(x)}}(x) - w\big|_{E\cap\partial K}(x) \quad \forall\, x\in\mathrm{int}(E). \tag{1.10}$$

Here, $K_{\mathrm{out}}(x)\in\mathcal{T}_h$ is defined by $K_{\mathrm{out}}(x)\neq K$ and $x\in K\cap K_{\mathrm{out}}(x)$ (see 1.9), and for $\tilde K\in\mathcal{T}_h$ and $E\in\mathcal{E}_h$ with $E\subset\partial\tilde K$, the value $w\big|_{E\cap\tilde K}(x)$ for $x\in\mathrm{int}(E)$ is defined by

$$w\big|_{E\cap\partial\tilde K}(x) = \lim_{z\to x,\, z\in\mathrm{int}(\tilde K)} w(z).$$

Remark 1.

1. The jump $\left[\!\left| v \right|\!\right]$ is only defined on $\operatorname{int}(E)$ with $E \in \mathcal{E}_{h,\mathrm{in}}$. However, this is sufficient because it is only used in terms like $(1, \left[\!\left| v \right|\!\right])_{0,\partial K\setminus\Gamma}$ or $\| \left[\!\left| v \right|\!\right] \|_{0,\partial K\setminus\Gamma}$.
2. For a piecewise linear FE-space, $\Delta u_h\big|_{K\in\mathcal{T}_h} = 0$ holds, which simplifies η_K.
3. Sometimes, e.g. $h_K \|J_{h,\partial K\setminus\Gamma,\mathrm{n}}(u_h)\|^2_{0,\partial K\setminus\Gamma}$ is also given in the equivalent form

$$\sum_{E\in\mathcal{E}_h\cap(\partial K\setminus\Gamma)} h_E \|J_{h,\partial K\setminus\Gamma,\mathrm{n}}(u_h)\|^2_{0,E}$$

   with $\mathcal{E}_h$ being the set of all edges of the triangulation and h_E the length of E. To simplify matters, we do not distinguish between h_K and h_E, because under usual assumptions like the minimal angle condition these are equivalent.

In [8], the authors proved (1.6) using a Helmholtz decomposition of the space $[L^2(\Omega)]^2$ that we will also use.

Compared to the above norm-based measure, the DWR (dual-weighted residual) method as given in [5] is a more general approach using other measures of accuracy. In the following, we repeat the main ideas of this approach for the conforming FEM defined by: Find $u_h \in V_h^{\mathrm{c}} (\subset V)$ such that

$$a(u_h, v_h) = F(v_h) \quad \forall\, v_h \in V_h^{\mathrm{c}}.$$

A linear error function $\mathcal{J}(\cdot)\colon V \to \mathbb{R}$ is used to control the error e_h, i.e., instead of the usual $\|e_h\|_\Omega$ we use $|\mathcal{J}(e_h)|$ to measure the accuracy.

In [5], various examples for the choice of $\mathcal{J}$ are given some of which yield error estimates with respect to norms like the H^1-seminorm or the L^2-norm.

With the unique solution $z \in V$ of the dual problem

$$a(v, z) = \mathcal{J}(v) \quad \forall\, v \in V, \tag{1.11}$$

we obtain $\mathcal{J}(e_h) = a(e_h, z)$ and, because of the Galerkin orthogonality $a(e_h, v_h) = 0$,

$$\mathcal{J}(e_h) = a(e_h, z - v_h) \quad \forall\, v_h \in V_h^{\mathrm{c}}.$$

Partial integration over $K \in \mathcal{T}_h$ yields the error identity

$$\mathcal{J}(e_h) = \sum_{K\in\mathcal{T}_h} \{(R_K(u_h), z - v_h)_K - (r_{\partial K\setminus\Gamma}(u_h), z - v_h)_{\partial K\setminus\Gamma}\} \quad \forall\, v_h \in V_h^{\mathrm{c}} \tag{1.12}$$

with the local computable cell residuals $R_K(u_h) = (f + \Delta u_h)\big|_K$ and edge residuals $r_{\partial K\setminus\Gamma}(u_h) = \frac{1}{2} J_{\partial K\setminus\Gamma,\mathrm{n}}(u_h)$ with $J_{\partial K\setminus\Gamma,\mathrm{n}}(u_h) = [\mathrm{n}_K]^{\mathrm{T}} \left[\!\left| \operatorname{\mathbf{grad}} u_h \right|\!\right]$ (compare to (1.8)).

Now, in the derivation and also in the practical computation of local a posteriori error indicators we have to distinguish two cases.

For some $\mathcal{J}$, local a posteriori error indicators like (1.6) are computable without knowing the solution z of (1.11). This is possible, if the following two conditions hold:

- The solution z of (1.11) satisfies $z \in H^{\beta}(\Omega)$ for some $\beta \geqslant 1$, and
- for an interpolation operator $I_h\colon V \to V_h^{\mathrm{c}}$ a local interpolation error estimate

$$\|v - I_h v\|_{0,K} + h_K^{1/2} \|v - I_h v\|_{0,\partial K} \leqslant C h_K^{\beta} |v|_{\beta,\omega_K} \quad \forall\, K \in \mathcal{T}_h,\ v \in V \tag{1.13}$$

  is satisfied, where ω_K is a suitable and possibly small neighbourhood of K.

