Ivona Svobodová

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ON AN INTERACTION OF TWO ELASTIC BODIES: ANALYSIS AND ALGORITHMS

Ivona Svobodová, Ostrava

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Abstract. The paper deals with existence and uniqueness results and with the numerical solution of the nonsmooth variational problem describing a deflection of a thin annular plate with Neumann boundary conditions. Various types of the subsoil and the obstacle which influence the plate deformation are considered. Numerical experiments compare two different algorithms.

Keywords: obstacle problem, variational formulation, semi-coercive problem, finite elements, semismooth Newton method, method of successive approximations

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1. Introduction

A contact between two elastic bodies without penetration is one of the frequently solved problems arising in the practise of the engineering. Its mathematical model consists of the system of two equlibrium equations (each corresponds to one of the bodies), which are connected by a contact condition. In the literature, a lot of attention has been paid to these problems, see for example [5] and the references therein. A possible simplification of the problem is to focus on the main body. Then the problem is reduced into one equilibrium equation whose resulting deformations are affected by the second body. The influence of the second body is described by an extra term in the equilibrium equation.

This paper deals with elastic annular axisymmetric thin plate as the main body, which occupies the region

$$Q = \Big\{(x,y,z) \in \mathbb{R}^3 \colon\ a < \sqrt{x^2+y^2} < b,\ -\frac{t}{2} < z < \frac{t}{2}\Big\},$$

where a, b, t are given positive constants, $a < b$ and $t \ll 1$. The mathematical model is described by the theory of linear elasticity. Taking into account the rotational symmetry, the problem may be rewritten as a boundary value problem for ODE (ordinary differential equation) of fourth order. The variational formulation of the problem consists of the minimization of the total potential energy through a space of virtual displacements, which is an appropriate subspace of a finite energy space. In view of the fact that the bodies and the loads are axisymmetrical it is suitable to formulate the problem in cylindrical coordinates. Therefore, the space of functions with finite energy is defined as a weighted Sobolev space [4].

The starting point describing the influence of the second body is the so-called unilateral Winkler's foundation model. Here, the extra term in the equilibrium equation is the positive or negative part of a solution. The influence of the second body is described by the operator ψ defined by

$$\psi\, u = \sum_{i=1}^{m} k_{N_i}(u - L_{+i})^{+} - \sum_{j=1}^{n} k_{P_j}(u + L_{-j})^{-}, \tag{1.1}$$

where the functions k_{N_i}, k_{P_j} describe the response of ith upper and jth lower layer of the second body, respectively, and L_{+i}, L_{-j} are their distances from the main body. The operator ψ enables us to characterize a wider class of various mechanical problems. After including ψ into the equlibrium equation, the energy functional will become nonsmooth. Therefore, additional assumptions guaranteeing the existence of the solution are needed. The uniqueness of the solution may be proved only for special cases of ψ. The analysis of similar problems for a circular plate or a beam on an elastic foundation can be found in [7], [6], and [8].

The finite element approach leads to an algebraic problem that is nonsmooth again. Therefore, a special attention must be paid to the choice of a suitable computational method. We present two algorithms. The first one uses the nonsmooth variant of the Newton method with the superlinear rate of convergence [2]. However, this convergence result exhibits a local character, i.e. a sufficiently accurate initial iterate is needed. The second algorithm is based on the method of successive approximations that seems to be globally convergent but a correct proof of this fact is an open problem. We will compare results obtained by both algorithms.

The outline of the paper is as follows. In Section 2 we introduce the two-dimensional problem based on Kirchhoff's theory. Owing to the symmetry, the problem is simplified into the Neumann boundary value problem for ODE of the fourth order that is reformulated in terms of the variations. Section 3 contains the main existence and uniquess results. In Section 4 there is described the discretization of the problem and there are given two computational algorithms. Finally, Section 5 presents results of numerical experiments.

2. Setting of the problem

We describe deformations of the thin plate Q by a vector field of the displacement. According to Kirchhoff's theory, we suppose that this vector field $\boldsymbol{U} = (U_r, U_\varphi, U_z)$ is of the form

$$U_r = -z\frac{\partial}{\partial r}w(r,\varphi), \quad U_\varphi = -z\frac{1}{r}\frac{\partial}{\partial\varphi}w(r,\varphi), \quad U_z = w(r,\varphi) \tag{2.1}$$

for $(r,\varphi,z) \in (a,b)\times(-\pi,\pi)\times(-t/2,t/2)$, where w is a deflection function and r, φ, z are the cylindrical coordinates.

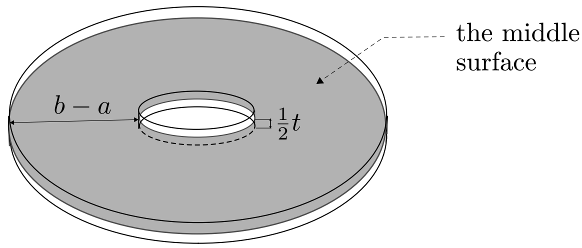

Figure 1. The middle surface of the plate.

Let the set $\{(x,y) \in \mathbb{R}^2 \colon a < \sqrt{x^2+y^2} < b\}$ be the so called middle surface of the plate Q, see Fig. 1. In the polar coordinates (r,φ) the middle surface is characterizated by the Cartesian product $(a,b)\times(-\pi,\pi)$ and its boundary by $\{a,b\}\times(-\pi,\pi)$. The equilibrium equation of the thin circular plate is a partial differential equation

$$D_0 t^2 \Delta_c^2 w(r,\varphi) + \psi\, w(r,\varphi) = f(r,\varphi), \quad (r,\varphi) \in (a,b)\times(-\pi,\pi),$$

where Δ_c is the Laplace operator in the polar coordinates, the map ψ describes the influence of the second body, $D_0 := E/(12(1-\sigma^2))$ with Young's modulus $E > 0$ and Poisson's ratio $\sigma \in (0,1/2)$, and f describes the given volume forces, which are perpendicular to the middle surface. We choose the boundary conditions of Neumann's type

$$\left.\begin{aligned} \mathcal{M}_n w(r,\varphi) &= m_n(r,\varphi) \\ \mathcal{T}_n w(r,\varphi) + \frac{1}{r}\frac{\partial}{\partial s}[r\mathcal{M}_{ns} w(r,\varphi)] &= p_n(r,\varphi) \end{aligned}\right\} \quad (r,\varphi) \in \{a,b\}\times(-\pi,\pi),$$

where the boundary differential operators $\mathcal{M}_n$, $\mathcal{M}_{ns}$, and $\mathcal{T}_n$ are the bending moments, the twisting moments and the shearing forces, respectively, for the outer normal vector n and the directional vector s. The functions m_n and p_n are given.

2.1. Classical formulation. Since the plate is *axisymmetrical*, the displacement vector field $\boldsymbol{U}$ simplifies to

$$U_r = -zu'(r), \qquad U_\varphi = 0, \qquad U_z = u(r), \tag{2.2}$$

where $r \in (a,b)$, $z \in (-t/2, t/2)$, $u = u(r)$ is the deflection function and the prime means the derivative with respect to r. Notice that the function $w = w(r,\varphi)$ is replaced by $u = u(r)$. Similary, we will write $f = f(r)$ instead of $f = f(r,\varphi)$, see Fig. 2. Consequently, the equilibrium equation reduces to ODE.

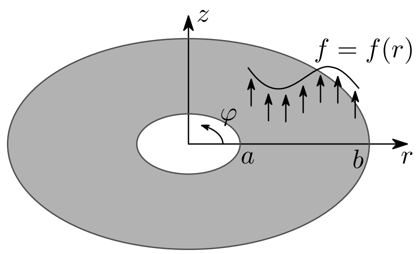

Figure 2. The cut $(a,b) \times \{0\} \times \{0\}$ of the plate middle surface.

The classical solution u satisfies the equilibrium equation

$$D_0 t^2 \Delta_c^2 u(r) + \psi\, u(r) = f(r), \quad r \in (a,b), \tag{2.3}$$

where $\Delta_c^2 u(r) = \frac{1}{r}[r[\frac{1}{r}[r \cdot u'(r)]']']'$. The operator ψ is of the form (1.1), where $m, n \in \mathbb{N}$, the functions k_{N_i}, k_{P_j} and the costants L_{+i}, L_{-j} are nonnegative. For a function $u = u(r)$, u^+ and u^- stands for the positive and negative part of u, respectively, i.e.

$$u^+ := \frac{1}{2}(|u| + u) \quad \text{and} \quad u^- := \frac{1}{2}(|u| - u).$$

In addition the prescribed Neumann's boundary conditions reduce to

$$\mathcal{M}u(r) = m_r \quad \text{and} \quad \mathcal{T}u(r) = p_r \quad \text{for } r \in \{a,b\}, \tag{2.4}$$

given values m_a, m_b, p_a, and p_b and operators $\mathcal{T}$ and $\mathcal{M}$ which represent the shear forces and bending moments on the boundary, respectively, defined by

$$\mathcal{T}u := D_0 t^2 \Big(ru''' + u'' - \frac{1}{r}\, u' \Big) \quad \text{and} \quad \mathcal{M}u := D_0 t^2 (ru'' + \sigma u').$$

E.g. the conditions (2.4) with $m_a = m_b = p_a = p_b = 0$ correspond to the so-called "free" plate.

