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GENERATED FUZZY IMPLICATIONS AND FUZZY PREFERENCE STRUCTURES

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The notion of a construction of a fuzzy preference structures is introduced. The properties of a certain class of generated fuzzy implications are studied. The main topic in this paper is investigation of the construction of the monotone generator triplet (p, i, j), which is the producer of fuzzy preference structures. Some properties of mentioned monotone generator triplet are investigated.

Keywords: generated fuzzy implication, fuzzy preference structure, fuzzy implications, t-norm

Classification: 60A05, 08A72, 28E10

1. PRELIMINARIES

First we recall notations and basic definitions used in the paper. We also briefly mention some important properties and results in order to make this work self-contained. We start with the basic logic connectives.

Definition 1.1. (see e. g. in Fodor and Roubens [3], Definition 1.1) A decreasing function $N : [0, 1] \to [0, 1]$ is called a *fuzzy negation* if for each $a, b \in [0, 1]$ it satisfies the following conditions

(i) $a < b \Rightarrow N(b) \leq N(a)$,

(ii) $N(0) = 1, N(1) = 0$.

Remark 1.2. A fuzzy negation N is called *strict* if N is strictly decreasing and continuous for arbitrary $x, y \in [0, 1]$. In a classical logic we have that $(A')' = A$. In multivalued logic this equality is not satisfied for each fuzzy negation. The fuzzy negations with this equality are called *involutive negations.* The strict fuzzy negation is *strong* if and only if it is involutive.

Some examples of strict and/or strong fuzzy negations are included in the following example. More examples of fuzzy negations can be found in [3].

Example 1.3. The next functions are fuzzy negations on $[0,1]$.

- $N_s(a) = 1 - a$ strong fuzzy negation, standard negation.
- $N(a) = 1 - a^2$ strict, but not strong fuzzy negation.
- $N(a) = \sqrt{1 - a^2}$ strong fuzzy negation.

Remark 1.4. Commonly used fuzzy negation in applications is *the standard negation N_s*.

Definition 1.5. An increasing mapping $C : [0,1]^2 \to [0,1]$ is called a *fuzzy conjunction* if

1. $C(x,y) = 0$ whenever $x = 0$ or $y = 0$, and
2. $C(1,1) = 1$.

Commonly used fuzzy conjunctions in fuzzy logic are the triangular norms.

Definition 1.6. (Klement et al. [5], Definition 1.1) A *triangular norm* (*t-norm* for short) is a binary operation on the unit interval $[0,1]$, i. e., a function $T : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$, the following four axioms are satisfied:

(T1) *Commutativity* $T(x,y) = T(y,x)$,

(T2) *Associativity* $T(x, T(y,z)) = T(T(x,y), z)$,

(T3) *Monotonicity* $T(x,y) \leq T(x,z)$ whenever $y \leq z$,

(T4) *Boundary Condition* $T(x,1) = x$.

Three most common continuous t-norms are:

- *Minimum t-norm* $T_M(x,y) = \min(x,y)$,
- *Product t-norm* $T_P(x,y) = x \cdot y$,
- *Łukasiewicz t-norm* $T_L(x,y) = \max(0, x + y - 1)$.

Remark 1.7. Let $s \in]0,\infty[- \{1\}$. Then function $T^s : [0,1]^2 \to [0,1]$, which is given as follows

$$T^s(x,y) = \log_s \left(1 + \frac{(s^x - 1)(s^y - 1)}{s - 1}\right)$$

is called *Frank t-norm with parameter s*. The limit cases are $T^0 = T_M$, $T^1 = T_P$ and $T^\infty = T_L$.

Remark 1.8. Note that the dual operator to a fuzzy conjunction C, defined by $D(x,y) = 1 - C(1-x, 1-y)$, is called a *fuzzy disjunction*. Commonly used fuzzy disjunctions in fuzzy logic are the *triangular conorms*. A triangular conorm (also called a $t-$conorm) is a binary operation S on the unit interval $[0,1]$ which, for all $x, y, z \in [0,1]$, satisfies $(T1) - (T3)$ and $(S4)$ $S(x,0) = x$. For more information, see [5].

Dual t-conorms to T_M, T_P and T_L are:

- *Maximum t-conorm* $S_M(x,y) = \max(x,y)$,
- *Probabilistic sum* $S_P(x,y) = x + y - x \cdot y$,
- *Łukasiewicz t-conorm* $S_L(x,y) = \min(1, x+y)$.