Then a computable local a posteriori error indicator is given by

$$\eta_K = h_K^{\beta} \Big\{ \|f + \Delta u_h\|_{0,K}^2 + \frac{1}{2}\, h_K^{-1} \|J_{\partial K \setminus \Gamma, \mathrm{n}}(u_h)\|_{0,\partial K \setminus \Gamma}^2 \Big\}^{1/2},$$

and the global a posteriori error indicator η defined by (1.2) obviously satisfies

$$|\mathcal{J}(e_h)| \leqslant C\eta \quad \text{(reliability)}. \tag{1.14}$$

This follows with the change from the error identity (1.12) to the error estimate

$$\begin{aligned} |\mathcal{J}(e_h)| \leqslant \sum_{K \in \mathcal{T}_h} \Big\{ & \|f + \Delta u_h\|_{0,K} \|z - v_h\|_{0,K} \\ & + \frac{1}{2} \|J_{\partial K \setminus \Gamma, \mathrm{n}}(u_h)\|_{0,\partial K \setminus \Gamma} \|z - v_h\|_{0,\partial K \setminus \Gamma} \Big\} \end{aligned} \tag{1.15}$$

by using the local interpolation error estimate (1.13).

This case includes such $\mathcal{J}$ which lead to estimates with respect to the H^1-seminorm or to the L^2-norm. For all other $\mathcal{J}$, different techniques have to be used (see [5]).

A proof of a property like the efficiency is not known in the general context of the DWR method. But, as mentioned in [5], it seems to be clear that two conditions are necessary:

- The change from identity (1.12) to (1.15) is not too bad, and
- the estimate (1.13) is satisfied with an optimal β.

Now, let us consider a nonconforming FEM. In some papers (e.g. [12], [10], [14]), the DWR method is already studied for nonconforming FEM. In that case, because of $V_h^{\mathrm{nc}} \not\subset V$, an error function

$$\mathcal{J}(\cdot)\colon V \oplus V_h^{\mathrm{nc}} \to \mathbb{R} \tag{1.16}$$

has to be used.

In [12], first of all the triangular linear nonconforming FEM is considered. The use of the space $V_h^{\mathrm{c}} = V_h^{\mathrm{nc}} \cap V$ with V_h^{c} from the triangular linear conforming FEM is possible, so that the results follow with the same techniques as in the conforming case.

Further, in [12] on a partition into quadrilaterals a nonconforming Q_1-element is considered, namely the Rannacher-Turek element (see [13]). For this FEM, the same ideas as in [10] are applied and yield the additional term $(\left[\!\left| u_h \right|\!\right], \partial_{\mathrm{n}_K} z)_{\partial K}$ on the right-hand side of (1.12).

In [10] on a partition into quadrilaterals a linear nonconforming FEM is studied (for details see [10]). But in contrast to the nonconforming triangular P_1-element in [12], it is not possible to use $V_h^{\mathrm{nc}} \cap V$, because this space is too small (i.e., for a given norm in a local interpolation error estimate like (1.13) β is too bad). Therefore, in [10] V_h^{nc} is first expanded and then restricted to a conforming space V_h^{c}. That yields the additional term $a_h(e_h, v_h^{\mathrm{c}})$ for $v_h^{\mathrm{c}} \in V_h^{\mathrm{c}}$ on the right-hand side of (1.12).

In [14] for a given partition $\mathcal{T}_h$ a class of nonconforming FEM is considered based on polynomial elements. The nonconforming FE-space V_h^{nc} has to satisfy $\int_E \left[\!\left| v \right|\!\right] \,\mathrm{d}s = 0$ for all $v \in V_h^{\mathrm{nc}}$ and $E \in \mathcal{E}_{h,\mathrm{in}}$. The approach is based on two spaces $V_h^{\mathrm{c}} \subset V$ and $V_h^{\mathrm{c},1} = V \cap V_h^{\mathrm{nc}}$ as well as on two operators $R_h\colon V_h^{\mathrm{nc}} \oplus V_h^{\mathrm{c}} \to V_h^{\mathrm{c}}$ and $I_h^{\mathrm{c},1}\colon V \to V_h^{\mathrm{c},1}$. The result is the error identity

$$\begin{aligned}\mathcal{J}(e_h) = \sum_{K \in \mathcal{T}_h} &\Big\{ (f + \Delta u_h, z - I_h^{\mathrm{c},1} z)_K - \frac{1}{2} (J_{\partial K \setminus \Gamma, \mathrm{n}}(u_h), z - I_h^{\mathrm{c},1} z)_{\partial K \setminus \Gamma} \Big\} \\ &+ \eta^{(p)}(u_h, z)\end{aligned}$$

with $\eta^{(p)}(u_h, z) = a_h(u_h - R_h u_h, z) - J(u_h - R_h u_h)$, where z is the solution of the dual problem (1.11).

In some other papers ([6], [3], [2]) interesting approaches and ideas are given, however only for a posteriori error indicators with respect to the discrete H^1-seminorm.

In [6], where a general class of nonconforming FEM is considered, a space $V_h^{\mathrm{c}} \subset V$ and a suitable linear operator $\Pi\colon V \to V_h^{\mathrm{nc}}$ have to be known. Then in contrast to our approach, for the estimation of $a_h(u_h, v - v_h)$ not all $v_h \in V_h^{\mathrm{nc}}$ are used, but only the ones for which $v_h = \Pi \tilde{v}_h$ holds with $\tilde{v}_h \in V_h^{\mathrm{c}}$ (for details see [6]). However, in [6] it is shown that (1.6) is also a local a posteriori error indicator for the four nonconforming FEM considered in Subsection 4.2.

In [3] also a class of nonconforming FEM is considered, however only based on a triangular partition and on P_l-elements. All these FEM satisfy our assumptions (A1) from Subsection 2.1 and (A2) from Section 3 with $k = l - 1$.

As in [2], where only the quadrilateral Rannancher-Turek element is considered, an additional space $V_h^{\mathrm{c}} \subset V$ is not needed, but in contrast to our indicator it is more difficult to calculate the local a posteriori error indicators given in [3] and [2]. However, because of Lemma 2 (see Section 3) it is not surprising that the error part $J_{h,\partial K\setminus\Gamma,\mathrm{n}}(u_h)$ known from (1.6) is not identifiable in the estimate in [3], since this part is not needed at all if a special interpolation operator is used.