R e m a r k 2.1. We now inspect (1.1) in detail. In this mathematical model we take into account the influence of the apriori unknown active part of the second elastic body to the plate. The operator ψ describes such environment (the second body), which has a piecewise linear convex change of the response. This problem arises in the engineering or geological mechanics. For example, the enviromental influence, which is composed by sequentially activated springs, see Fig. 3. Or the special geological cross-sections consisting of clay and rock. Generally, we describe the environment with m levels over the plate and n levels under it. The distance between the ith level and the plate is L_{+i} for the upper environment (the case of a hanging wall) and L_{-i} for the lower one (the case of a subsoil). If the deflection of the plate is from the interval $(L_{-(k+1)}, L_{-k}\rangle$ then the deformation of the plate is influenced by the first lower level as far as the kth one. The function $k_{N_i} = k_{N_i}(r)$ defines the response of the upper ith layer. The lower jth layer response is described by the function $k_{P_j} = k_{P_j}(r)$. We assume that L_{+i} and L_{-j} satisfy

$$(2.5)\qquad -L_{-n} \leqslant \ldots \leqslant -L_{-2} \leqslant -L_{-1} \leqslant 0 \leqslant L_{+1} \leqslant L_{+2} \leqslant \ldots \leqslant L_{+m}.$$

Then the magnitude L_{+1} means the distance of the upper elastic environment from the plate. Actually, for $L_{+1} > 0$ it is the upper obstacle. The magnitude L_{-1} is the distance of the lower obstacle, see Fig. 3.

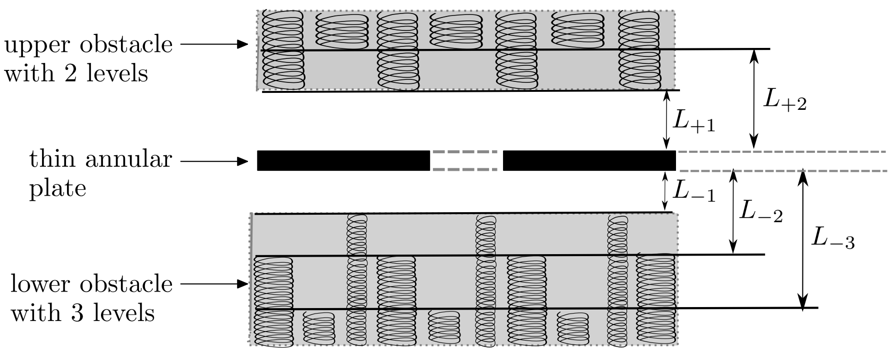

Figure 3. The plate with the lower and the upper elastic obstacles.

By ψ we can describe for example the following cases.

(1) The linear environment influence $\psi\, u = ku$, if we put $k_{N_1} = k_{P_1} = k$ on (a, b), $L_{+1} = L_{-1} = 0$ and $k_{N_i} \equiv k_{P_j} \equiv 0$, $i, j \geqslant 2$.

(2) The unilateral foundation of Winkler's type influence $\psi\, u = -k_{P_1} u^-$, if we put $L_{-1} = 0$, $k_{N_i} \equiv k_{P_j} \equiv 0$ for $i \geqslant 1$ and $j \geqslant 2$.

(3) The elastical upper obstacle influence $\psi\, u = k_{N_1}(u - L_{+1})^+$, if we put $L_{+1} > 0$, $k_{N_i} \equiv k_{P_j} \equiv 0$ for $i \geqslant 2$ and $j \geqslant 1$.

2.2. Variational formulation. In order to generalize the problem formulation we introduce the finite energy function space. Let us denote the weighted Lebesgue space by $L^2_{\varrho(r)}((a,b))$ with the inner product $(u,v)_{\varrho(r)} := \int_a^b u(r)v(r)\varrho(r)\,\mathrm{d}r$ and the induced norm $|\cdot|_{\varrho(r)}$. We consider the weighted Sobolev space

$$H^2((a,b);r,1/r,r) := \{v = v(r)\colon\ v, v'' \in L^2_r((a,b)),\ v' \in L^2_{1/r}((a,b))\}$$

with the inner product $(\!(u,v)\!)_{[r,1/r,r]} := (u,v)_r + (u',v')_{1/r} + (u'',v'')_r$ and the induced norm $\|\cdot\|_{[r,1/r,r]}$. Both spaces are Hilbert spaces, for more details see [4].

The space $H^2((a,b);r,1/r,r)$ is *the virtual displacement space* for the weak formulation of the problem (2.3)–(2.4). Therefore, we put

$$V = H^2((a,b);r,1/r,r).$$

The following assumptions on the expressions from ψ (see (1.1)) are the consequences of its physical meaning, see Remark 2.1. We assume that

$$\text{(2.6)}\qquad \begin{cases} k_{N_i} \in L^\infty((a,b)),\ k_{N_i} \geqslant 0 \text{ a.e. } (a,b),\ L_{+i} \in \mathbb{R},\ L_{+i} \geqslant 0,\ i = 1,2,\dots m, \\ k_{P_j} \in L^\infty((a,b)),\ k_{P_j} \geqslant 0 \text{ a.e. } (a,b),\ L_{-j} \in \mathbb{R},\ L_{-j} \geqslant 0,\ j = 1,2,\dots n. \end{cases}$$

The constants L_{+i} and L_{-j} are ordered as mentioned in Remark 2.1.

For $w, v \in V$ we introduce the following forms

$$\text{(2.7)}\qquad \begin{cases} a_0(w,v) := D_0 t^2((w',v')_{1/r} + (w'',v'')_r + \sigma(w'',v')_1 + \sigma(w',v'')_1), \\ a_\psi(w,v) := (\psi\, w, v)_r, \\ \mathcal{F}(w) := (f,w)_r - \langle p_r, w\rangle_{\{a,b\};r} + \langle m_r, w'\rangle_{\{a,b\};r} \end{cases}$$

for given $f \in L^2_r((a,b))$, Poisson's ratio σ and the magnitudes $p_a, p_b, m_a, m_b \in \mathbb{R}$, which are from the dualities, i.e.

$$\langle p_r, w\rangle_{\{a,b\};r} = p_b w(b) b - p_a w(a) a \quad \text{and} \quad \langle m_r, w'\rangle_{\{a,b\};r} = m_b w'(b) b - m_a w'(a) a.$$

The functional $\mathcal{P}_\psi\colon V \mapsto \mathbb{R}$ defined by

$$\text{(2.8)}\qquad \mathcal{P}_\psi(v) := \frac{1}{2} a_0(v,v) + \frac{1}{2} a_\psi(v,v) - \mathcal{F}(v)$$

is the potential of the problem. The function $u \in V$ is called *a variational solution* of the problem (2.3), (2.4) whenever

$$\text{(2.9)}\qquad u = \arg\min_{v\in V} \mathcal{P}_\psi(v).$$

The potential $\mathcal{P}_\psi$ is convex and differentiable on V, so that the variational solution u of (2.9) is characterised by the associated Euler-Lagrange equation

$$\text{(2.10)}\qquad a_0(u,v) + a_\psi(u,v) = \mathcal{F}(v) \quad \forall\, v \in V.$$

3. Existence of the variational solution

In this section we prove the existence and uniqueness results to the variational problem (2.9).

First of all we set up notation and formulate some general results of the calculus of variations. Let V be a Hilbert space and $\mathcal{P}\colon V \mapsto \mathbb{R}$ a continuous functional. Recall that the functional $\mathcal{P}$ is

- weakly lower semicontinuous on V, if for every sequence $\{u_n\}_n \subset V$, $u_n \rightharpoonup u$ we have $\liminf\limits_{n\to\infty} \mathcal{P}(u_n) \geqslant \mathcal{P}(u)$;
- coercive on V, if $\lim\limits_{\|u\|_V \to \infty} \mathcal{P}(u) = \infty$;
- semi-coercive on V, if there exists C closed subspace of V such that $\mathcal{P}(u) = \mathcal{P}(u+c)$ for all $u \in V$ and $c \in C$ and the functional $\mathcal{P}$ is coercive on the factorspace $V/_C$.