Definition 1.9. Let $\varphi : [0,1] \to [0,1]$ be an order-automorphism. Then

$$\begin{aligned} T_\varphi(x,y) &= \varphi^{-1}(T(\varphi(x),\varphi(y))) \\ S_\varphi(x,y) &= \varphi^{-1}(S(\varphi(x),\varphi(y))) \\ (N_s)_\varphi(x) &= \varphi^{-1}(1-\varphi(x)) \end{aligned}$$

are called *φ-transformations* of T, S, and N_s, respectively.

In the literature, we can find several different definitions of fuzzy implications. In this paper we will use the following one, which is equivalent with the definition introduced by Fodor and Roubens in [3]. The readers can find more details in [2, 6].

Definition 1.10. A function $I : [0,1]^2 \to [0,1]$ is called a *fuzzy implication* if it satisfies the following conditions:

(I1) I is decreasing in its first variable,

(I2) I is increasing in its second variable,

(I3) $I(1,0) = 0$, $I(0,0) = I(1,1) = 1$.

Our constructions of fuzzy implications will use extensions of the classical inverse of a function. It can be extended as follows.

Definition 1.11. (Klement et al. [5], Corollary 3.3) Let $\varphi : [0,1] \to [0,\infty]$ be an increasing and non-constant function. The function $\varphi^{(-1)}$ defined by

$$\varphi^{(-1)}(x) = \sup\{z \in [0,1]; \varphi(z) < x\}$$

is called the *pseudo-inverse* of φ, with the convention $\sup \emptyset = 0$.

Definition 1.12. (Klement et al. [5], Corollary 3.3) Let $f : [0,1] \to [0,\infty]$ be a decreasing and non-constant function. The function $f^{(-1)}$ defined by

$$f^{(-1)}(x) = \sup\{z \in [0,1]; f(z) > x\}$$

is called the pseudo-inverse of f, with the convention $\sup \emptyset = 0$.

It is well-known that it is possible to generate t-norms from one variable functions. This means that it is enough to consider a one variable function instead of a two-variable function. Moreover, we can generate fuzzy implications in a similar way as t-norms. Hence, we can get so called I_f and I^g fuzzy implications.

Theorem 1.13. (Hliněná and Biba [4], Smutná [7]) Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$. Then the function $I_f : [0,1]^2 \to [0,1]$ given by

$$I_f(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ f^{(-1)}(f(y^+) - f(x)) & \text{otherwise,} \end{cases}$$

where $f(y^+) = \lim\limits_{y \to y^+} f(y)$ and $f(1^+) = f(1)$, is a fuzzy implication.

For strictly increasing functions g it is possible to construct a fuzzy implication I^g as follows.

Theorem 1.14. (Smutná [7] without proof) Let $g : [0,1] \to [0,\infty]$ be a strictly increasing function such that $g(0) = 0$. Then the function $I^g : [0,1]^2 \to [0,1]$ given by

$$I^g(x,y) = g^{(-1)}(g(1-x) + g(y))$$

is a fuzzy implication.

Proof. In [7] this assertion is not proved. Here we present a complete proof. We will proceed by points of Definition 1.10.

(I1) Let $x_1, x_2, y \in [0,1]$ and $x_1 \le x_2$. Function g is increasing and therefore $g(1-x_1) \ge g(1-x_2)$ and $g(1-x_1)+g(y) \ge g(1-x_2)+g(y)$. Pseudo-inverse $g^{(-1)}$ of function g is increasing too and $g^{(-1)}(g(1-x_1)+g(y)) \ge g^{(-1)}(g(1-x_2)+g(y))$. Therefore $I^g(x_1,y) \ge I^g(x_2,y)$ and it means that function I^g is decreasing in its first variable.

(I2) Let $x, y_1, y_2 \in [0,1]$ and $y_1 \le y_2$. Function g is increasing and therefore $g(y_1) \le g(y_2)$ and $g(1-x)+g(y_1) \le g(1-x)+g(y_2)$. Pseudo-inverse $g^{(-1)}$ of function g is increasing too and $g^{(-1)}(g(1-x)+g(y_1)) \le g^{(-1)}(g(1-x)+g(y_2))$. Therefore $I^g(x,y_1) \le I^g(x,y_2)$ and it means that function I^g is increasing in its second variable.