The remainder of the paper deals with the DWR method for the nonconforming FEM and is organized as follows. First, Section 2 gives necessary preliminaries including a small modification of the dual problem. In Section 3, new error identities for some nonconforming FEM are presented comparable to (1.12). In Section 4 applications to some nonconforming FEM are given. On the one hand, the error identities are satisfied for arbitrary error functions $\mathcal{J}$. On the other hand, the choice of a special error function yields a posteriori error indicators with respect to the discrete H^1-seminorm, which are new in certain cases.

2. Necessary preliminaries

2.1. The class of nonconforming FEM

For all $K \in \mathcal{T}_h$, let $\mathcal{R}_K \subset C^2(K)$ be a given finite dimensional function space. We consider nonconforming FEM whose FE-space V_h^{nc} is a subspace of

$$\{v \in L^2(\Omega)\colon\ v\big|_K \in \mathcal{R}_K\ \forall\, K \in \mathcal{T}_h\}.$$

Further, we introduce the following assumption

(A1) Let $k \in \mathbb{N}$ be fixed, and for all $v \in V_h^{\mathrm{nc}}$ let (as in [14])

$$(p, \big[\!\big|v\big|\!\big])_E = 0 \quad \forall\, E \in \mathcal{E}_{h,\mathrm{in}},\ p \in P_k(E)$$

and

$$(p, v)_E = 0 \quad \forall\, E \in \mathcal{E}_h \setminus \mathcal{E}_{h,\mathrm{in}},\ p \in P_k(E).$$

That is, with the space

$$\begin{aligned}\Upsilon_h^{\mathrm{nc}} = \big\{&v \in L^2(\Omega)\colon\ v\big|_K \in \mathcal{R}_K\ \forall\, K \in \mathcal{T}_h \text{ and}\\ &(p, \big[\!\big|v\big|\!\big])_E = 0\ \forall\, E \in \mathcal{E}_{h,\mathrm{in}},\ p \in P_k(E)\big\}\end{aligned} \tag{2.1}$$

the FE-space V_h^{nc} has to satisfy

$$V_h^{\mathrm{nc}} \subset \{v \in \Upsilon_h^{\mathrm{nc}}\colon\ (p, v)_E = 0\ \forall\, E \in \mathcal{E}_h \setminus \mathcal{E}_{h,\mathrm{in}},\ p \in P_k(E)\}. \tag{2.2}$$

R e m a r k 2. First, $\big[\!\big|v\big|\!\big]$ was defined on $\partial K \setminus \Gamma$ with respect to K (see (1.10)). However, in (A1) it is used as if it was defined on $E \in \mathcal{E}_{h,\mathrm{in}}$ which is possible because $(p, \big[\!\big|v\big|\!\big])_E = 0$ for $E \in \mathcal{E}_{h,\mathrm{in}}$.

2.2. The DWR method for nonconforming FEM

We use an error function $\mathcal{J}$ with (1.16), but in contrast to (1.11) we consider the following dual problem: Find $z \in V \oplus V_h^{\mathrm{nc}}$ such that

$$a_h(v, z) = \mathcal{J}(v) \quad \forall\, v \in V \oplus V_h^{\mathrm{nc}} \tag{2.3}$$

which gives as measure of accuracy

$$\mathcal{J}(e_h) = a_h(e_h, z).$$

This different, yet self-evident choice of the dual problem has two advantages. On the one hand, no additional terms arise which cannot be estimated directly (for details see below). On the other hand, the Galerkin orthogonality is not needed.

2.3. A suitable Helmholtz decomposition and its application

As in [8] (and also in [6], [2], [3]) we use a suitable Helmholtz decomposition of the space $[L^2(\Omega)]^2$. More precisely: We use a suitable Helmholtz decomposition of the discrete gradient $\mathbf{grad}_h\, \tilde{v} \in [L^2(\Omega)]^2$ with $\tilde{v} = v + v_h$, $v \in V$ and $v_h \in V_h^{\mathrm{nc}}$, where $\mathbf{grad}_h\, \tilde{v}$ is defined by

$$(\mathbf{grad}_h\, \tilde{v})\big|_K = \mathbf{grad}\, (\tilde{v}\big|_K) \quad \forall\, K \in \mathcal{T}_h. \tag{2.4}$$

Denoting

$$\mathbf{rot}\, w = \Big[-\frac{\partial w}{\partial y}, \frac{\partial w}{\partial x}\Big]^{\mathrm{T}}$$

(as in [8]), we obtain the orthogonal decomposition

$$\mathbf{grad}_h\, \tilde{v} = \mathbf{grad}\, w_{\tilde{v}} + \mathbf{rot}\, \phi_{\tilde{v}}$$

with $w_{\tilde{v}} \in V = H_0^1(\Omega)$, $\phi_{\tilde{v}} \in \Phi = H^1(\Omega)$ for our model problem, where the function $\tilde{v}$ as an index in $w_{\tilde{v}}$ and $\phi_{\tilde{v}}$ shall indicate the dependence of w and ϕ on $\tilde{v}$.

The orthogonality is understood in the sense $(\mathbf{grad}\, w, \mathbf{rot}\, \phi)_\Omega = 0$ for all $w \in V$, $\phi \in \Phi$, which implies

$$\|\mathbf{grad}_h\, \tilde{v}\|_{0,\Omega}^2 = |\tilde{v}|_{1,\Omega,h}^2 = |w_{\tilde{v}}|_{1,\Omega}^2 + |\phi_{\tilde{v}}|_{1,\Omega}^2. \tag{2.5}$$

Remark 3.