Remark 3.1. If the derivative $\mathrm{d}\mathcal{P}$ is monotone, i.e.

$$(\mathrm{d}\mathcal{P}(u) - \mathrm{d}\mathcal{P}(v), u - v)_V \geqslant 0 \quad \forall\, u, v \in V,$$

then $\mathcal{P}$ is weakly lower semicontinuous.

Theorem 3.1. *If a functional $\mathcal{P}\colon V \mapsto \mathbb{R}$ is weakly lower semicontinuous and coercive on V, then $\mathcal{P}$ achieves its minimum on V. If the functional $\mathcal{P}$ is moreover strictly convex, then the minimizer is unique.*

Proof. For the proof see [3]. □

3.1. Properties of the potential. We will prove the existence of the variational solution using Theorem 3.1.

Theorem 3.2. *The potential $\mathcal{P}_\psi$ is weakly lower semicontinuous on V.*

Proof. It is enough to show that the functionals $\frac{1}{2}a_0(u,u) - \mathcal{F}(u)$ and $\frac{1}{2}a_\psi(u,u)$ are weakly lower semicontinuous (since the set of weakly lower semicontinuous operators forms a cone, the sum of these functionals preserves this property).

Step 1. The part $\frac{1}{2}a_0(u,u) - \mathcal{F}(u)$ is convex and quadratic. Indeed, the bilinear mapping a_0 is bounded and the functional $\mathcal{F}$ is from the dual space V^*. In the calculus of variations there is a standard outcome that $\frac{1}{2}a_0(u,u) - \mathcal{F}(u)$ is weakly lower semicontinuous.

Step 2. The mapping $a_\psi\colon V\times V\mapsto\mathbb{R}$ is nonlinear in the first argument. By a standard computation we obtain

$$\Bigl(\mathrm{d}\Bigl[\frac{1}{2}a_\psi(u,u)\Bigr],v\Bigr)_r=(\psi\,u,v)_r.$$

In view of Remark 3.1 it remains to prove that ψ is monotone. Since the operator ψ is a sum of the terms $k_{N_i}(u-L_{+i})^+$ and $-k_{P_j}(u+L_{-j})^-$, it is enough to prove that $u\mapsto k_{N_i}u^+$ and $u\mapsto -k_{P_j}u^-$ are monotone. Obviously,

$$\begin{aligned}(k_{N_i}u^+-k_{N_i}v^+,u-v)_r&\geqslant \operatorname{ess\,inf}(k_{N_i})(u^+-v^+,u-v)_r\\&=\operatorname{ess\,inf}(k_{N_i})(|u^+-v^+|_r^2-(u^+-v^+,u^--v^-)_r)\\&\geqslant \operatorname{ess\,inf}(k_{N_i})(-(u^+-v^+,u^--v^-)_r)\\&=\operatorname{ess\,inf}(k_{N_i})((u^+,v^-)_r+(v^+,u^-)_r)\geqslant 0\end{aligned}$$

and analogously for $u\mapsto -k_{P_j}u^-$. □

Before proving the coercivity of the potential $\mathcal{P}_\psi$, we introduce auxiliary results. For the sake of simplicity we identify the constant function defined on (a,b) with its value, i.e. the function $v(r)=c$ for $c\in\mathbb{R}$ and for all $r\in(a,b)$ is denoted by c. The symbol $\mathcal{R}$ stands for the set of all constant functions defined on (a,b). For $d\in\mathcal{R}$ we denote $\mathcal{R}_d^+=\{c\in\mathcal{R}\colon\ c\geqslant d\}$, $\mathcal{R}_d^-=\{c\in\mathcal{R}\colon\ c\leqslant d\}$ and for a set V of functions defined on (a,b) we introduce the operations $V\pm d=\{w\colon\ w=v\pm d,\ v\in V\}$.

Lemma 3.1. *There exists $C_P>0$ such that*

$$C_P(|v''|_r^2+|v'|_{1/r}^2+(v,1)_r^2)\geqslant\|v\|_{[r,1/r,r]}^2 \tag{3.1}$$

holds for all $v\in H^2((a,b);r,1/r,r)$.

Proof. From the well-known *Poincaré inequality* it follows that there exists $c>0$ such that

$$c(|v'|^2_{L^2((a,b))}+(v,1)^2_{L^2((a,b))})\geqslant|v|^2_{L^2((a,b))}+|v'|^2_{L^2((a,b))}$$

for $v\in H^1((a,b))=\{v=v(r)\colon\ v\in L^2((a,b)),\ v'\in L^2((a,b))\}$. Since for $w\in L^2((a,b))$

$$\frac{1}{b}|w|^2_{L^2((a,b))}\leqslant|w|^2_{1/r}\leqslant\frac{1}{a}|w|^2_{L^2((a,b))},\quad |w|_r^2\leqslant b|w|^2_{L^2((a,b))},$$

and

$$(w,1)_r^2\geqslant a^2(w,1)^2_{L^2((a,b))},$$

it follows that

$$c_1(|v'|^2_{1/r} + (v,1)^2_r) \geqslant |v|^2_r + |v'|^2_{1/r}$$

for $c_1 = c\max\{1/a, b\}\max\{1/a^2, b\} > 0$, $v \in H^1((a,b))$. In particular, this inequality is satisfied for all $v \in H^2((a,b); r, 1/r, r)$. Adding the expression $|v''|^2_r$ to both sides we get the inequality (3.1) for an appropriate positive constant C_P. □

The following three theorems give sufficient conditions for the coercivity of the potential $\mathcal{P}_\psi$.

Theorem 3.3. *Let ψ be of the form*

$$\psi\, u = \sum_{i=1}^{m} k_{N_i}(u - L_{+i})^+, \tag{3.2}$$

and $k_{N_1} > 0$ a.e. on (a,b). If

$$\mathcal{F}(1) > 0, \tag{3.3}$$

then $\mathcal{P}_\psi$ is coercive.

Proof. The proof is divided into four steps.

Step 1 (Decomposition of the space V). The set of small displacements of the problem is

$$\{p \in V\colon\ a_0(p,p) + a_\psi(p,p) = 0\} = \mathcal{R}_0^- + L_{+1}.$$

The set $\mathcal{R}_0^- + L_{+1}$ is a convex, closed subset of V. The set $\mathcal{R}_0^-$ is a closed convex cone, i.e. $c_- + d_- \in \mathcal{R}_0^-$ and $\lambda c_- \in \mathbb{R}_0^-$ for all $c_-, d_- \in \mathcal{R}_0^-$, and constants $\lambda \in \mathbb{R}$, $\lambda > 0$. It is known (see [1]) that

for a closed convex cone $\mathcal{C}$, which is a subset of a Hilbert space V, every $u \in V$ may be decomposed uniquely as $u = p + \bar{u}$, where $p \in \mathcal{C}$ and $\bar{u}$ is in the corresponding negative polar cone

$$\mathcal{C}^\ominus := \{\overline{v} \in V\colon\ (\overline{v}, p)_V \leqslant 0\ \forall\, p \in \mathcal{C}\}.$$

The decomposition is orthogonal, i.e. $(p, \bar{u})_V = 0$. Moreover, the orthogonal projector $P_\mathcal{C}$ onto $\mathcal{C}$ such that $p = P_\mathcal{C} u$ is the minimum-distance projector, i.e. $\|u - P_\mathcal{C} u\|_V = \inf\limits_{q \in \mathcal{C}} \|u - q\|_V$.