(I3) For $I^g(0,0)$ and $I^g(1,1)$ we get $I^g(0,0) = I^g(1,1) = g^{(-1)}(g(1)) = 1$. For $I^g(1,0)$ we have

$$I^g(1,0) = g^{(-1)}(2 \cdot g(0)) = \sup\{z \in [0,1]; g(z) < 2 \cdot g(0)\} = 0.$$

□

Remark 1.15. Previous theorem can be further generalized using general fuzzy negation $N(x)$ instead of $N_s(x) = 1-x$, see [7]. Let $g : [0,1] \to [0,\infty]$ be a strictly increasing function, $g(0) = 0$ and $N : [0,1] \to [0,1]$ be a fuzzy negation, then

$$I^g_N(x,y) = g^{(-1)}(g(N(x)) + g(y))$$

is a fuzzy implication.

2. FUZZY PREFERENCE STRUCTURES

A preference structure is a basic concept of preference modelling. In a classical preference structure (PS), a decision-maker makes one of three decisions for each pair (a, b) from the set $\boldsymbol{A}$ of all alternatives. His decision defines a triplet P, I, J of crisp binary relations on $\boldsymbol{A}$:

1) a is preferred to $b \Leftrightarrow (a, b) \in P$ (strict preference)
2) a and b are indifferent $\Leftrightarrow (a, b) \in I$ (indifference)
3) a and b are incomparable $\Leftrightarrow (a, b) \in J$ (incomparability).

A *preference structure (PS)* on a set $\boldsymbol{A}$ is a triplet (P, I, J) of binary relations on $\boldsymbol{A}$ such that

(ps1) I is reflexive, while P and J are irreflexive,

(ps2) P is asymmetric, while I and J are symmetric,

(ps3) $P \cap I = P \cap J = I \cap J = \emptyset$,

(ps4) $P \cup I \cup J \cup P^t = \mathbf{A} \times \mathbf{A}$ where $P^t(x, y) = P(y, x)$.

Using characteristic mappings [8] a minimal definition of (PS) can be formulated as a triplet (P, I, J) of binary relations on $\boldsymbol{A}$ such that

- I is reflexive and symmetric,
- $P(a, b) + P^t(a, b) + I(a, b) + J(a, b) = 1$ for all $(a, b) \in \boldsymbol{A}^{\mathbf{2}}$.

A preference structure can be characterized by the reflexive relation $R = P \cup I$ called the *large preference relation*. The relation R can be interpreted as

$$(a, b) \in R \Leftrightarrow a \text{ is prefered to } b \text{ or } a \text{ and } b \text{ are indifferent.}$$

It can be easily proved that

$$co(R) = P^t \cup J,$$

where $co(R)$ is the complement of R and

$$P = R \cap co(R^t),\ I = R \cap R^t,\ J = co(R) \cap co(R^t).$$

This allows us to construct a preference structure (P, I, J) from a reflexive binary operation R only.

To define *fuzzy preference structure (FPS)* it is necessary to consider some fuzzy connectives. First we recall some special fuzzy relations:

Definition 2.1. (Zadeh [9]) Fuzzy relation R is

- *reflexive,* if $\forall x \in X; \mu_R(x, x) = 1$,
- *irreflexive,* if $\forall x \in X; \mu_R(x, x) = 0$,

- *symmetric,* if $\forall x, y \in X; \mu_R(x,y) = \mu_R(y,x)$.

We shall consider a continuous De Morgan triplet (T, S, N) consisting of a continuous t-norm T, continuous t-conorm S and a strong fuzzy negation N such that $T(x,y) = N(S(N(x), N(y)))$. The main problem lies in the fact that the completeness condition (ps4) can be written in many forms, e. g.:

$$co(P \cup P^t) = I \cup J, P = co(P^t \cup I \cup J), P \cup I = co(P^t \cup J).$$

Note that it was proved in [3, 8] that reasonable constructions of fuzzy preference structure (FPS) should use a nilpotent t-norm only. Since any nilpotent t-norm (t-conorm) is isomorphic to the Łukasiewicz t-norm (t-conorm), it is enough to restrict our attention to De Morgan triplet $(T_L, S_L, 1-x)$. Then we can define (FPS) as the triplet of binary fuzzy relations (P, I, J) on the set of alternatives $\boldsymbol{A}$ satisfying:

- I is reflexive and symmetric,
- $\forall (a,b) \in \boldsymbol{A^2}$, $P(a,b) + P^t(a,b) + I(a,b) + J(a,b) = 1$.