1. In [8] this decomposition is used in the case of more general boundary conditions.
2. Similarly to the conforming case, in the nonconforming one the uniform boundedness of both $|w_{\tilde{v}}|_{1,\Omega}$ and $|\phi_{\tilde{v}}|^2_{1,\Omega}$ is needed to deduce a local a posteriori error indicator from an error identity like (1.12) (uniform, because our choice $\tilde{v} = z$ yields a dependence on h). To prove this, we have to verify the uniform boundedness of $|z|_{1,\Omega,h}$ taking advantage of (2.5).

In our context, we use the Helmholtz decomposition of $\mathbf{grad}_h\, \tilde{v}$ with $\tilde{v} = z$, where $z \in V \oplus V_h^{\mathrm{nc}}$ is the solution of (2.3). That yields

$$\mathcal{J}(e_h) = a_h(u - u_h, z) = a(u, w_z) - a_h(u_h, w_z) - b_h(u - u_h, \phi_z)$$

with

$$b_h(v, \phi) = -\sum_{K \in \mathcal{T}_h} b_K(v, \phi) \quad \text{and} \quad b_K(v, \phi) = (\mathbf{grad}\, v, \mathbf{rot}\, \phi)_K.$$

Now, the orthogonality of w and ϕ mentioned above can be written equivalently as

$$b_h(w, \phi) = 0 \quad \forall\, w \in V,\ \phi \in \Phi. \tag{2.6}$$

Taking into account $a(u, w) = F(w)$ for all $w \in V$ (see (1.1)) and $a_h(u_h, w_h) = F(w_h)$ for all $w_h \in V_h^{\mathrm{nc}}$ (see (1.4)), we obtain

$$\begin{aligned} a(u, w) - a_h(u_h, w) &= F(w) - a_h(u_h, w) \pm a_h(u_h, w_h) \\ &= F(w - w_h) - a_h(u_h, w - w_h) \quad \forall\, w_h \in V_h^{\mathrm{nc}}. \end{aligned}$$

Due to (2.6), this yields the error identity

$$\mathcal{J}(e_h) = F(w_z - w_h) - a_h(u_h, w_z - w_h) + b_h(u_h, \phi_z) \quad \forall\, w_h \in V_h^{\mathrm{nc}}.$$

Lemma 3.4 from [6] gives for our model problem for all $\Phi_h^{\mathrm{c}} \subset \Phi$

$$b_h(v_h, \phi_h) = 0 \quad \forall\, v_h \in V_h^{\mathrm{nc}},\ \phi_h \in \Phi_h^{\mathrm{c}}. \tag{2.7}$$

Due to (2.7) it follows for w_z, ϕ_z (which originate from the Helmholtz decomposition of $\mathbf{grad}_h\, z$, where $z \in V \oplus V_h^{\mathrm{nc}}$ is the solution of (2.3)), and for all $\Phi_h^{\mathrm{c}} \subset \Phi$ that

$$\mathcal{J}(e_h) = F(w_z - w_h) - a_h(u_h, w_z - w_h) + b_h(u_h, \phi_z - \phi_h) \quad \forall\, w_h \in V_h^{\mathrm{nc}},\ \phi_h \in \Phi_h^{\mathrm{c}}, \tag{2.8}$$

which is the starting point for our further studies.

R e m a r k 4. For $\mathcal{J}$ from (2.3) the Galerkin orthogonality is not needed for inserting an arbitrary $w_h \in V_h^{\mathrm{nc}}$. Moreover, so far we have no additional terms in the error identity as in [10] or [14].

3. Main results

Lemma 1. *Let $\Phi_h^{\mathrm{c}} \subset \Phi$ be a given finite dimensional space.*
Then for all $w \in V$ and $\phi \in \Phi$ and for all $v_h, w_h \in V_h^{\mathrm{nc}}$ and $\phi_h \in \Phi_h^{\mathrm{c}}$ we have

$$\begin{aligned}&F(w-w_h) - a_h(v_h, w-w_h) + b_h(v_h, \phi-\phi_h)\\&\quad = \sum_{K\in\mathcal{T}_h} \{(f+\Delta v_h, w-w_h)_K - (\partial_{\mathrm{n}_K} v_h, w-w_h)_{\partial K} + (\partial_{\mathrm{t}_K} v_h, \phi-\phi_h)_{\partial K}\}.\end{aligned}$$

P r o o f. For $v \in H^2(K)$, $w \in H^1(K)$ partial integration yields

$$a_K(v,w) = \iint_K [\mathbf{grad}\, v]^{\mathrm{T}}\, \mathbf{grad}\, w = -\iint_K w\Delta v + \oint_{\partial K} w\, [\mathrm{n}_K]^{\mathrm{T}}\, \mathbf{grad}\, v$$

and therefore for all $w \in V$ and for all $v_h, w_h \in V_h^{\mathrm{nc}}$

$$\begin{aligned}&F(w-w_h) - a_h(v_h, w-w_h)\\&\quad = \sum_{K\in\mathcal{T}_h} \{(f+\Delta v_h, w-w_h)_K - (\partial_{\mathrm{n}_K} v_h, w-w_h)_{\partial K}\}.\end{aligned} \tag{3.1}$$

Further, because of

$$[\mathbf{grad}\, v]^{\mathrm{T}}\, \mathbf{rot}\, w = -\operatorname{div}\, (w\, \mathbf{rot}\, v) \quad \forall\, w \in H^1(K),\ v \in H^2(K)$$