Let us consider the space $V - L_{+1} = \{v_{L_{+1}}\colon\ v_{L_{+1}} = v - L_{+1},\ v \in V\}$ with its orthogonal decomposition into the cones

$$V - L_{+1} = \mathcal{R}_0^- \oplus (\mathcal{R}_0^-)^\ominus \tag{3.4}$$

for the negative polar cone

$$
\begin{aligned}
(\mathcal{R}_0^-)^{\ominus} := \{\overline{v}_- \in V\colon\ (\!(\overline{v}_-, p_-)\!)_{[r,1/r,r]} \leqslant 0\ \forall\, p_- \in \mathcal{R}_0^-\} \\
= \{\overline{v}_- \in V\colon\ (\overline{v}_-, 1)_r \geqslant 0\}.
\end{aligned}
$$

In view of (3.4) every $u_{L_{+1}} \in V - L_{+1}$ can be uniquely decomposed, i.e. $u_{L_{+1}} = p_- + \bar{u}_-$ for $p_- \in \mathcal{R}_0^-$ and $\bar{u}_- \in (\mathcal{R}_0^-)^{\ominus}$. Then for every $u \in V$ we can write

$$
u = u_{L_{+1}} + L_{+1} = \bar{u}_- + p_- + L_{+1}. \tag{3.5}
$$

Step 2 (Estimation of $\mathcal{P}_\psi$ from below). In view of (3.5) we get

$$
\begin{aligned}
\mathcal{P}_\psi(u) &= \frac{1}{2}a_0(u,u) + \frac{1}{2}a_\psi(u_{L_{+1}} + L_{+1}, u_{L_{+1}} + L_{+1}) - \mathcal{F}(u_{L_{+1}} + L_{+1}) \\
&= \frac{1}{2}a_0(u,u) + (k_{N_1}u_{L_{+1}}^+, u_{L_{+1}} + L_{+1})_r - \mathcal{F}(u_{L_{+1}} + L_{+1}) \\
&\quad + \sum_{i=2}^{M}(k_{N_i}(u_{L_{+1}} + L_{+1} - L_{+i})^+, u_{L_{+1}} + L_{+1} \pm L_{+i})_r
\end{aligned}
$$

for $u \in V$. Since

$$
\begin{aligned}
\sum_{i=2}^{M}(k_{N_i}(u_{L_{+1}} + L_{+1} - L_{+i})^+, u_{L_{+1}} + L_{+1} \pm L_{+i})_r \geqslant 0, \\
(k_{N_1}u_{L_{+1}}^+, u_{L_{+1}} + L_{+1})_r \geqslant \operatorname{ess\,inf}(k_{N_1})|u_{L_{+1}}^+|_r^2
\end{aligned}
$$

and

$$
a_0(u,u) = (1-\sigma)D_0t^2(|u''|_r + |u'|_{1/r} + \sigma D_0 t^2|\Delta_c u|_r, \tag{3.6}
$$

where $\sigma D_0 t^2 |\Delta_c u|_r \geqslant 0$, we get

$$
\begin{aligned}
\mathcal{P}_\psi(u) \geqslant D_0t^2(1-\sigma)(|(\bar{u}_-)''|_r^2 + |(\bar{u}_-)'|_{1/r}^2) + \operatorname{ess\,inf}(k_{N_1})|u_{L_{+1}}^+|_r^2 \\
- \mathcal{F}(\bar{u}_-) + (|p_-| - L_{+1})\mathcal{F}(1).
\end{aligned} \tag{3.7}
$$

According to the orthogonality of (3.4), only two cases for u from (3.5) are possible. Either

$$
p_- = 0 \quad \text{and} \quad (\bar{u}_-, 1)_r \geqslant 0 \tag{3.8 a}
$$

or

$$
p_- \leqslant 0 \quad \text{and} \quad (\bar{u}_-, 1)_r = 0. \tag{3.8 b}
$$

Case A. Let us assume that (3.8 a) is satisfied, i.e. $u_{L_{+1}} \equiv \bar{u}_-$ and $(\bar{u}_-, 1)_r \geqslant 0$. From (3.7) we get

$$\mathcal{P}_\psi(u) \geqslant D_0 t^2 (1-\sigma)(|(\bar{u}_-)''|_r^2 + |(\bar{u}_-)'|_{1/r}^2) \\ + \text{ess inf}(k_{N_1})|(\bar{u}_-)^+|_r^2 - \mathcal{F}(\bar{u}_-) - L_{+1}\mathcal{F}(1).$$

Since

$$0 \leqslant (\bar{u}_-, 1)_r \leqslant ((\bar{u}_-)^+, 1)_r \leqslant c_3 |(\bar{u}_-)^+|_r$$

for some constant $c_3 > 0$, we get

$$(\bar{u}_-, 1)_r^2 \leqslant c_4 |(\bar{u}_-)^+|_r^2$$

for $c_4 > 0$. Further, we use the last inequality together with (3.1). We get

$$\begin{aligned} \mathcal{P}_\psi(u) &\geqslant c_1(|(\bar{u}_-)''|_r^2 + |(\bar{u}_-)'|_{1/r}^2 + (\bar{u}_-, 1)_r^2) - \|F\|^* \|\bar{u}_-\|_{[r,1/r,r]} - L_{+1}\mathcal{F}(1) \\ &\geqslant c_5 \|\bar{u}_-\|_{[r,1/r,r]}^2 - \|F\|^* \|\bar{u}_-\|_{[r,1/r,r]} - L_{+1}\mathcal{F}(1) \\ &\geqslant \mathcal{C}_\mathcal{K}(\|\bar{u}_-\|_{[r,1/r,r]}) \|\bar{u}_-\|_{[r,1/r,r]} - L_{+1}\mathcal{F}(1), \end{aligned}$$

where $\mathcal{C}_\mathcal{K}$ is a real nonnegative function such that $\lim_{t\to\infty} \mathcal{C}_\mathcal{K}(t) = \infty$. Therefore, the functional $\mathcal{P}_\psi$ is coercive.

Case B. Let us assume that (3.8 b) holds, i.e. $u_{L_{+1}} \equiv p_- + \bar{u}_-$, where $p_- \in \mathcal{R}_0^-$ and $(\bar{u}_-, 1)_r = 0$. From (3.7) we get

$$\begin{aligned} \mathcal{P}_\psi(u) &\geqslant D_0 t^2 (1-\sigma)(|(\bar{u}_-)''|_r^2 + |(\bar{u}_-)'|_{1/r}^2 + (\bar{u}_-, 1)_r^2) - \mathcal{F}(\bar{u}_-) + (|p_-| - L_{+1})\mathcal{F}(1) \\ &\geqslant c_5 \|\bar{u}_-\|_{[r,1/r,r]}^2 - \|\mathcal{F}\|^* \|\bar{u}_-\|_{[r,1/r,r]} + (|p_-| - L_{+1})\mathcal{F}(1) \\ &\geqslant \mathcal{C}_\mathcal{K}(\|\bar{u}_-\|_{[r,1/r,r]}) \|\bar{u}_-\|_{[r,1/r,r]} + |p_-|\mathcal{F}(1) - L_{+1}\mathcal{F}(1), \end{aligned}$$

where $\mathcal{C}_\mathcal{K}$ is the same as in Case A. From the triangular inequality

$$\|u\|_{[r,1/r,r]}^2 \leqslant |p_-|_r^2 + \|\bar{u}_-\|_{[r,1/r,r]}^2 + |L_{+1}|_r^2$$

it follows that:

$$\text{if } \|u\|_{[r,1/r,r]} \to \infty \quad \text{then} \quad |p_-|_r \to \infty \text{ or } \|\bar{u}_-\|_{[r,1/r,r]} \to \infty.$$

Therefore, if (3.3) is satisfied then the functional $\mathcal{P}_\psi$ is coercive. □

Theorem 3.4. *Let ψ be of the form*

$$\psi\, u = -\sum_{j=1}^{n} k_{P_j}(u + L_{-j})^{-}, \tag{3.9}$$

and $k_{P_1} > 0$ a.e. on (a, b). If

$$\mathcal{F}(1) < 0, \tag{3.10}$$

then $\mathcal{P}_\psi$ is coercive.

P r o o f. The process is analogous to the previous proof. We sum up the following steps.

Step 1. The orthogonal decomposition of the space $V + L_{-1}$ into the cones is

$$V + L_{-1} = \mathcal{R}_0^+ \oplus (\mathcal{R}_0^+)^{\ominus}, \tag{3.11}$$

where $\mathcal{R}_0^+$ is the convex cone of all nonnegative constants and $(\mathcal{R}_0^+)^{\ominus}$ is the negative polar cone, i.e. $(\mathcal{R}_0^+)^{\ominus} = \{\overline{v}_+ \in V\colon\ (\overline{v}_+, 1)_r \leqslant 0\}$. The unique decomposition of $u \in V$ is

$$u = \bar{u}_+ + p_+ - L_{-1}. \tag{3.12}$$

Step 2. The initial lower estimation of $\mathcal{P}_\psi$ is

$$\begin{aligned}\mathcal{P}_\psi(u) \geqslant{}& D_0 t^2(1-\sigma)(|(\bar{u}_-)''|_r^2 + |(\bar{u}_-)'|_{1/r}^2) + \operatorname{ess\,inf}(k_{N_1})|u_{L+1}^+|_r^2 \\ &- \mathcal{F}(\bar{u}_+) - (|p_+| - L_{-1})\mathcal{F}(1).\end{aligned} \tag{3.13}$$

For the decomposition (3.12) only two cases are possible. Either

$$p_+ = 0 \quad \text{and} \quad (\bar{u}_+, 1)_r \leqslant 0 \tag{3.14 a}$$

or

$$p_+ \geqslant 0 \quad \text{and} \quad (\bar{u}_+, 1)_r = 0. \tag{3.14 b}$$

Case A. If (3.14 a) holds then from (3.13) it follows that

$$\mathcal{P}_\psi(u) \geqslant \mathcal{C}_{\mathcal{K}}(\|\bar{u}_+\|_{[r,1/r,r]})\|\bar{u}_+\|_{[r,1/r,r]} + L_{-1}\mathcal{F}(1).$$

Therefore, $\mathcal{P}_\psi$ is coercive.