It has been mentioned, that it is possible to construct preference structure from a large preference relation R in the classical case, however, in fuzzy case this is not possible. This fact was proved by Alsina in [1] and later by Fodor and Roubens in [3]:

Proposition 2.2. (Fodor and Roubens [3], Proposition 3.1) There is no continuous de Morgan triplet (T, S, N) such that $R = P \cup_S I$ holds with $P(a,b) = T(R(a,b), N(R(b,a)))$ and $I(a,b) = T(R(a,b), R(b,a))$.

Because of this negative result, Fodor and Roubens (among others) proposed axiomatic construction. Assume that we deal with the Łukasiewicz triplet $(T_L, S_L, 1-x)$.

(R1) Independence of Irrelevant Alternatives:
For any two alternatives a, b the values of $P(a,b), I(a,b), J(a,b)$ depend only on the values $R(a,b), R(b,a)$. I.e., there exist functions $p, i, j : [0,1]^2 \to [0,1]$ such that, for any $a, b \in \mathbf{A}$,

$$P(a,b) = p(R(a,b), R(b,a)),$$

$$I(a,b) = i(R(a,b), R(b,a)),$$

$$J(a,b) = j(R(a,b), R(b,a)).$$

(R2) Positive Association Principle:
Functions $p(x, 1-y)$, $i(x,y)$, $j(1-x, 1-y)$ are increasing in x and y.

(R3) Symmetry:
$i(x,y)$ and $j(x,y)$ are symmetric functions.

(R4) $(P,\ I,\ J)$ is (FPS) for any reflexive relation R on a set $\boldsymbol{A}$ such that

$$S_L(P, I) = R, \quad S_L(P, J) = 1 - R^t.$$

It was proved ([3], Theorem 3.1) that for all $x, y \in [0,1]$ it holds:

$$T_L(x,y) \ \leq \ p(x, 1-y), i(x,y), j(1-x, 1-y) \ \leq \ T_M(x,y).$$

The mentioned triplet (p, i, j) is called *the monotone generator triplet.* Summarizing, the monotone generator triplet is a triplet (p, i, j) of mappings $[0,1]^2 \to [0,1]$ such that

(gt1) $p(x, 1-y),\ i(x,y),\ j(1-x, 1-y)$ are increasing in both coordinates,

(gt2) $T_L(x,y) \ \leq \ p(x, 1-y), i(x,y), j(1-x, 1-y) \ \leq \ T_M(x,y)$,

(gt3) $i(x,y) = i(y,x)$,

(gt4) $p(x,y) + p(y,x) + i(x,y) + j(x,y) = 1$,

(gt5) $p(x,y) + i(x,y) = x$.

Using these properties, one may show that also $j(x,y) = j(y,x)$ and $p(x,y) + j(x,y) = 1 - y$. Therefore the axiom (R4) can be expressed as a system of functional equations:

(R4')

$$p(x,y) + i(x,y) = x,$$
$$p(x,y) + j(x,y) = 1 - y.$$

Remark 2.3. It is possible to formulate similar axioms in the framework of more general De-Morgan triplet $(T_L)_\varphi, (S_L)_\varphi, (N_s)_\varphi)$, which is a φ−transformation of $(T_L, S_L, 1-x)$. The solution is then expressed as $(p, i, j)_\varphi$.

Fuzzy implications are closely related to generators of a strict preference. The following proposition can be found in [3]. Fodor and Roubens supposed general triplet $(T_\varphi, S_\varphi, N_\varphi)$:

Proposition 2.4. (Fodor and Roubens [3], Proposition 3.5) Let $S : [0,1]^2 \to [0,1]$ be any continuous t-conorm and $N : [0,1] \to [0,1]$ be a strong fuzzy negation. If $(p, i, j)_\varphi$ is a solution of the system

$$S(p(x,y), i(x,y)) = x,$$
$$S(p(x,y), j(x,y)) = N(y),$$

then $I^{\rightarrow}(x,y) = N_\varphi(p(x,y))$ is a fuzzy implication such that

$$I^{\rightarrow}(1,x) = x \qquad \forall x \in [0,1],$$
$$I^{\rightarrow}(x,0) = N_\varphi(x) \quad \forall x \in [0,1].$$

Since we are dealing with Łukasiewicz triplet $(T_L, S_L, 1-x)$, this proposition can be simplified:

Proposition 2.5. Let (p, i, j) be a solution of the system in (R4'), then $I^{\rightarrow}(x,y) = 1 - p(x,y)$ is a fuzzy implication and

$$I^{\rightarrow}(1,x) = x \qquad \forall x \in [0,1],$$
$$I^{\rightarrow}(x,0) = 1 - x \quad \forall x \in [0,1].$$

3. PREFERENCE STRUCTURES GIVEN BY GENERATED FUZZY IMPLICATIONS

First we will turn our attention to I_f fuzzy implications. In the next example, we deal with the Łukasiewicz triplet $(T_L, S_L, 1-x)$:

Example 3.1. Let $f(x) = N_s(x)$. Note that fuzzy negation N_s satisfies assumptions of Theorem 1.13. We obtain fuzzy implication $I_{N_s}(x, y) = \min(1 - x + y, 1)$. For function p we have

$$p(x, y) = 1 - I_{N_s}(x, y) = \max(x - y, 0).$$

In order to satisfy (R4'), mappings i, j must be $i(x, y) = \min(x, y)$ and $j(x, y) = \min(1-x, 1-y)$. Obviously i and j are symmetric functions. Therefore (R3) is satisfied.

Now, we turn our attention to the properties (gt1)–(gt5). Axioms (R3) and (R4') imply properties (gt3) and (gt5). More, from (R3) and (R4') we have

$$p(x, y) + p(y, x) + i(x, y) + j(x, y) = p(x, y) + i(x, y) + p(y, x) + j(y, x) = x + 1 - x = 1.$$

Therefore property (gt4) again follows from (R3) and (R4').

It is obvious that in this example the properties (gt1) and (gt2) are satisfied, too. Therefore triplet (p, i, j) is the monotone generator triplet.

Remark 3.2. Note that the fuzzy implication $I_{N_s}(x, y) = \min(1 - x + y, 1)$ from the previous example is the well-known Łukasiewicz implication I_{T_L}.

The following proposition shows that the fuzzy implications I_{T_L} is the only one we can use:

Proposition 3.3. Let $f : [0, 1] \to [0, \infty]$ be a strictly decreasing function such that $f(1) = 0$ and $p(x, y) = 1 - I_f(x, y)$. Then triplet (p, i, j), where $i(x, y) = x - p(x, y)$ and $j(x, y) = 1 - y - p(x, y)$, satisfies (R3) and (R4') if and only if $I_f(x, y) = I_{T_L}$.

Proof. Let (p, i, j) satisfy (R3) and (R4'). Then by (R3), $i(x, y)$ is symmetric function. Since $p(x, y) = 1 - I_f(x, y)$, from (R4') we get

$$x - 1 + I_f(x, y) = y - 1 + I_f(y, x).$$

From the definition of I_f (see Theorem 1.13), either $I_f(x, y) = 1$ or $I_f(y, x) = 1$. Therefore by previous equality, either $I_f(y, x) = 1 - y + x$, or $I_f(x, y) = 1 - x + y$ in order to satisfy both (R3) and (R4') at the same time. The converse is obvious from previous example. □

Remark 3.4. Note that a fuzzy implication satisfies *the ordering property (OP)* if the following is true: $x \le y$ if and only if $I(x, y) = 1$. The previous proposition can be generalized for all fuzzy implications with (OP).

Proposition 3.5. Let $I : [0, 1]^2 \to [0, 1]$ be a fuzzy implication satisfying (OP), and $p(x, y) = 1 - I(x, y)$. Then the triplet (p, i, j) satisfies (R3) and (R4') if and only if $I(x, y) = I_{T_L}$.

P r o o f. Let the triplet (p, i, j) satisfy (R3) and (R4') and $p(x, y) = 1 - I(x, y)$. Using (R4') we get $i(x, y) = x - 1 + I(x, y)$ and from symmetry of $i(x, y)$ we have the equality

$$x + I(x, y) = y + I(y, x).$$

Since $I(x, y)$ satisfies (OP), we have $I(x, y) = 1$ or $I(y, x) = 1$, and therefore we get $I(x, y) = I_{T_L}$. The converse is similar to Example 3.1. □

Remark 3.6. Note, that the triplets mentioned in previous propositions satisfy also properties (gt1)–(gt5), this means they are monotone generator triplets.

Remark 3.7. Note, that it has been proved (see [4]) that continuity of function f at $x = 1$ is equivalent with (OP) for the fuzzy implication I_f.