Green's formula and (1.7) yield

$$b_K(v,w) = \iint_K [\mathbf{grad}\, v]^{\mathrm{T}}\, \mathbf{rot}\, w = -\oint_{\partial K} w\, [\mathrm{n}_K]^{\mathrm{T}}\, \mathbf{rot}\, v = -\oint_{\partial K} w\, [\mathrm{t}_K]^{\mathrm{T}}\, \mathbf{grad}\, v,$$

and therefore for all $\phi \in \Phi$ and for all $v_h \in V_h^{\mathrm{nc}}$ and $\phi_h \in \Phi_h^{\mathrm{c}}$

$$b_h(v_h, \phi-\phi_h) = \sum_{K\in\mathcal{T}_h} (\partial_{\mathrm{t}_K} v_h, \phi-\phi_h)_{\partial K}. \tag{3.2}$$

Identities (3.1) and (3.2) prove the lemma. □

In addition to (A1), let us introduce the following assumption:

(A2) For the same $k \in \mathbb{N}$ as in (A1) we have

$$\partial_{\mathrm{n}_K} w\big|_{E \cap \partial K} \in P_k(E) \quad \forall\, w \in V_h^{\mathrm{nc}},\ K \in \mathcal{T}_h,\ \text{and } E \in \mathcal{E}_h \cap \partial K. \tag{3.3}$$

Remark 5. All nonconforming FEM in Section 4 satisfy (A1) and (A2).

Theorem 1. *Assume that the FE-space V_h^{nc} satisfies* (A1) *and* (A2)*. Further, let $\Phi_h^{\mathrm{c}} \subset \Phi$ be a given finite dimensional space.*

Then for all $w \in V$ and $\phi \in \Phi$ and for all $v_h, w_h \in V_h^{\mathrm{nc}}$ and $\phi_h \in \Phi_h^{\mathrm{c}}$ we have

$$\begin{aligned} F(w - w_h) &- a_h(v_h, w - w_h) + b_h(v_h, \phi - \phi_h) \\ &= \sum_{K \in \mathcal{T}_h} \Big\{ (f + \Delta v_h, w - w_h)_K - \frac{1}{2} (J_{h,\partial K \setminus \Gamma, \mathrm{n}}(v_h), w - w_h)_{\partial K \setminus \Gamma} \\ &\qquad + \frac{1}{2} (J_{h,\partial K \setminus \Gamma, \mathrm{t}}(v_h), \phi - \phi_h)_{\partial K \setminus \Gamma} + (\partial_{\mathrm{t}_K} v_h, \phi - \phi_h)_{\partial K \cap \Gamma} \Big\}. \end{aligned}$$

Proof. Because of Lemma 1, obviously we only have to prove

$$\sum_{K \in \mathcal{T}_h} (\partial_{\mathrm{n}_K} v_h, w - w_h)_{\partial K} = \frac{1}{2} \sum_{K \in \mathcal{T}_h} (J_{h,\partial K \setminus \Gamma, \mathrm{n}}(v_h), w - w_h)_{\partial K \setminus \Gamma} \tag{3.4}$$

and

$$\sum_{K \in \mathcal{T}_h} (\partial_{\mathrm{t}_K} v_h, \phi - \phi_h)_{\partial K \setminus \Gamma} = \frac{1}{2} \sum_{K \in \mathcal{T}_h} (J_{h,\partial K \setminus \Gamma, \mathrm{t}}(v_h), \phi - \phi_h)_{\partial K \setminus \Gamma}. \tag{3.5}$$

Proof of (3.4). For $E \in \mathcal{E}_h \setminus \mathcal{E}_{h,\mathrm{in}}$ because of the homogeneous boundary condition and (A1) it follows for all $w \in V$ and $w_h \in V_h^{\mathrm{nc}}$

$$(p, w)_E = (p, w_h)_E = 0 \quad \forall\, p \in P_k(E),$$

which due to (3.3) implies

$$(\partial_{\mathrm{n}_K} v_h, w - w_h)_{\partial K \cap E} = 0 \quad \forall\, E \in \mathcal{E}_h \setminus \mathcal{E}_{h,\mathrm{in}}.$$

For $E \in \mathcal{E}_{h,\mathrm{in}}$ with $K_1, K_2 \in \mathcal{T}_h$, $K_1 \neq K_2$, and $E = K_1 \cap K_2$ (see (1.9)), we study

$$\begin{aligned} \sum_{i=1}^{2} (\partial_{\mathrm{n}_{K_i}} v_h, w - w_h)_{\partial K_i \cap E} &= (\partial_{\mathrm{n}_{K_1}} v_h, w - w_h)_{\partial K_1 \cap E} \\ &\quad + (\partial_{\mathrm{n}_{K_2}} v_h, w - w_h)_{\partial K_2 \cap E}. \end{aligned}$$

Because of (A1), $(p, \left[\!\left| w_h \right|\!\right])_E = 0$ holds for all $w_h \in V_h^{\rm nc}$, $E \in \mathcal{E}_{h,\rm in}$, and $p \in P_k(E)$, which gives

$$(p, \tilde{w})_{\partial K_1 \cap E} = (p, \tilde{w})_{\partial K_2 \cap E} \quad \text{for } \tilde{w} = w_h. \tag{3.6}$$

Obviously, (3.6) also holds for $\tilde{w} = w \in V$.