Case B. If (3.14 b) holds then from (3.13) it follows that

$$\mathcal{P}_\psi(u) \geqslant \mathcal{C}_{\mathcal{K}}(\|\bar{u}_+\|_{[r,1/r,r]})\|\bar{u}_+\|_{[r,1/r,r]} - |p_+|\mathcal{F}(1) + L_{-1}\mathcal{F}(1).$$

If (3.10) is satisfied then $\mathcal{P}_\psi$ is coercive. □

R e m a r k 3.2. The potentials $\mathcal{P}_\psi$ from Theorems 3.3 and 3.4 are semi-coercive on V whenever $\mathcal{F}(1)=0$. This follows from Case B of the proofs. The potentials are coercive on the factorspaces $V/_{\mathcal{R}^-_{L_{+1}}}$ and $V/_{\mathcal{R}^+_{-L_{-1}}}$, respectively.

Theorem 3.5. *Let ψ be of the form* (1.1), $k_{N_1}>0$ *and* $k_{P_1}>0$ *a.e. on* (a,b) *and* $L_{+1}>0$, $L_{-1}>0$. *If*

$$\mathcal{F}(1)\neq 0, \tag{3.15}$$

then $\mathcal{P}_\psi$ is coercive.

P r o o f. From Step 1 of the two previous proofs we get two possible ways how to decompose the space V namely

$$V=[\mathcal{R}_0^-\oplus(\mathcal{R}_0^-)^\ominus]+L_{+1}\quad\text{and}\quad V=[\mathcal{R}_0^+\oplus(\mathcal{R}_0^+)^\ominus]-L_{-1}.$$

See (3.4) and (3.11), respectively. Thus we can write V as the union

$$V=((\mathcal{R}_0^-)^\ominus+L_{+1})\cup W\cup((\mathcal{R}_0^+)^\ominus-L_{-1}). \tag{3.16}$$

Consequently, we estimate $\mathcal{P}_\psi$ from below separately on each set from (3.16). The procedure for $u\in(\mathcal{R}_0^-)^\ominus+L_{+1}$ and for $u\in(\mathcal{R}_0^+)^\ominus-L_{-1}$ is analogous to the Case A in the proofs of Theorems 3.3 and 3.4, respectively.

If we inspect the remaining part W, we obtain that any $u\in W$ may be uniquely decomposed in two ways. Either

$$u=\bar u_-+p_-+L_{+1}\quad\text{for }(\bar u_-,1)_r=0\text{ and }p_-\leqslant 0 \tag{3.17 a}$$

or

$$u=\bar u_++p_+-L_{-1}\quad\text{for }(\bar u_+,1)_r=0\text{ and }p_+\geqslant 0. \tag{3.17 b}$$

If $\mathcal{F}(1)>0$ we use the decomposition (3.17 a) and then we decompose $\bar u_-$ by means of (3.17 b). We get

$$u=\overline{(\bar u_-)}_++p_+-L_{-1}+p_-+L_{+1}$$

for $\overline{(\bar u_-)}_+\in(\mathcal{R}_0^+)^\ominus$. Moreover, if we substitute for $\bar u_-$ in the condition $(\bar u_-,1)_r=0$ its decomposition in accordance with (3.17 b), i.e.

$$(\overline{(\bar u_-)}_++p_+-L_{-1},1)_r=0,$$

then in view of $(\overline{(\bar u_-)}_+,1)_r=0$ we get $p_+=L_{-1}$. Therefore,

$$u=\overline{(\bar u_-)}_++p_-+L_{+1}.$$

Next procedure is the same as in Case B in the proof of Theorem 3.3.

If $\mathcal{F}(1)<0$ we proceed analogously.

In both cases (i.e. $\mathcal{F}(1)>0$ and $\mathcal{F}(1)<0$) the potential $\mathcal{P}_\psi$ is coercive. $\square$

R e m a r k 3.3. The potential $\mathcal{P}_\psi$ from Theorem 3.5 is semi-coercive on V whenever $\mathcal{F}(1) = 0$. It is coercive only on the factorspace $V/_{\mathcal{R}^+_{-L_{-1}} \cap \mathcal{R}^-_{L_{+1}}}$. On the other hand, the potential $\mathcal{P}_\psi$ from Theorem 3.5 is unconditionally coercive for $L_{+1} = L_{-1} = 0$ and $k_{N_1} \not\equiv 0$ and $k_{P_1} \not\equiv 0$ a.e. on (a, b).

3.2. Existence and uniqueness results. Now we present the main existence and uniquess results for the variational solution of (2.9).

Theorem 3.6. *Let the operator ψ be of the form* (3.2).

(i) *If there exists a variational solution of* (2.9), *then* $\mathcal{F}(1) \geqslant 0$.

(ii) *If $\mathcal{F}(1) > 0$ holds, then there exists a unique variational solution of* (2.9).

P r o o f. We prove the conclusions separately.

(i) The starting equation is the Euler-Lagrange equation (2.10). We take $v \in V$ as a constant $c \in \mathbb{R}$. We get

$$\sum_{i=1}^{m} (k_{N_i}(u - L_{+i})^+, 1)_r - \mathcal{F}(1) = 0$$

and then

$$\mathcal{F}(1) = \sum_{i=1}^{m} (k_{N_i}(u - L_{+i})^+, 1)_r \geqslant 0.$$

Indeed, the functions $(u - L_{+i})^+$ and k_{N_i} are nonnegative.

(ii) *Step 1* (Existence). The assumptions of Theorem 3.1 are satisfied. That follows from Theorems 3.2 and 3.3. Therefore there exists at least one variational solution of (2.9).

Step 2 (Uniqueness—by contradiction). Let us assume that there are two different solutions u_1 and u_2 of the problem (2.9). If we put u_1 and u_2 into (2.10), subtract the equations and put $v = u_1 - u_2$, we get

$$a_0(u_1 - u_2, u_1 - u_2) + (\psi\, u_1 - \psi\, u_2, u_1 - u_2)_r = 0. \tag{3.18}$$

The map a_0 could be expressed as in (3.6) and the map ψ is monotone (see the proof of Theorem 3.2). Since both terms in (3.18) are nonnegative, we get

$$a_0(u_1 - u_2, u_1 - u_2) = 0 \quad \text{and} \quad (\psi\, u_1 - \psi\, u_2, u_1 - u_2)_r = 0.$$

From the first equation it follows that $u_1'' = u_2''$ and $u_1' = u_2'$, thus $u_1 - u_2$ is constant, i.e.

$$u_2(r) = u_1(r) + c \quad \text{for} \quad c \in \mathbb{R}. \tag{3.19}$$

Without loss of generality we suppose that $c > 0$, thus $u_1 < u_2$ on (a, b).

Substituting for $v \in V$ any nonzero constant function in (2.10), we get

$$(\psi\, u_i, 1)_r = \mathcal{F}(1) \quad \text{for} \quad i = 1, 2.$$

From (3.3) we get

$$(\psi\, u_1, 1)_r = (\psi\, u_2, 1)_r > 0. \tag{3.20}$$

We set

$$\overline{M}(w) := \{r \in (a, b)\colon\ w(r) > L_{+1}\}.$$

From the inequalities $(\psi\, u_1, 1)_r = (\psi\, u_2, 1)_r > 0$ and $u_1 < u_2$ it follows that

$$\emptyset \neq \overline{M}(u_1) \subset \overline{M}(u_2).$$

Then

$$\begin{aligned}
(\psi\, u_1, 1)_r &= \int_{(a,b)} \psi\, u_1 r \,\mathrm{d}r = \int_{\overline{M}(u_1)} \psi\, u_1 r \,\mathrm{d}r \\
&= \int_{\overline{M}(u_1)} k_{N_1}(u_1 - L_{+1}) r \,\mathrm{d}r + \sum_{i \geqslant 2} \int_{\overline{M}(u_1)} k_{N_i}(u_1 - L_{+i})^+ r \,\mathrm{d}r \\
&< \int_{\overline{M}(u_1)} k_{N_1}(u_2 - L_{+1}) r \,\mathrm{d}r + \sum_{i \geqslant 2} \int_{\overline{M}(u_1)} k_{N_i}(u_2 - L_{+i})^+ r \,\mathrm{d}r \\
&\leqslant \int_{\overline{M}(u_2)} k_{N_1}(u_2 - L_{+1}) r \,\mathrm{d}r + \sum_{i \geqslant 2} \int_{\overline{M}(u_2)} k_{N_i}(u_2 - L_{+i})^+ r \,\mathrm{d}r \\
&= (\psi u_2, 1)_r.
\end{aligned}$$

The inequality $(\psi\, u_1, 1)_r < (\psi\, u_2, 1)_r$ gives the contradiction with (3.20) and consequently $u_1 = u_2$ on (a, b). □

The following existence and uniqueness conclusions can be proved analogously.