In the next example, we will assume de Morgan triplet $((T_L)_\varphi, (S_L)_\varphi, (N_s)_\varphi)$:

Example 3.8. Let φ be an order-automorphism and $f(x) = 1 - \varphi(x)$, then

$$I_f(x, y) = \begin{cases} 1 & x \leq y, \\ \varphi^{-1}(1 - \varphi(x) + \varphi(y)) & x > y. \end{cases}$$

The triplet $(p, i, j)_\varphi$ such that $p(x, y) = (N_s)_\varphi(I_f(x, y))$, $i(x, y) = \varphi^{-1}(\varphi(x) - 1 + \varphi(I_f(x, y)))$, and $j(x, y) = \varphi^{-1}(\varphi(I_f(x, y)) - \varphi(y))$, satisfies axioms (R3) and (R4'): After plugging in $I_f(x, y)$, we get

$$p(x, y) = \varphi^{-1}\left(\max(\varphi(x) - \varphi(y), 0)\right),$$

$$i(x, y) = \varphi^{-1}\left(\min(\varphi(x), \varphi(y))\right),$$

$$j(x, y) = \varphi^{-1}\left(\min(1 - \varphi(x), 1 - \varphi(y))\right).$$

As we have mentioned, we assume de Morgan triplet $((T_L)_\varphi, (S_L)_\varphi, (N_s)_\varphi)$ in this example. In this case, a more general form of (R4') is needed:

$$(S_L)_\varphi\left(p(x, y), i(x, y)\right) = x, \quad (S_L)_\varphi\left(p(x, y), j(x, y)\right) = (N_s)_\varphi(y).$$

Obviously the mappings i, j are symmetric functions, i. e. (R3) is satisfied. The proof that axiom (R4') is also satisfied is simple, but lengthy.

For the triplet $((T_L)_\varphi, (S_L)_\varphi, (N_s)_\varphi)$ and fuzzy implications I_f we get a result similar to Proposition 3.3:

Proposition 3.9. Let φ be an order-automorphism. Let $f : [0, 1] \to [0, \infty]$ be a strictly decreasing function such that $f(1) = 0$, and

$$I_f(x, y) = \begin{cases} 1 & x \leq y, \\ f^{(-1)}(f(y^+) - f(x)) & x > y. \end{cases}$$

Then the system $(p, i, j)_\varphi$ where $p(x, y) = (N_s)_\varphi(I_f(x, y))$ satisfies (R3), and (R4') if and only if $I_f(x, y) = \min(\varphi^{-1}(1 - \varphi(x) + \varphi(y)), 1)$.

A proof of this fact is similar to the proofs of previous propositions.

Now we turn our attention to the fuzzy implications I^g and I^g_N. The partial mapping of $I^g(x,0)$ is $I^g(x,0) = 1-x$, and for an arbitrary fuzzy negation N we have $I^g_N(x,0) = N(x)$. On the other hand, Proposition 2.5 gives that $I^{\rightarrow}(x,0) = 1-x$, therefore we will investigate function $p(x,y) = 1 - I^g(x,y)$. Using (R4'), we get $i(x,y) = I^g(x,y) + x - 1$ and $j(x,y) = I^g(x,y) - y$. From (R3), the function i is symmetric, which leads to the equality

$$I^g(x,y) - I^g(y,x) = y - x \qquad \forall x,y \in [0,1]. \tag{1}$$

If this equality is fulfilled for some fuzzy implication I, then the described triplet (p,i,j) is a generator triplet.

We are looking for functions g, such that fuzzy implications I^g satisfy the equality (1). Several appropriate functions are given in the following examples.

Example 3.10. Let $g_1(x) = -\ln(1-x)$, then its pseudo-inverse function is $g_1^{(-1)}(x) = 1 - e^{-x}$. The fuzzy implication I^{g_1} is given by

$$I^{g_1}(x,y) = 1 - x + xy.$$

For the mentioned difference we get

$$I^{g_1}(x,y) - I^{g_1}(y,x) = (1-x+xy) - (1-y+xy) = y - x.$$

Equality (1) holds, and triplet (p,i,j), where $p(x,y) = x(1-y)$, $i(x,y) = xy$, $j(x,y) = (1-x)(1-y)$, satisfies axioms (R3)–(R4') and properties (gt1)–(gt5). Note that fuzzy implication I^{g_1} is the well-known Reichenbach implication which is not isomorphic with I_{T_L}.