Because of (3.3) and (3.6) it follows that

$$(\partial_{{\rm n}_{K_2}} v_h, w - w_h)_{\partial K_2 \cap E} = (\partial_{{\rm n}_{K_2}} v_h\big|_{E \cap \partial K_2}, w - w_h)_{\partial K_1 \cap E}$$

and therefore,

$$\sum_{i=1}^{2} (\partial_{{\rm n}_{K_i}} v_h, w - w_h)_{\partial K_i \cap E} \\ = (\partial_{{\rm n}_{K_1}} v_h\big|_{E \cap \partial K_1} + \partial_{{\rm n}_{K_2}} v_h\big|_{E \cap \partial K_2}, w - w_h)_{\partial K_1 \cap E}.$$

In the same way we obtain

$$\sum_{i=1}^{2} (\partial_{{\rm n}_{K_i}} v_h, w - w_h)_{\partial K_i \cap E} \\ = (\partial_{{\rm n}_{K_1}} v_h\big|_{E \cap \partial K_1} + \partial_{{\rm n}_{K_2}} v_h|_{E \cap \partial K_2}, w - w_h)_{\partial K_2 \cap E},$$

which implies

$$\sum_{i=1}^{2} (\partial_{{\rm n}_{K_i}} v_h, w - w_h)_{E \cap (\partial K_i \setminus \Gamma)} = (J_{h, \partial K_j \setminus \Gamma, {\rm n}}(v_h), w - w_h)_{E \cap (\partial K_j \setminus \Gamma)}, \quad j = 1, 2.$$

Altogether this proves (3.4).

P r o o f of (3.5). Because of $\Phi_h^{\rm c} \subset \Phi$ and the fact that (3.6) is obviously true for $\tilde{w} = \phi \in \Phi$, we can follow the last part of the proof of (3.4), which proves (3.5) and therefore the theorem. $\square$

Conclusion 1. *Assume that the nonconforming FEM* (1.4) *satisfies* (A1) *and* (A2), *and let $\mathcal{J}$ be an error function with* (1.16). *Further, let $\Phi_h^{\rm c} \subset \Phi$ be a given finite dimensional space, let z be the solution of the dual problem* (2.3) *and let (w_z, ϕ_z) be the corresponding pair from the Helmholtz decomposition of* $\mathbf{grad}_h\, \tilde{v}$ *with $\tilde{v} = z$ as described in Subsection 2.2 (i.e. in particular $w_z \in V$ and $\phi_z \in \Phi$).*

Then for all $w_h \in V_h^{\rm nc}$ and $\phi_h \in \Phi_h^{\rm c}$ we have the error identity

$$\mathcal{J}(e_h) = \sum_{K \in \mathcal{T}_h} \Big\{ (f + \Delta u_h, w_z - w_h)_K - \frac{1}{2} (J_{h, \partial K \setminus \Gamma, {\rm n}}(u_h), w_z - w_h)_{\partial K \setminus \Gamma} \\ + \frac{1}{2} (J_{h, \partial K \setminus \Gamma, {\rm t}}(u_h), \phi_z - \phi_h)_{\partial K \setminus \Gamma} + (\partial_{{\rm t}_K} u_h, \phi_z - \phi_h)_{\partial K \cap \Gamma} \Big\}. \tag{3.7}$$

P r o o f. Conclusion 1 is an easy consequence of (2.8) and Theorem 1. $\square$

Conclusion 1 gives the error identity (3.7) which holds for all $w_h \in V_h^{\mathrm{nc}}$ and $\phi_h \in \Phi_h^{\mathrm{c}}$.

However, for the practical use to measure the accuracy it is necessary to choose two particular $w_h \in V_h^{\mathrm{nc}}$ and $\phi_h \in \Phi_h^{\mathrm{c}}$, e.g. to use an interpolant of w_z and ϕ_z, respectively. Now, if we choose w_h to be a special interpolant of w_z then the second summand on the right-hand side of (3.7) will be equal to zero. In fact, this technique has already been used before, e.g. in [3].

As an interpolation operator $I_{h,V}\colon\ V \to V_h^{\mathrm{nc}}$, we use one which is an element of

$$\hat{I}_h = \{I_h\colon\ H^1(\Omega) \to \Upsilon_h^{\mathrm{nc}}\colon\ (p, v - I_h v)_E = 0\ \forall\, E \in \mathcal{E}_h,\ p \in P_k(E)\}, \tag{3.8}$$

however, restricted to V instead of $H^1(\Omega)$ (i.e. $I_{h,V} \in \hat{I}_h\big|_V$), such that the image is V_h^{nc} instead of Υ_h^{nc} because of the homogeneous boundary conditions and (A1).

Remark 6. On the one hand, the set $\hat{I}_h$ of interpolation operators in (3.8) is well-defined because of (A1). On the other hand, the choice of the set $\hat{I}_h$ originates from (A1) in a self-evident way.

Lemma 2. *Assume that the FE-space V_h^{nc} satisfies* (A1) *and* (A2). *Further, let $I_{h,V}\colon\ V \to V_h^{\mathrm{nc}}$ be given with $I_{h,V} \in \hat{I}_h\big|_V$.*

Then for all $v_h \in V_h^{\mathrm{nc}}$ and $w \in V$ with the corresponding $w_h = I_{h,V} w \in V_h^{\mathrm{nc}}$ we have

$$\sum_{K \in \mathcal{T}_h} (J_{h,\partial K\setminus\Gamma,\mathrm{n}}(v_h), w - w_h)_{\partial K\setminus\Gamma} = 0.$$

Proof. Lemma 2 is an easy consequence of (3.3), (1.8), and $I_{h,V} \in \hat{I}_h\big|_V$. □

Conclusion 2. *Under the same assumptions as in Conclusion* 1 *the following is true.*

Let $I_{h,V}\colon\ V \to V_h^{\mathrm{nc}}$ be given with $I_{h,V} \in \hat{I}_h\big|_V$.