Theorem 3.7. *Let the operator ψ be of the form* (3.9).

(i) *If there exists a variational solution of* (2.9)*, then* $\mathcal{F}(1) \leqslant 0$.

(ii) *If* $\mathcal{F}(1) < 0$ *holds, then there exists a unique variational solution of* (2.9).

Theorem 3.8. *Let the operator ψ be of the form* (1.1)*. If*

$$\mathcal{F}(1) \neq 0, \tag{3.21}$$

then there exists the unique variational solution of problem (2.9).

R e m a r k 3.4. Notice that there is no necessary condition in Theorem 3.8. Indeed, if the solution u of problem (2.9) exists, then $\mathcal{F}(1)$ can be arbitrary.

Theorem 3.9. *Let the operator ψ be of the form* (1.1). *If $k_{N_1} > 0$ and $k_{P_1} > 0$ a.e. on (a, b) and $L_{+1} = L_{-1} = 0$, then there exists a unique variational solution of* (2.9).

R e m a r k 3.5. From Remarks 3.2 and 3.3 it follows that

(1) the solution of the problem from Theorem 3.8 is unique on the factorspace $V/\langle -L_{-1}, L_{+1}\rangle$, whenever $\mathcal{F}(1) = 0$;
(2) the solution of the problem from Theorem 3.7 is unique on the factorspace $V/\langle -L_{-1}, \infty)$, whenever $\mathcal{F}(1) = 0$;
(3) the solution of the problem from Theorem 3.6 is unique on the factorspace $V/(-\infty, L_{+1}\rangle$, whenever $\mathcal{F}(1) = 0$.

4. Discretization and algebraical formulation

We will approximate problem (2.3)–(2.4) by the finite element method. Let a discretization of the interval $\langle a, b\rangle$ be

$$\mathcal{N}_h := \{r_i\colon\ i = 1, \dots N+1\},$$

where

$$a = r_1 < r_2 < \dots < r_N < r_{N+1} = b, \quad N \in \mathbb{N}. \tag{4.1}$$

The parameter h stands for the norm of the discretization, i.e. $h = \max\limits_i\{r_{i+1} - r_i\}$. The following finite element space

$$V_h := \{v_h \in C^1((a,b))\colon\ v_h|_{\langle r_i, r_{i+1}\rangle} \text{ is a cubic polynomial for } i = 1, \dots N\} \tag{4.2}$$

is associated with the discretization $\mathcal{N}_h$. On V_h we choose the basis functions denoted by $\{\varphi_k\}_{k=1}^{2N+2}$. Let $v = (v_k)_{k=1}^{2N+2}$ be the vector of coordinates of $v_h \in V_h$ according to this basis, i.e. $v_{2i-1} = v_h(r_i)$, $v_{2i} = v_h'(r_i)$ for $i = 1 \dots N+1$.

We search for the discrete solution $u_h \in V_h$ satisfying the approximation of (2.10), i.e.

$$a_0(u_h, \varphi_k) + a_\psi(u_h, \varphi_k) = \mathcal{F}(\varphi_k) \quad \text{for } k = 1, 2, \dots, 2N+2. \tag{4.3}$$

The existence and uniqueness of the discrete solution follows from the fact that V_h is a finite-dimensional subspace of V, in which a_0, a_ψ, and $\mathcal{F}$ satisfy the same assumptions as in the continuous setting.

After the inspection of the map a_ψ from (4.3), we get the following expression

$$a_\psi(u_h, \varphi_k) = (\psi\, u_h, \varphi_k)_r.$$

Let us note that $\psi\, u_h$ is not an element of V_h. Therefore, we will seek for the suitable algebraic formulation of (4.3). The next two chapters contain two different approaches leading to two different algorithms.

4.1. Algorithm based on the semismooth Newton method. As we want to use the semismooth Newton method, we focus our attention on the map a_ψ from (4.3). We use the trapezoidal rule to approximate

$$a_\psi(u_h, \varphi_k) = \int_{a_k}^{b_k} \psi\, u_h(r)\varphi_k(r) r \,\mathrm{d}r \quad \text{for } k = 1, 2, \ldots, 2N+2,$$

where the interval $(a_k, b_k) \subset (a, b)$, $a_k, b_k \in \mathcal{N}_h$, is the support of φ_k. We get new mapping a_ψ^h such that

$$a_\psi^h(u_h, \varphi_k) =: \begin{cases} \xi_k(\zeta_k - \xi_k)\varphi_k(\zeta_k - \xi_k)\psi\, u_h(\zeta_k - \xi_k) & \text{for } k = 1, 2, \\ \xi_k\zeta_k\varphi_k(\zeta_k)\psi\, u_h(\zeta_k) & \text{for } k = 3, 4, \ldots, 2N-1, 2N, \\ \xi_k(\zeta_k + \xi_k)\varphi_k(\zeta_k + \xi_k)\psi\, u_h(\zeta_k + \xi_k) & \text{for } k = 2N+1, 2N+2, \end{cases}$$

where $\zeta_k := \frac{1}{2}(b_k + a_k)$ and $\xi_k := \frac{1}{2}(b_k - a_k)$. As ζ_k and $\zeta_k \pm \xi_k$ are elements of $\mathcal{N}_h$, each of the values $u_h(\zeta_k)$, $u_h(\zeta_k + \xi_k)$, and $u_h(\zeta_k - \xi_k)$ corresponds to the respective component of the vector $u = (u_k)_{k=1}^{2N+2}$. Since the function values of the basis functions φ_k at the nodes are known, therefore,

$$a_\psi^h(u_h, \varphi_k) = \begin{cases} \xi_1 a\psi^h u_1 & \text{for } k = 1, \\ \xi_k\zeta_k\psi^h u_k & \text{for } k = 3, 5, 7, \ldots, 2N-1, \\ \xi_{2N+1} b\psi^h u_{2N+1} & \text{for } k = 2N+1, \\ 0 & \text{for } k \text{ even}, \end{cases} \tag{4.4}$$

and for the mapping $\psi^h\colon\ \mathbb{R} \mapsto \mathbb{R}$ we have

$$\psi^h u_k = \sum_{i=1}^{m} k_{N_i}(r_l)(u_k - L_{+i})^+ - \sum_{j=1}^{n} k_{P_j}(r_l)(u_k + L_{-j})^-, \quad l = \left[\frac{k}{2}\right] + 1.$$

Thus we have derived the finite-dimensional approximation of the nonlinear mapping (1.1). We arrive at the following algebraical representation of (4.3)

$$\text{find } u \in \mathbb{R}^{2N+2} \text{ such that}$$
$$Ku + \sum_{i=1}^{m} B_{+i}(u - L_{+i})^+ - \sum_{j=1}^{n} B_{-j}(u + L_{-j})^- = f, \tag{4.5}$$

where $K \in \mathbb{R}^{(2N+2)\times(2N+2)}$ is the stiffness matrix with respect to a_0, $f \in \mathbb{R}^{2N+2}$ is the load vector, B_{+i}, B_{-j} are the diagonal matrices, whose elements consist of the coefficients ξ_k, ζ_k, a, b from (4.4) and the appropriate response, which is described by the functions k_{N_i} resp. k_{P_j}. The terms $(u-L_{+i})^+$ and $(u+L_{-j})^-$ for $L_{+i}, L_{-j} \in \mathbb{R}^+$ are understood componentwisely.

We will apply the semismooth Newton method to the problem (4.5), see [2]. The main idea is based on the usage of the so called *slanting functions* instead of the classical Jacobian.