Example 3.11. Let $g_2(x) = x$. The pseudo-inverse of function g_2 is given by $g_2^{(-1)}(x) = \min(x,1)$ and therefore the fuzzy implication I^{g_2} is given by

$$I^{g_2}(x,y) = \min(1-x+y, 1) = I_{T_L}(x,y).$$

As we know from example 3.1, the triplet

$$p(x,y) = 1 - I^{g_2}(x,y) = \max(x-y, 0),$$

$$i(x,y) = \min(x,y), \quad j(x,y) = \min(1-x, 1-y),$$

satisfies axioms (R3)–(R4') and properties (gt1)–(gt5). Equality (1) again holds.

The last example presents fuzzy implication which is related to mentioned Frank t-norms.

Example 3.12. Let $g_3 = \ln \frac{2}{3^{1-x}-1}$, then the fuzzy implication I^{g_3} is given by

$$I^{g_3}(x,y) = 1 - \log_3\left(\frac{(3^x-1)\cdot(3^{1-y}-1)}{2} + 1\right).$$

Note that the function g_3 is generator of Frank t-conorm and this fuzzy implication I^{g_3} is not isomorphic with I_{T_L}. For the mentioned difference we get

$$I^{g_3}(x,y) - I^{g_3}(y,x) = \log_s \frac{(3^{1-x}-1)\cdot(3^y-1)+2}{(3^x-1)\cdot(3^{1-y}-1)+2}$$

$$= \log_3 \frac{3^{y-x+1}-3^{1-x}-3^y+3}{3^{x-y+1}-3^{1-y}-3^x+3} = \log_3 \frac{\frac{3^{y+1}-3-3^{y+x}+3^{x+1}}{3^x}}{\frac{3^{x+1}-3-3^{x+y}+3^{y+1}}{3^y}} = \log_3 \frac{3^y}{3^x} = y-x.$$

Since the equality is satisfied, related triplet (p,i,j) is a generator triplet.

The following proposition is a generalization of the previous example. We present special class of fuzzy implications with the equality (1). This class of fuzzy implications is not isomorphic with I_{T_L} for arbitrary $s \in]0,\infty[-\{1\}$.

Proposition 3.13. Let $s \in]0,\infty[-\{1\}$ and $g_s(x) = \ln \frac{s-1}{s^{1-x}-1}$. Then the fuzzy implication I^{g_s} satisfies equality $I(x,y) - I(y,x) = y - x$.

Proof. Let g be the function as described in the proposition. After substituting out $I^{g_s}(x,y)$, $I^{g_s}(y,x)$ and rearranging the terms, we get

$$I^{g_s}(x,y) - I^{g_s}(y,x) = \log_s \frac{(s^{1-x}-1)\cdot(s^y-1)+(s-1)}{(s^x-1)\cdot(s^{1-y}-1)+(s-1)}$$

$$= \log_s \frac{s^{y-x+1}-s^{1-x}-s^y+s}{s^{x-y+1}-s^{1-y}-s^x+s} = \log_s \frac{\frac{s^{y+1}-s-s^{y+x}+s^{x+1}}{s^x}}{\frac{s^{x+1}-s-s^{x+y}+s^{y+1}}{s^y}} = \log_s \frac{s^y}{s^x} = y-x.$$

□

Corollary 3.14. Let $s \in]0,\infty[-\{1\}$. If

$$I^{g_s}(x,y) = 1 - \log_s\left(\frac{(s^x-1)\cdot(s^{1-y}-1)}{s-1}+1\right),$$

then there exists a triplet of generators (p,i,j), such that $p(x,y) = 1 - I^{g_s}(x,y)$.

We have investigated the case, when $p(x,y) = 1 - I^g(x,y)$. A more general formula is $p(x,y) = N^{(-1)}(I^g_N(x,y))$. In this case, the condition for the generator of triplet is

$$N^{(-1)}(I^g_N(y,x)) - N^{(-1)}(I^g_N(x,y)) = y - x.$$

The contribution of this paper is in presentation of some special class of generated fuzzy implications and construction methods of monotone generators for fuzzy preference structures. We have shown that among the class of I_f fuzzy implications only those that are isomorphic to the Łukasiewicz implication are appropriate for FPS, while there seem to exist non Łukasiewicz implications from the class of I^g implications. We plan to characterize classes of I^g fuzzy implications such that satisfy the equality (1).

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