Then for all $\phi_h \in \Phi_h^{\mathrm{c}}$ we have the error identity

$$\begin{aligned} \mathcal{J}(e_h) = \sum_{K \in \mathcal{T}_h} \Big\{ &(f + \Delta u_h, w_z - I_{h,V} w_z)_K \\ &+ \frac{1}{2}(J_{h,\partial K\setminus\Gamma,\mathrm{t}}(u_h), \phi_z - \phi_h)_{\partial K\setminus\Gamma} + (\partial_{\mathrm{t}_K} u_h, \phi_z - \phi_h)_{\partial K\cap\Gamma} \Big\}. \end{aligned} \tag{3.9}$$

Proof. Conclusion 2 is an easy consequence of Conclusion 1 and Lemma 2. □

4.1. Preliminary

4.1.1. About nonconforming triangular P_l-elements. Here we have $\mathcal{R}_K = P_l$ in the definition of the nonconforming FE-space $V_h^{\rm nc}$ (see Subsection 2.1). Further, we choose $k = l - 1$; then assumption (A2) obviously holds.

In view of the application to these elements in Subsection 4.3, let us point out the result in [3] which can be written with our notation in the following form:

If l is odd (and $k = l - 1$) then the corresponding subclass of nonconforming FEM defined by (A1) is not empty and the following set of degrees of freedom is P_l-unisolvent:

(i) On $E \in \mathcal{E}_h$, the values of $v \mapsto (p_{i,E}, v)_E$, for $i = 0, \dots, k = l - 1$, where $p_{i,E}$ are scaled orthogonal Legendre polynomials of degree i on E, and

(ii) in $K \in \mathcal{T}_h$, the values of $v \mapsto (p_{i,K}, v)_E$, for $i = 0, \dots, \dim(P_{l-3})$, where $p_{i,K}$ are arbitrary basis functions of the space P_{l-3}.

The choice in (i) originates from (A1) in a self-evident way.

Remark 7. If l is even, the construction of the nonconforming FE-space $V_h^{\rm nc}$ is possible, but more difficult (see [3]).

4.1.2. Preparation for the application to a special error function. For arbitrary error functions $\mathcal{J}$, the difficulty is to get good approximations for w_z and ϕ_z. Here, further investigations are necessary. Therefore, for our application of the error identities (3.7) and (3.9) we restrict ourselves to the special error function

$$\mathcal{J}(v) = \frac{1}{\sqrt{a_h(e_h, e_h)}}\, a_h(v, e_h) \tag{4.1}$$

(see [5]) which implies $\mathcal{J}(e_h) = |e_h|_{1,\Omega,h}$, and to the case, where local a posteriori error indicators are computable without knowing the solution z of (2.3).

The last is possible, if the following two conditions hold (compare to the summary of the main ideas of the DWR method for conforming FEM in Section 1):

- For the solution z of (2.3) $w_z, \phi_z \in H^\beta(\Omega)$ holds for some $\beta \geqslant 1$, and
- for two different interpolation operators $I_{h,V}\colon\ V \to V_h^{\rm nc}$ and $I_{h,\varPhi}\colon\ \varPhi \to \varPhi_h^{\rm c}$ the local interpolation error estimates

$$\|v - I_{h,V}v\|_{0,K} + h_K^{1/2}\|v - I_{h,V}v\|_{0,\partial K} \leqslant Ch_K^\beta |v|_{\beta,\omega_K} \quad \forall\, K \in \mathcal{T}_h,\ v \in V, \tag{4.2}$$

and

$$\|\varphi - I_{h,\varPhi}\varphi\|_{0,K} + h_K^{1/2}\|\varphi - I_{h,\varPhi}\varphi\|_{0,\partial K} \leqslant Ch_K^\beta |\varphi|_{\beta,\omega_K} \quad \forall\, K \in \mathcal{T}_h, \varphi \in \varPhi, \tag{4.3}$$

respectively, are satisfied (compare to (1.13)), where ω_K is again a suitable and possibly small neighbourhood of K.

Then, on the one hand, we are able to compare our results to known a posteriori error indicators with respect to the discrete H^1-seminorm (1.5). On the other hand, the reliability follows in the same way as in (1.14) for conforming FEM.

Remark 8. Because of (2.7) the choice $\Phi_h^{\rm c} \subset \Phi$ is determined only by the fact that (4.3) holds with the same β as that in (4.2). Therefore, in our applications the choice of $\Phi_h^{\rm c}$ is simple.

4.2. The case $k = 0$

4.2.1. The nonconforming triangular P_1-element. For this element, in (2.1) we have $\mathcal{R}_K = P_1 = \operatorname{span}\{1, x, y\}$.

If $\mathcal{M}_h$ denotes the set of all midpoints of the edges $E \in \mathcal{E}_h$, then obviously $\Upsilon_h^{\rm nc}$ from (2.1) is equivalent to

$$(4.4)\quad \Upsilon_h^{\rm nc} = \{v \in L^2(\Omega)\colon\ v\big|_K \in P_1\ \forall\, K \in \mathcal{T}_h \text{ and } v \text{ is continuous at } P \in \mathcal{M}_h \cap \Omega\}.$$

Further, $V_h^{\rm nc}$ from (2.2) is uniquely defined and equivalent to

$$V_h^{\rm nc} = \{v \in \Upsilon_h^{\rm nc}\colon\ v(P) = 0\ \forall\, P \in \mathcal{M}_h \cap \Gamma\}.$$

Therefore, in this case the function values at $P \in \mathcal{M}_h$ can be used as degrees of freedom. Obviously, (A1) and (A2) are satisfied. Further, we choose

$$\Phi_h^{\rm c} = \{v \in C(\overline{\Omega})\colon\ v\big|_K \in P_1\ \forall\, K \in \mathcal{T}_h\}. \tag{4.5}$$

Then for the triangular linear nonconforming FEM and for $\Phi_h^{\rm c}$ defined by (4.5) Conclusion 1 yields (3.7) and Conclusion 2 yields (3.9). This is the first result for this FEM and holds for arbitrary error functions $\mathcal{J}$.