Definition 4.1. Let X and Y be Banach spaces, $\mathcal{O} \subseteq X$ be an open set and $L(X,Y)$ be the set of all bounded linear mappings from X to Y. The function $F\colon \mathcal{O} \mapsto Y$ is called

- slantly differentiable at $u \in \mathcal{O}$ if there exists a mapping $F^\circ\colon \mathcal{O} \mapsto L(X,Y)$ so that the family $\{F^\circ(u+v)\colon v \text{ sufficiently small}\}$ is uniformly bounded in the operator norm and
$$\lim_{v\to 0} \frac{1}{\|v\|_X} \|F(u+v) - F(u) - F^\circ(u+v)v\|_Y = 0,$$
where F° is called a slanting function for F in u;
- slantly differentiable in $\mathcal{O}$ if there exists $F^\circ\colon \mathcal{O} \mapsto L(X,Y)$ such that F° is slanting function for F at every $u \in \mathcal{O}$. We say that F° is slanting function for F on $\mathcal{O}$.

Theorem 4.1 (see [2])**.** *Let a function F be slantly differentiable in $\mathcal{O}$ with the slanting function F°. Suppose that $u^* \in \mathcal{O}$ is a solution of the equation*

$$F(u) = 0.$$

If $F^\circ(u)$ is nonsingular on $\mathcal{O}$ and $\{\|F^\circ(u)^{-1}\|\colon u \in \mathcal{O}\}$ is bounded, then the semismooth Newton iterations

$$u^{(k+1)} = u^{(k)} - F^\circ(u^{(k)})^{-1} F(u^{(k)})$$

converge superlinearly to u^* for sufficiently small $\|u^{(0)} - u^*\|_X$.

E x a m p l e 4.1. For $F(u) = u^+$, $u \in \mathbb{R}$, the slanting function is

$$F^\circ(u) := \begin{cases} 1, & u > 0, \\ \delta, & u = 0,\ \delta \in \mathbb{R} \text{ arbitrary}, \\ 0, & u < 0. \end{cases}$$

For the application of the semismooth Newton method we introduce the mapping

$$F(u) := Ku + \sum_{i=1}^{m} B_{+i}(u - L_{+i})^+ - \sum_{j=1}^{n} B_{-j}(u + L_{-j})^- - f. \tag{4.6}$$

A solution of (4.5) satisfies $F(u) = 0$ with the slanting function

$$F^\circ(u) := K + \sum_{i=1}^{m} B_{+i} D(\mathcal{A}_{+i}(u)) - \sum_{j=1}^{n} B_{-j} D(\mathcal{A}_{-j}(u)),$$

where the diagonal matrices $D(\mathcal{A}_{+i}(u))$, $D(\mathcal{A}_{-j}(u))$ correspond to the active sets

$$\mathcal{A}_{+i}(u) := \{k\colon\ u_k > L_{+i},\ k \text{ odd}\}, \quad \mathcal{A}_{-j}(u) := \{k\colon\ u_k < -L_{-j},\ k \text{ odd}\},$$

respectively. Thus the diagonal entries are

$$D(\mathcal{A}_{+i}(u))_{kk} = \begin{cases} 1, & k \in \mathcal{A}_{+i}(u), \\ 0, & \text{otherwise}, \end{cases} \qquad D(\mathcal{A}_{-j}(u))_{kk} = \begin{cases} 1, & k \in \mathcal{A}_{-j}(u), \\ 0, & \text{otherwise}. \end{cases}$$

The semismooth Newton iterations can be reformulated as follows

$$\Bigl(K + \sum_{i=1}^{m} B_{+i} D(\mathcal{A}_{+i}(u^{(k)})) - \sum_{j=1}^{n} B_{-j} D(\mathcal{A}_{-j}(u^{(k)}))\Bigr) u^{(k+1)} = f. \tag{4.7}$$

We arrive at the implementation of the semismooth Newton method (SSNM) in terms of the active set terminology.

Algorithm SSNM:

1. Set matrices K, B_{+i}, B_{-j}, the vector f and the precission ε.
2. Choose the initial approximation $u^{(0)}$.
3. For the kth iteration $u^{(k)}$ repeat:
    a) set the active sets $\mathcal{A}_{+i}(u^{(k)})$ and $\mathcal{A}_{-j}(u^{(k)})$,
    b) solve (4.7),
    c) evaluate the terminating criterion, $E_k = (\|u^{(k+1)} - u^{(k)}\|)/(\|u^{(k)}\|) \leqslant \varepsilon$.

R e m a r k 4.1. If the algorithm SSNM converges, then it finds a solution of the discretizated problem $F(u)=0$ for F defined by (4.6) in a finite number of iterations. The reason is the fact that the number of different active sets is finite.

4.2. Algorithm based on the method of successive approximations. Another way how to compute a solution of our problem is the method of successive approximations (SAM). To this end we reformulate the solution of the discretizated problem as the fixed point of a mapping.

Let us denote $X=\{\psi\, v_h\colon\ v_h\in V_h\}$, $X\subset C^0((a,b))$ and $\mathcal{K}^{\pm}$ the set of all mappings from V_h into X of the form

$$v_h\mapsto\sum_{i=1}^{m}k_{1,i}(v_h-L_i)^+-\sum_{j=1}^{n}k_{2,j}(v_h+L_j)^-$$

for constants L_i, L_j and functions $k_{1,i},k_{2,j}\in L^\infty((a,b))$.

Proposition 4.1. *For each mapping $\psi\in\mathcal{K}^{\pm}$ there exists $\psi_0\in\mathcal{K}^{\pm}$, $\psi_0\neq-\psi$, a linear operator $\ell\colon\ V_h\mapsto V_h$ and a function $c\in L^\infty((a,b))$ such that*

$$\psi\, v_h+\psi_0\, v_h=\ell v_h-c$$

holds for all $v_h\in V_h$.

E x a m p l e 4.2. If we put $\psi\, v_h=k_N(v_h-L_1)^+-k_P(v_h+L_2)^-$, then

$$\begin{aligned}\psi_0\, v_h&=\ -k_N(v_h-L_1)^-+k_P(v_h+L_2)^+,\\ \ell v_h&=(k_N+k_P)v_h\end{aligned}$$

and

$$c=k_NL_1-k_PL_2.$$

Applying Proposition 4.1 in (4.3), we obtain

$$a_0(u_h,\varphi_k)+(\ell u_h,\varphi_k)_r=\mathcal{F}(\varphi_k)+(\psi_0\, u_h,\varphi_k)_r+(c,\varphi_k)_r.$$

Let us define $\mathcal{S}\colon\ V_h\to V_h$ as the operator assigning to every $w_h\in V_h$ the solution u_h of the equation

$$a_0(u_h,\varphi_k)+(\ell\, u_h,\varphi_k)_r=\mathcal{F}(\varphi_k)+(\psi_0\, w_h,\varphi_k)_r+(c,\varphi_k)_r.$$

Then the discrete solution $u_h \in V_h$ is a fixed point of $\mathcal{S}$. Consequently, we can use the SAM that leads to the following iterative process

$$(K+L)u^{(n+1)} = f + f_{\psi_0}^{(n)} + c, \tag{4.8}$$

where K and f are the same as before, the elements of $L \in \mathbb{R}^{(2N+2)\times(2N+2)}$ are $L_{lk} = (\ell\varphi_l, \varphi_k)_r$, the vector $f_{\psi_0}^{(n)} \in \mathbb{R}^{2N+2}$ is such that $f_{\psi_0,k}^{(n)} = (\psi_0\, u^{(n)}, \varphi_k)_r$ for all $k, l = 1, \ldots, 2N+2$, and $c \in \mathbb{R}^{2N+2}$, $c_k = (c, \varphi_k)_r$.

Algorithm SAM:

1. Set matrices K, L, vectors f, c and the precission ε.
2. Choose the initial approximation $u^{(0)}$.
3. For the kth iteration $u^{(k)}$ repeat:
   a) set the vector $f_{\psi_0}^{(n)}$,
   b) solve (4.8),
   c) evaluate the same terminating criterion as in the algorithm SSNM.

R e m a r k 4.2. Note that $K+L$ is a positive-definite matrix which does not change during the iterative process. Therefore, we have reached a uniquely solvable stable scheme for any starting vector $u^{(0)}$.

The convergence of the SAM will be tested numerically.