Further, it is known that for $\Upsilon_h^{\rm nc}$ from (4.4) the interpolation operator $\hat{I}_h$ from (3.8) is uniquely defined and satisfies the local interpolation error estimate (4.2) with $\beta = 1$ under weak assumptions (see e.g. [1]). If we now choose $\mathcal{J}$ from (4.1), then (3.9) and these properties of $\hat{I}_h$ together with the analogous properties of the interpolation operator $I_{h,\Phi}\colon \Phi \to \Phi_h^{\rm c}$, which are also known, obviously result in the local a posteriori error indicator with respect to the discrete H^1-seminorm

$$\eta_K^* = \Big\{h_K^2\|f\|_{0,K}^2 + \frac{1}{2}h_K\|J_{h,\partial K\setminus\Gamma,\rm t}(u_h)\|_{0,\partial K\setminus\Gamma}^2 + h_K\|\partial_{{\rm t}_K} u_h\|_{0,\partial K\cap\Gamma}^2\Big\}^{1/2} \tag{4.6}$$

due to $\Delta u_h = 0$ on K, which is the second result for this FEM.

Remark 9.

1. For the local error indicators η_K^* from (4.6) and η_K from (1.6), we obviously have $\eta_K^* \leqslant \eta_K$, so that the known efficiency of η_K implies that of η_K^*.
2. The local a posteriori error indicator (4.6) was already given in [8], but deduced in another way.

4.2.2. Nonconforming quadrilateral elements. It is easy to verify that the following nonconforming FEM also considered in [6], namely the ones based on parallelograms and

- on the Rannacher-Turek element (see [13]) with $\mathcal{R}_{\tilde{K}} = \operatorname{span}\{1, x, y, x^2 - y^2\}$ or
- on the Han element (see [11]) with $\mathcal{R}_{\tilde{K}} = \operatorname{span}\{1, x, y, 3x^2 - 5x^4, 3y^2 - 5y^4\}$ or
- on the DSSY element (see [9]) with $\theta_1(t) = 3t^2 - 5t^4$, $\theta_2(t) = 6t^2 - 25t^4 + 21t^6$, and $\mathcal{R}_{\tilde{K}} = \operatorname{span}\{1, x, y, \theta_1(x) - \theta_1(y), \theta_2(x) - \theta_2(y)\}$,

(the unit square $\tilde{K} = (-1, 1)^2$ being the reference element), satisfy (A1) and (A2) if the degree of freedom on $E \in \mathcal{E}_h$ is chosen again as $(1, v)_E$ for $v \in V_h^{\mathrm{nc}}$. If we use

$$\Phi_h^{\mathrm{c}} = \{v \in C(\overline{\Omega})\colon\ v\big|_K \in Q_1\ \forall\, K \in \mathcal{T}_h\},$$

then also Conclusion 1 yields (3.7) and Conclusion 2 yields (3.9) for these three nonconforming quadrilateral FEM, which is again the first result for these FEM.

Remark 10.

1. For $\mathcal{J}$ from (4.1) and assuming the above properties of the interpolation operators $I_{h,V}$ and $I_{h,\Phi}$, with the same arguments as in Subsubsection 4.2.1 it is possible to show that

$$\eta_K^* = \Big\{h_K^2\|f + \Delta u_h\|_{0,K}^2 + \frac{1}{2}h_K\|J_{h,\partial K\setminus\Gamma,\mathrm{t}}(u_h)\|_{0,\partial K\setminus\Gamma}^2 + h_K\|\partial_{\mathrm{t}_K} u_h\|_{0,\partial K\cap\Gamma}^2\Big\}^{1/2} \tag{4.7}$$

   is a local a posteriori error indicator with respect to the discrete H^1-seminorm for these nonconforming FEM, which is a new one and which is the second result for these FEM.
2. Again, for the local error indicators η_K^* from (4.7) and η_K from (1.6), we obviously have $\eta_K^* \leqslant \eta_K$, so that the known efficiency of η_K (see [6]) implies that of η_K^*.
3. For the Rannacher-Turek element the above properties of the corresponding interpolation operator $\hat{I}_h$ can be found in [2].

4. For the Han and for the DSSY element, respectively, one additional degree of freedom on $K \in \mathcal{T}_h$ has to be chosen, which can be the value of

$$(1, v)_K \quad \text{for } v \in V_h^{\mathrm{nc}}.$$

Then $I_{h,V} \in \hat{I}_h^*\big|_V$ with the interpolation operator

$$\hat{I}_h^* = \{I_h \in \hat{I}_h\colon\ (1, v - I_h v)_K = 0\ \forall\, K \in \mathcal{T}_h\}$$

can be used, which is uniquely defined.

4.3. The case $k = 2$ and one nonconforming triangular P_3-element

We use the P_3-element described in Subsubsection 4.1.1 including the choice of the degrees of freedom there, such that the corresponding nonconforming cubic triangular FEM is uniquely defined and (A1) and (A2) are satisfied. Further, we use

$$\Phi_h^{\mathrm{c}} = \{v \in C(\overline{\Omega})\colon\ v\big|_K \in P_3\ \forall\, K \in \mathcal{T}_h\}. \tag{4.8}$$

Then for that nonconforming cubic triangular FEM and for Φ_h^{c} defined by (4.8) Conclusion 1 yields (3.7) and Conclusion 2 yields (3.9).

R e m a r k 11. In the same way applications are possible to nonconforming triangular FEM for arbitrary even k.

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Author's address: *R. Vanselow*, Institut für Numerische Mathematik, Technische Universität, Dresden, Germany, e-mail: `Reiner.Vanselow@tu-dresden.de`.