5. Numerical examples

In this section we will experimentaly compare solutions computed by the SAM and the SSNM algorithms. The terminating tolerance will be the same in both cases $\varepsilon = 10^{-5}$. Moreover, we compute the relative residuum norm $\mathtt{R}$ of (4.7). The solutions computed by the SSNM and the SAM algorithms will be denoted by u_h^{SSNM} and u_h^{SAM}, respectively, and we will evaluate

$$\mathtt{M} = \max_i |u_h^{\mathrm{SSNM}}(r_i) - u_h^{\mathrm{SAM}}(r_i)|.$$

E x a m p l e 5.1. We compute the deflection of a steel plate ($E = 2.14 \cdot 10^{11}$, $\sigma = 0.29$) with $a = 1$, $b = 5$, $t = 0.01$. It is simply supported and loaded by the bending moments on both edges, i.e. $f(r) = 0$ on $r \in (a, b)$ so that the mixed boundary conditions are

$$\mathcal{M}u(a) = 5.5 \cdot 10^4, \quad u(a) = 0.0,$$
$$\mathcal{M}u(b) = 9.0 \cdot 10^4, \quad u(b) = 0.0.$$

The operator ψ is given as $\psi u = 5.0 \cdot 10^6 u^-$. The previous theory yields that the problem has a unique solution. The resulting functions for the deflection and its derivative are shown in Fig. 4.

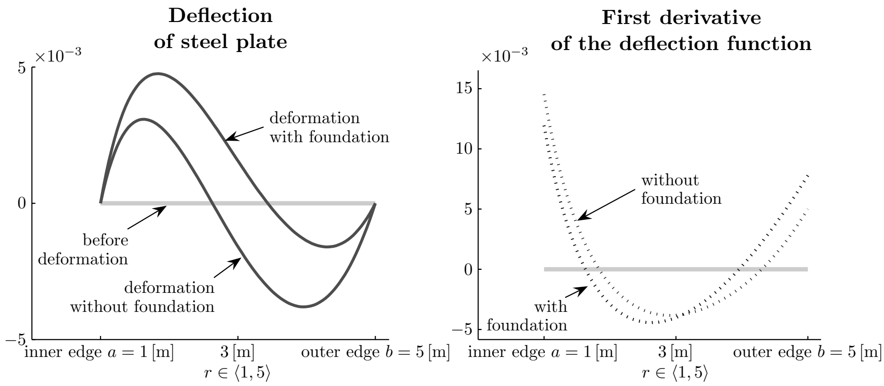

Figure 4. The computed deflection with and without foundation.

Tab. 1 presents results for increasing size of the discrete problem N. The small values of `R` for the SSNM algorithm are due to the finite terminating property mentioned in Remark 4.1. When the active set corresponding to the solution is found, then the problem simplifies to the linear one and its solution is obtained from an appropriate system of linear equations. In accordance to check the SAM convergence we will focus to the last two columns in the table. The value of the SAM residuum decreases as well as the value of `M`.

N	iteration		residuum `R`		SSNM vs. SAM
	SSNM	SAM	SSNM	SAM	`M`
2	3	14	2.23060e−017	8.88931e−003	7.604785e−004
10	6	21	7.09753e−018	2.64937e−004	1.584678e−005
20	6	21	9.64389e−018	1.68153e−005	1.042335e−007
50	7	22	1.18721e−017	4.23683e−007	2.948362e−007
200	7	22	9.78820e−018	1.64048e−009	5.394307e−008
500	7	22	3.73446e−017	4.19235e−011	4.188592e−008

Table 1. Comparison of the SSNM and the SAM algorithm for increasing N.

Example 5.2. We compute the deflection of a steel plate as in Example 5.1 with different boundary conditions. It is loaded by the bending moments on both edges so that Neumann's boundary condition are

$$\mathcal{M}u(a) = 5.5 \cdot 10^4, \qquad \mathcal{T}u(a) = 0.0,$$
$$\mathcal{M}u(b) = 9.0 \cdot 10^4, \qquad \mathcal{T}u(b) = 0.0,$$

and $f(r) = 0$ for all $r \in (a, b)$. The operator ψ is $\psi\, u = 5.0 \cdot 10^3 u^+$. Let us note that the solution of this problem is not unique due to semi-coercivity and $\mathcal{F}(1) = 0$.

For the SSNM and the SAM iteration histories see Tabs. 2 (a) and 2 (b), respectively. Comparing these tables, we can check typical behaviours of the SSNM and the SAM iterations in semi-coercive cases. The number of the SAM iterations increases when the value of the environment response increases (the value of the coefficient $5.0 \cdot 10^3$ from the operator ψ). The last rows of Tab. 2 (a) demonstrate again the finite terminating property of the SSNM algorithm. The problem has infinitely many

SSNM			SAM		
iter	error $E(k)$	residuum R	iter	error $E(k)$	residuum R
1		7.44980e − 003	1	2.73600e−001	8.57589e−009
2	5.37924e+001	7.24627e−008	2	1.32280e−001	4.76288e−009
3	6.99420e−001	1.60708e−008	3	7.90095e−002	3.08345e−009
4	2.39883e−001	7.87751e−009	4	5.33184e−002	2.14118e−009
5	9.48147e−002	3.73564e−009		⋮	⋮
6	4.84808e−002	1.60091e−009			
7	6.02343e−002	2.20696e−017	153	1.03461e−005	5.07242e−010
8	0.00000e+000	2.20696e−017	154	9.91946e−006	5.07205e−010

(a) The SSNM iterations (b) The SAM iterations

Table 2. Iteration history of the SSNM and the SAM (for $N = 20$)

solutions which are equal modulo constant function with a value from the interval $(-\infty, 0\rangle$, see Remark 3.5. The SSNM and the SAM difference is

$$\begin{aligned} u_h^{\mathrm{SSNM}}(r_i) - u_h^{\mathrm{SAM}}(r_i) &= \; -0.0023, \\ \frac{\mathrm{d}}{\mathrm{d}r}\, u_h^{\mathrm{SSNM}}(r_i) - \frac{\mathrm{d}}{\mathrm{d}r}\, u_h^{\mathrm{SAM}}(r_i) &= \quad 0.0 \end{aligned}$$

for all $i = 1, 2 \ldots N+1$. The graphs of the resulting deflection functions and their derivatives are shown in Fig. 5.

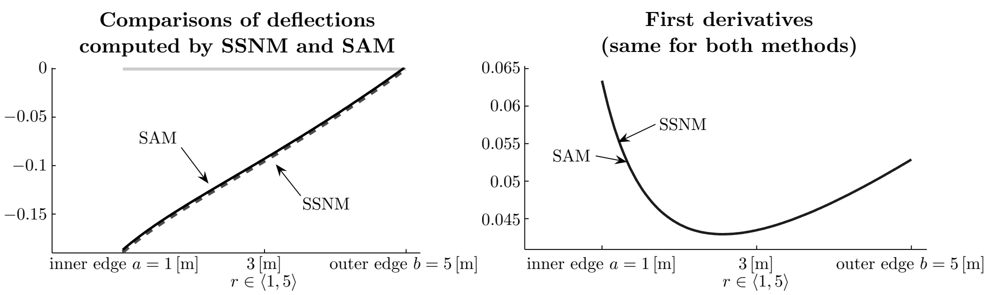

Figure 5. The computed deflections for both numerical methods.

R e m a r k 5.1. In the first iteration of the SSNM there is a positive semi-definite matrix on the left-hand side whenever all the active sets are empty. Thus if the initial approximation $u^{(0)}$ is not suitably chosen, the algorithm can fail. Theorem 4.1 indicates that we need the initial guess to be "sufficiently close" to the discrete solution. In the numerical examples 5.1 and 5.2 the initial approximation $u^{(0)}$ satisfies the boundary conditions in the sense of the finite differences and it approximately respects the prescribed volume forces. According to Remark 4.2 and the numerical experiments, the choice of $u^{(0)}$ does not cause any divergent behaviour of the SAM algorithm. Thus it is choosen $u^{(0)} = 0 \in \mathbb{R}^{2N+2}$.

6. Comments and conclusions

We presented the coercivity proof for the convex nonsmooth functional $\mathcal{P}_\psi$ representing the deflection of a thin annular plate influenced by an elastic obstacle. The main tool is the cone decompositon theorem in the Hilbert space. Consequently, we arrived at the existence and uniqueness results of the variational formulation of the problem.

After the discretization we derived two computational algorithms based on the semismooth Newton method (SSNM) and the method of successive approximations (SAM). Their comparison was realized by the numerical examples. It is experimentally demonstrated that the SSNM is faster than the SAM, especially for the semi-coercive problems. However, the SSNM algorithm requires a sufficiently accurate initial approximation, as is indicated by the convergence theorem (Theorem 4.1). Furthermore, when an iteration leads to empty active sets, the corresponding slanting function value is a singular matrix so that the SSNM may fail. Fortunately, such behaviour was observed only in the first iteration. Note that a suitable initial approximation for the SSNM can be generated by a few steps of SAM.

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Author's address: *I. Svobodová*, Department of Mathematics and Descriptive Geometry, VŠB TU, 17. listopadu 15/2172, 708 33 Ostrava-Poruba, Czech Republic, e-mail: ivona.svobodova@vsb.cz.